Homework 10 - Predictive Modeling in Finance and Insurance

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1. Building a decision tree

a. Determining roots and first split

The split at the root node has to put at least one data point into each child node. Since the heights have the possible values $\{1.4, 1.5, 1.6, 1.7, 1.8\}$, there are 4 possible splits on height, and since gender has the possible values $\{M, F\}$, there is 1 possible split on gender. I examine each of these and their corresponding $SSE = SSE_l + SSE_r$:

- Splitting on Height ≤ 1.4: In this case, the left child has one data point, data point 6; the mean response is just $\bar{y}_l = y_6 = 55$, so $SSE_l = 0$. The other side contains the other 6 points; here the mean response is $\bar{y}_r = \frac{88+82+60+73+77+80}{6} = \frac{230}{3}$. Thus, $SSE_r = \sum_{i \in R}(y_i - \bar{y}_r)^2 = 459.33$. Therefore, $SSE = 0 + 459.33 = 459.33$.
- Splitting on Height ≤ 1.5: In this case, the left child has 3 data points: 3, 4, and 6. Therefore, the mean response is $\bar{y}_l = \frac{60+77+55}{3} = 64$. Therefore, $SSE_l = \sum_{i \in L}(y_i - \bar{y}_l)^2 = 266$. The other side contains the other 4 data points, and the mean response is $\bar{y}_r = \frac{88+82+73+80}{4} = 80.75$. Here, $SSE_R = \sum_{i \in R}(y_i - \bar{y}_r)^2 = 114.75$. So, $SSE = 266 + 114.75 = 380.75$.
- Splitting on Height ≤ 1.6: In this case, the left child has 4 data points: 1,3,5, and 6. The mean response is $\bar{y}_l = \frac{88+60+77+55}{4} = 70$. Consequently, $SSE_l = \sum_{i \in L}(y_i - \bar{y}_l)^2 = 698$. Then, the right child has 3 data points: 2, 5, and 7. The mean response is $\bar{y}_r = \frac{82+73+80}{3} = \frac{235}{3}$. So, $SSE_r = \sum_{i \in R}(y_i - \bar{y}_r)^2 = 44.67$. Finally, $SSE = 698 + 44.67 = 742.67$.
- Splitting on Height ≤ 1.7: In this case, the left child has 6 data points: 1,2,3,5,6,7. The mean response is $\bar{y}_l = \frac{88+82+60+77+55+80}{6} = \frac{221}{3}$. Then, $SSE_l = \sum_{i \in L}(y_i - \bar{y}_l)^2 = 861.33$. The right child only has one data point: point 4. So, the mean response is $\bar{y}_r = y_4 = 73$, and $SSE_r = 0$. Finally, $SSE = 861.33 + 0 = 861.33$.
- Splitting on Gender < 0.5: Encode $\{M : 0, F : 1\}$. Thus, all males (points 1,4,5, 7) are in the left child, and all females (points 2,3, 6) are in the right child. Thus, $\bar{y}_l = \frac{88+73+77+80}{4} = 79.5$, and $SSE_l = \sum_{i \in L}(y_i - \bar{y}_l)^2 = 121$. Similarly, $\bar{y}_r = \frac{82+60+55}{3} = 65.67$ and $SSE_r = \sum_{i \in R}(y_i - \bar{y}_r)^2 = 412.67$. Thus, $SSE = 121 + 412.67 = 533.67$.

Summarizing in a table:

Split	Points in L	Points in R	SSE_l	SSE_r	SSE
Height ≤ 1.4	6	1,2,3,4,5,7	0	459.33	459.33
Height ≤ 1.5	3,5,6	1,2,4,7	266	114.75	380.75
Height ≤ 1.6	1,3,5,6	2,4,7	698	44.67	742.67
Height ≤ 1.7	1,2,3,5,6,7	4	861.33	0	861.33
Gender < 0.5	1,4,5,7	2,3,6	121	412.67	533.67

Therefore, the root node is split on Height $\leq$ **1.5**, the left leaf contains $3, 5, 6$, the right leaf contains $1, 2, 4, 7$, and their SSE's are 266 and 114.75 respectively, with an association SSE of 380.75.

b. Determining second split

At this juncture, it is possible that either leaf is subject to the next split, as both still have more than 2 data points. However, $SSE_l > SSE_r$; if I were to find a split that reduced SSE_l by more than the totality of SSE_r, I can verify that L would be split next, as this would reduce total SSE by more than any potential split in R. In leaf L, I notice that the heights are $\{1.4, 1.5\}$ and the genders are $\{M, F\}$, so there are two possible splits (I'll refer to the left child as L_1 and the right as L_2):

- Splitting on Height ≤ 1.4: In this case, only data point 6 is in L_1, so $\bar{y}_{L_1} = y_6 = 55$ and $SSE_{L_1} = 0$. Meanwhile, data points 3 and 5 are in L_2, so $\bar{y}_{L_2} = \frac{60+77}{2} = 68.5$ and $SSE_{L_2} = \sum_{i \in L_2}(y_i - \bar{y}_{L_2})^2 = 144.5$. So, $SSE = SSE_{L_1} + SSE_{L_2} = 144.5$.
- Splitting on Gender < 0.5: In this case, the males, or only data point 5, are in L_1; therefore, $\bar{y}_{L_1} = y_5 = 77$ and $SSE_{L_1} = 0$. Then, the females, or data points 3 and 6, are in L_2; therefore, $\bar{y}_{L_2} = \frac{60+55}{2} = 57.5$ and $SSE_{L_2} = \sum_{i \in L_2}(y_i - \bar{y}_{L_2})^2 = 12.5$. So, $SSE = SSE_{L_1} + SSE_{L_2} = 12.5$.

Note that, when the left leaf is split on gender, the new SSE on that side of the tree is 12.5. Note that $SSE_L - SSE = 266 - 12.5 = 243.5$, a reduction in SSE greater than the entirety of the SSE in the right leaf. Thus, as this split maximizes reduction in SSE of the data points in the left leaf, and the reduction is greater than the entirety of the SSE in the right leaf, we conclude that:

- The second split is on the left leaf, and the split is Gender < 0.5.
- Data point 5 is in L_1, and data points 3 and 6 are in L_2.
- $SSE_{L_1} = 0$ and $SSE_{L_2} = 12.5$
- $SSE = SSE_{L_1} + SSE_{L_2} = 12.5$; $SSE_{\text{total}} = 12.5 + 144.75 = 157.25$.

c. Complete decision tree build

Note that L_1 has 1 data point, and L_2 has 2 data points. Therefore, both have reached the minimum of data points in a leaf, and cannot be split any more. However, leaf R has 4 data points (1,2,4,7); therefore, it can be split once again. The heights $\{1.6, 1.7, 1.8\}$ and the genders are $\{M, F\}$ among the 4 data points; therefore, there can be $2 + 1 = 3$ possible splits (calling the left child R_1 and right child R_2).

- Splitting on Height ≤ 1.6: Here, data point 1 is in R_1, so $\bar{y}_{R_1} = y_1 = 88$ and $SSE_{R_1} = 0$. Then, data points 2,4, and 7 are in R_2, so $\bar{y}_{R_2} = \frac{82+73+80}{3} = \frac{235}{3} = 78.33$; thus, $SSE_{R_2} = \sum_{i \in R_2}(y_i - \bar{y}_{R_2})^2 = 44.67$. So $SSE_R = 0 + 44.67 = 44.67$.
- Splitting on Height ≤ 1.7: Here, data points 1,2, and 7 are in R_1; therefore, $\bar{y}_{R_1} = \frac{88+82+80}{3} = \frac{250}{3} = 83.33$. So, $SSE_{R_1} = \sum_{i \in R_1}(y_i - \bar{y}_{R_1})^2 = 34.67$. The only data point remaining is data point 4, which is in R_2; so $\bar{y}_{R_2} = y_4 = 73$ and $SSE_{R_2} = 0$. Thus, $SSE_R = 34.67 + 0 = 34.67$.
- Splitting Gender < 0.5: Here, all the males are in R_1, or data points 1,4, and 7. Here, $\bar{y}_{R_1} = \frac{88+73+80}{3} = \frac{241}{3} = 80.33$. Thus, $SSE_{R_1} = \sum_{i \in R_1}(y_i - \bar{y})^2 = 112.67$. There is only one female in R: data point 2. As a result, $\bar{y}_{R_2} = y_2 = 82$ and $SSE_{R_2} = 0$. So, $SSE_R = 112.67 + 0 = 112.67$.

The best split here would be on Height ≤ 1.7, as this minimizes the SSE_R. With this split, $SSE = SSE_L + SSE_R = 12.5 + 34.67 = \mathbf{47.17}$. However, R_1 still has 3 data points, so it must be split once more to reach the minimum. R_1 contains the points 1,2, and 7. The potential heights are $\{1.6, 1.7\}$ and genders are $\{M, F\}$, thus giving way to two final splits (I will call the left child R_{1_a} and R_{1_b}):

- Splitting on Height ≤ 1.6: Here, point 1 is the only point in R_{1_a}, so $\bar{y}_{R_{1_a}} = y_1 = 88$ and $SSE_{R_{1_a}} = 0$. Then, points 2 and 7 are in R_{1_b}, so $\bar{y}_{R_{1_b}} = \frac{82+80}{2} = 81$ and $SSE_{R_{1_b}} = \sum_{i \in R_{1_b}}(y_i - \bar{y}_{R_{1_b}})^2 = 2$. So, $SSE_{R_1} = 0 + 2 = 2$.
- Splitting on Gender < 0.5: All the males, or points 1 and 7, would be in R_{1_a}. Thus, $\bar{y}_{R_{1_a}} = \frac{88+80}{2} = 84$ and $SSE_{R_{1_a}} = \sum_{i \in R_{1_a}}(y_i - \bar{y}_{R_{1_a}})^2 = 32$. This would leave the only female in the subset, or data point 2, in R_{1_b}, so $\bar{y}_{R_{1_b}} = y_2 = 82$ and $SSE_{R_{1_b}} = 0$. Finally, $SSE_{R_1} = 0 + 32 = 32$

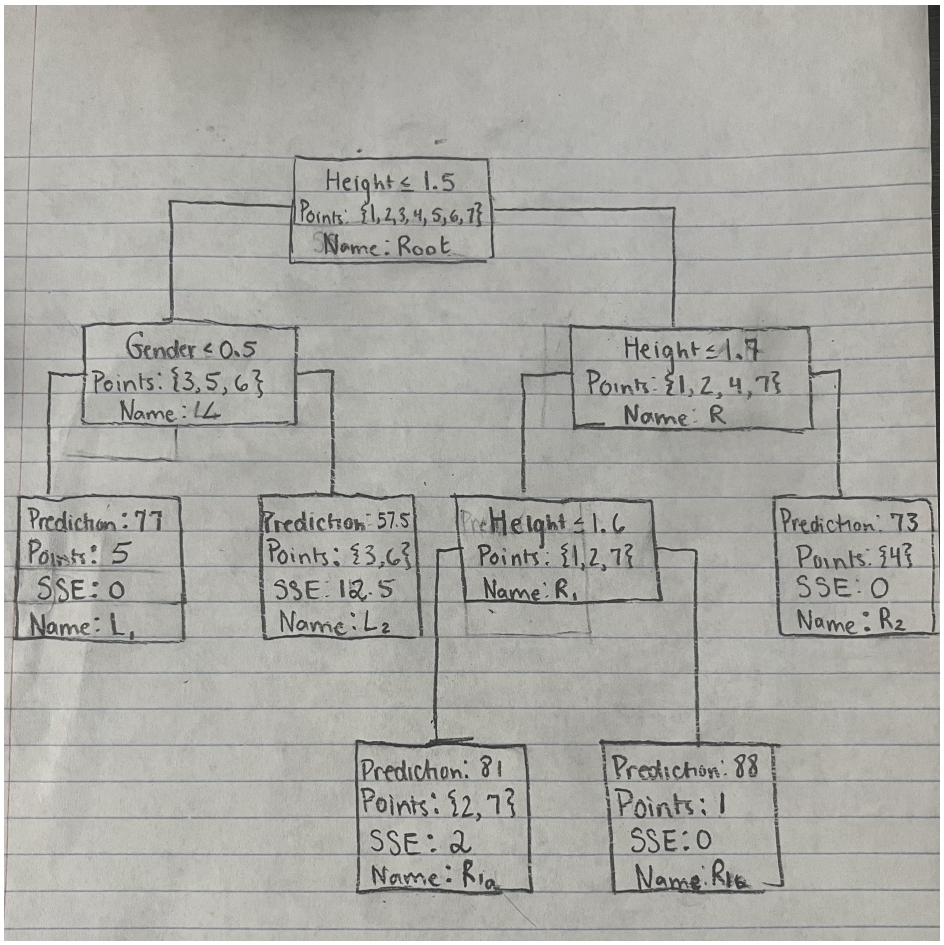
Clearly, splitting on Height ≤ 1.6 reduces the SSE the most. Therefore, this split is undertaken, and R_{1_a} has 1 data point while R_{1_b} has 2, so each leaf node has 2 or less data points. The tree is complete, with final $SSE = SSE_L + SSE_R = 12.5 + SSE_{R_1} + SSE_{R_2} = 12.5 + 2 + 0 = 14.5$. The leaves, the condition to be in those leaves, the prediction (sample mean), the points in the leaves, and the corresponding SSE can be found in the table below:

Leaf Name	Condition	Prediction	Points	SSE
L_1	Height $\leq 1.5 \wedge$ Gender = M	77	5	0
L_2	Height $\leq 1.5 \wedge$ Gender = F	57.5	3,6	12.5
R_{1_a}	Height $\in (1.5, 1.6]$	88	1	0
R_{1_b}	Height $\in (1.6, 1.7]$	81	2,7	2
R_2	Height > 1.7	73	4	0

d. depict entire tree

The tree is shown below:

```
knitr::include_graphics("tree.jpg")
```



e. Predicting weight for male, height 1.45 m

Using this tree, and the fact that male encodes for 0 in gender, I note that a male of height 1.45 meters falls into leaf L_1. Thus, the predicted weight for said male would be **77** kg.

f. What is sequence of α_T?

To determine the list of α_T values that prune nodes, I have to first know the total SSE after each split. Therefore, I list the trees in order after each split, the number of leaves they have, and the SSE at the current time:

Tree	Latest Split	SSE	Number of Leaves
1	N/A	861.71	1
2	Height ≤ 1.5	380.75	2
3	Gender < 0.5	157.25	3
4	Height ≤ 1.7	47.17	4
5	Height ≤ 1.6	14.5	5

The Tree score formula is $SSE + \alpha|T|$, where T is the number of leaves. Therefore, I pick tree 4 over 5 when:

$$47.17 + \alpha_1 * 4 < 14.5 + \alpha_1 * 5 \rightarrow \alpha_1 > 47.17 - 14.5 = 32.67$$

I do a similar procedure for the other consecutive trees:

$$157.25 + \alpha_2 * 3 < 47.17 + \alpha_2 * 4 \rightarrow \alpha_2 > 157.25 - 47.17 = 110.08$$
$$380.75 + \alpha_3 * 2 < 157.25 + \alpha_3 * 3 \rightarrow \alpha_3 > 380.75 - 157.25 = 223.5$$
$$861.71 + \alpha_4 * 1 < 380.75 + \alpha_4 * 2 \rightarrow \alpha_4 > 861.71 - 380.75 = 480.96$$

Rounding to the nearest whole number above the minimum for pruning a given leaf, I generate the following α_T values under which each tree is optimal, (starting from the most complex tree and iteratively pruning back to the root):

$$\alpha_T = \{0, 33, 111, 224, 481\}$$

2. Building Decision Tree with Train-Test Split

```
library(MASS)
library(randomForest)
library(rpart)
library(rpart.plot)
library(tree)
library(gbm)
```

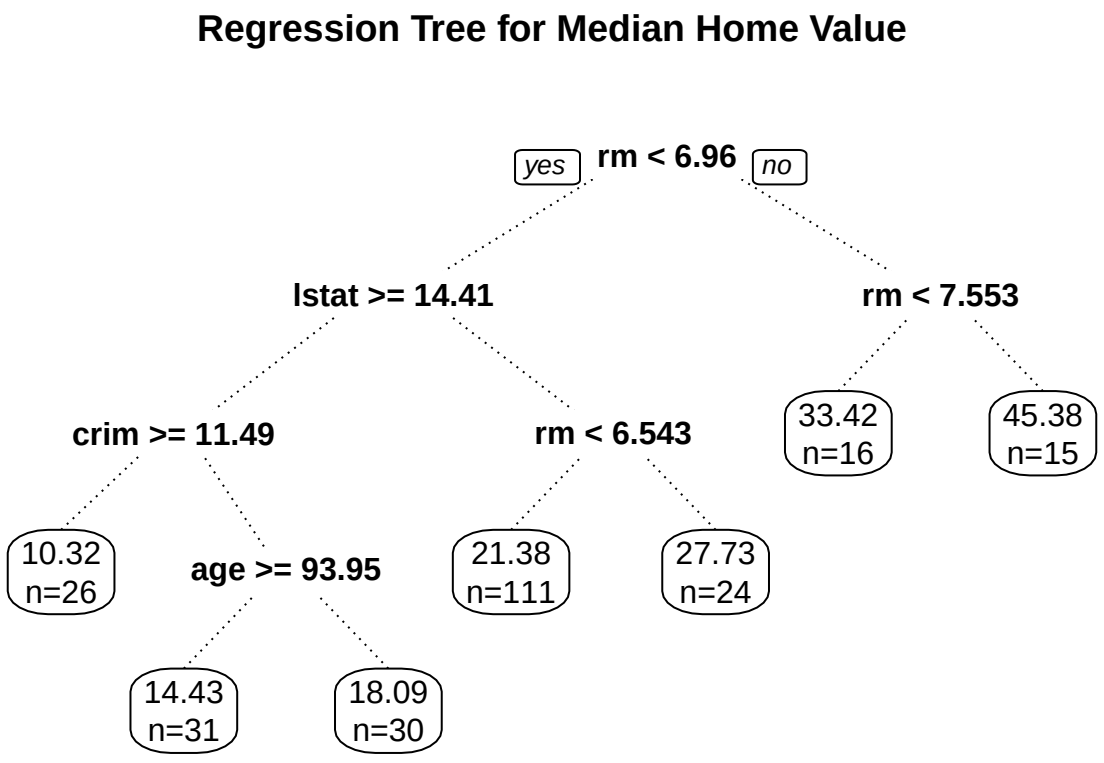
a. Splitting dataset

```
set.seed(1)
train_sample <- sort(sample(seq_len(nrow(Boston)),
                      size = floor(0.5 * nrow(Boston))))
trainData <- Boston[train_sample,]
testData <- Boston[-train_sample,]
```

b. Building decision tree

i. Fitting Regression Tree

```
rtree <- rpart(medv ~ ., data = trainData, method = "anova")
prp(rtree, main = "Regression Tree for Median Home Value", roundint = FALSE,
    extra = 1, digits = 4, branch.lty = 3)
```



Note that the number of rooms is split on first, as well as being the only variable split on multiple times. So,

rm is the most important variable. The other variables split on (ordered by depth in tree at which they are split on) are *lstat*, *crim*, and *age*. So the order of importance for variables is $1.rm, 2.lstat, 3.crim, 4.age$.

ii. Calculating test MSE

I predict using the fitted tree, and calculate test MSE:

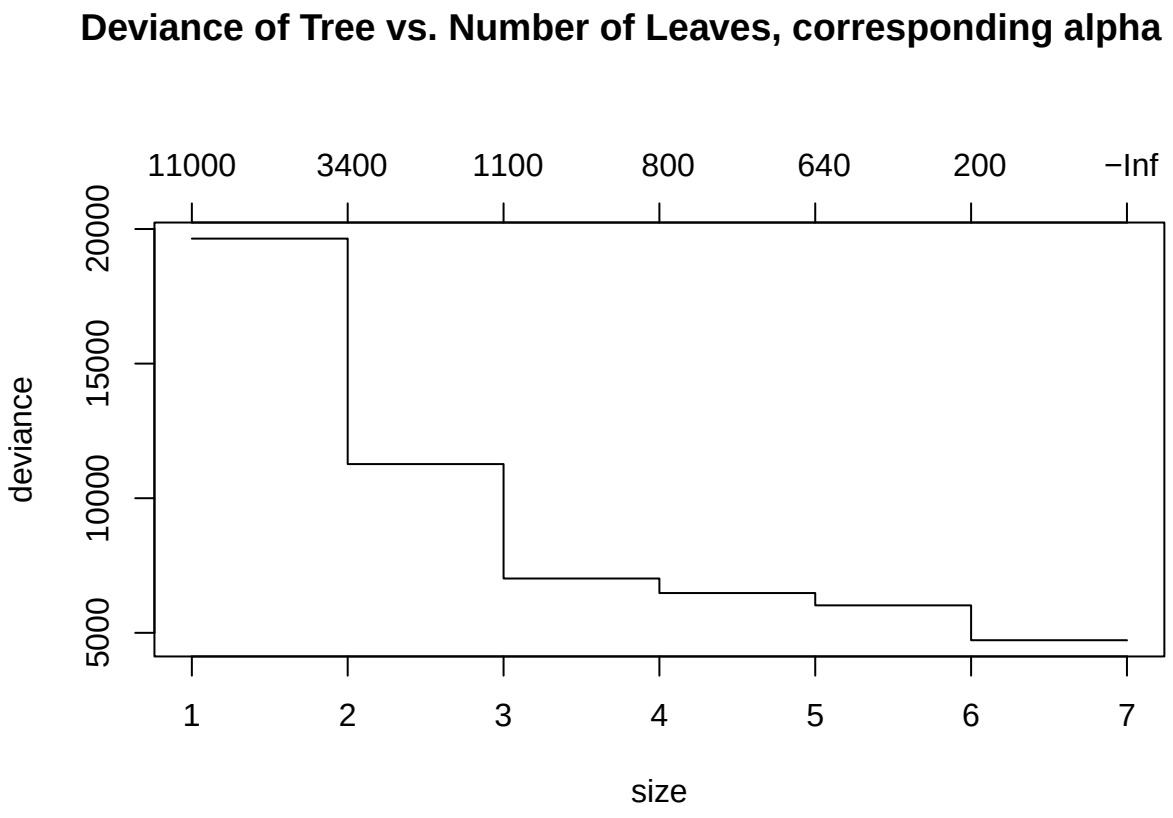
```
predictions <- predict(rtree, newdata = testData)
SSE <- sum((predictions - testData$medv)^2)
MSE = SSE/(length(predictions))
sprintf("Test MSE: %.5f", MSE)
```

```
## [1] "Test MSE: 35.28688"
```

iii. Cross-Validation for optimal Tree

I do 10-fold cross validation and plot the deviance (note that I use the tree method instead of rpart, as rpart plots better):

```
set.seed(1)
rtree <- tree(medv ~ ., data = trainData)
cv <- cv.tree(rtree, K = 10, FUN = prune.tree)
par(oma = c(0,0,2,0))
plot(cv)
title(main = "Deviance of Tree vs. Number of Leaves, corresponding alpha",
      outer = TRUE)
```



The deviance is very close between the tree with 6 leaves and 7 leaves. The seven leaf tree is the one already created; therefore, I examine the test error of the 6 leaf tree:

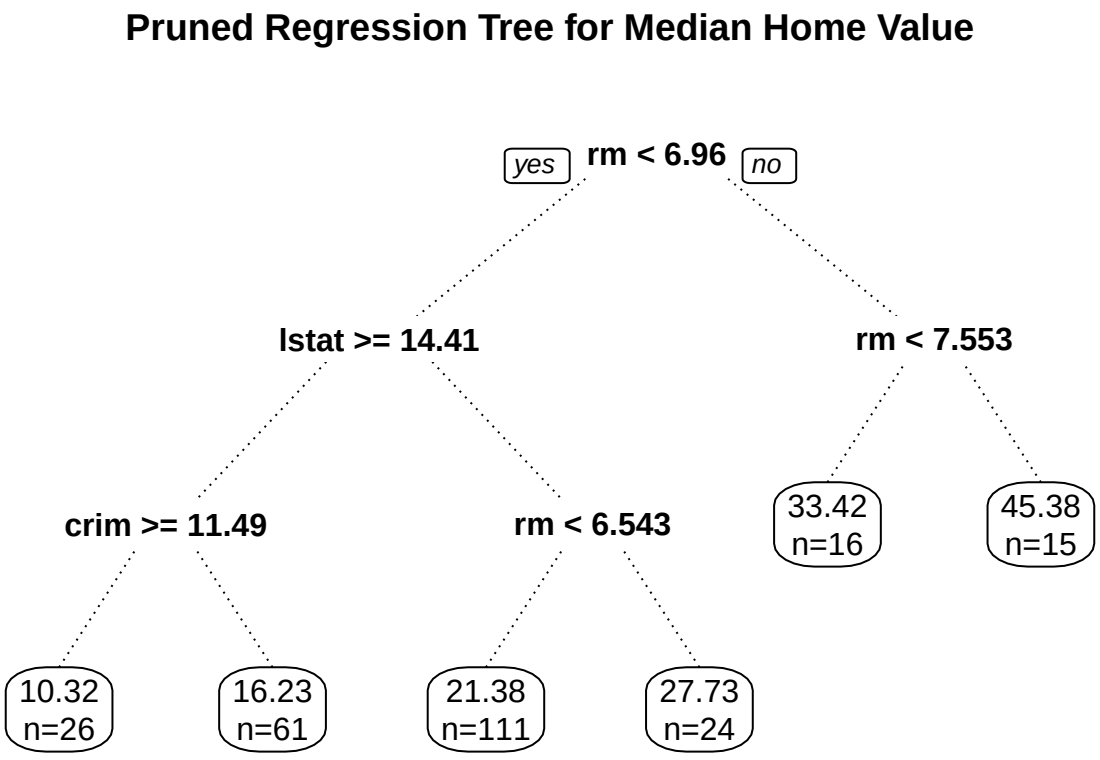
```
rtree6 <- prune.tree(rtree, best = 6)
pred6 <- predict(rtree6, newdata = testData)
SSE6 <- sum((pred6 - testData$medv)^2)
MSE6 = SSE6/(length(pred6))
sprintf("Test MSE of 6 leaf tree: %.5f", MSE6)
```

```
## [1] "Test MSE of 6 leaf tree: 35.16439"
```

Note that $\text{MSE}_6 = 35.16439 < 35.28688 = \text{MSE}_7$. Thus, the the tree with **6 leaves** is the best.

iv. Pruning the tree

```
replot <- rpart(medv ~ ., data = trainData, method = "anova")
replot <- prune(replot, cp = 0.02)
prp(replot, main = "Pruned Regression Tree for Median Home Value",
    roundint = FALSE, extra = 1, digits = 4, branch.lty = 3)
```



v. Test error

The test error is was calculated as: $\text{MSE}_6 = 35.16439$.

c. Building Bagging Model

I create a bagging model, and calculate the test MSE:

```
set.seed(1)
rB <- randomForest(medv ~., ntree = 500, mtry = dim(Boston)[2] - 1,
```

```
                  data = trainData, importance = TRUE)
predictionsrB <- predict(rB, newdata = testData)
MSErB <- mean((predictionsrB - testData$medv)^2)
sprintf("Test MSE of Bagging Model: %.5f", MSErB)
```

```
## [1] "Test MSE of Bagging Model: 23.27780"
```

I also return the top three variables in importance as measured by the mean decrease in MSE:

```
sortedImportance <- sort(rB$importance[,2], decreasing = TRUE)
sortedImportance[1:3]
```

```
##         rm      lstat       crim
## 11998.3365  4967.9514   814.6526
```

So, this suggests that *rm*, *lstat*, and *crim* are the 3 most important variables in decreasing MSE.

d. Building Random Forest

I create a random Forest, and calculate the test MSE:

```
set.seed(1)
rF <- randomForest(medv ~., ntree = 500, data = trainData, importance = TRUE)
predictionsrF <- predict(rF, newdata = testData)
MSErF <- mean((predictionsrF - testData$medv)^2)
sprintf("Test MSE of random Forest Model: %.5f", MSErF)
```

```
## [1] "Test MSE of random Forest Model: 18.73938"
```

e. Building boosted tree

i. Building Model, Calculating testMSE

I build the tree, and then calculate:

```
set.seed(1)
gbMod <- gbm(medv ~ ., data = trainData, distribution = "gaussian",
             n.trees = 1000, interaction.depth = 4)
predictionsgB <- predict(gbMod, newdata = testData)
MSEgB <- mean((predictionsgB - testData$medv)^2)
sprintf("Test MSE of gradient Boosted Model: %.5f", MSEgB)
```

```
## [1] "Test MSE of gradient Boosted Model: 18.47792"
```

ii. Determining optimal interaction depth d

I generate each of the gradient boosted models, and find the interaction depth parameter that yields the minimum MSE:

```
set.seed(1)
mseVec <- c()
for(i in 2:10){
  gbTest <- gbm(medv ~ ., data = trainData, distribution = "gaussian",
              n.trees = 1000, interaction.depth = i)
  predgBTest <- predict(gbTest, newdata = testData)
  MSEgBTest <- mean(((predgBTest - testData$medv)^2))
  mseVec <- append(mseVec, MSEgBTest)
}
```

```
names(mseVec) <- 2:10
optD <- as.numeric(names(which.min(mseVec)))
optDMSE <- mseVec[as.character(optD)]
sprintf("Optimal depth: %.0f, Test MSE: %.5f", optD, optDMSE)
```

```
## [1] "Optimal depth: 5, Test MSE: 17.60284"
```

iii. Determining optimal shrinkage parameter

I iterate over different possible shrinkage parameters, fitting the model with each and the optimal interaction depth parameter from above to find the minimum MSE:

```
set.seed(1)
mseVecL <- c()
for(l in seq(0.001, 0.2, by = 0.001)){
 gbTestL <- gbm(medv ~ ., data = trainData, distribution = "gaussian",
             n.trees = 1000, shrinkage = l, interaction.depth = optD)
 predgBTestL <- predict(gbTestL, newdata = testData)
 MSEgBTestL <- mean((predgBTestL - testData$medv)^2)
 mseVecL <- append(mseVecL, MSEgBTestL)
}
names(mseVecL) <- seq(0.001, 0.2, by = 0.001)
optL <- as.numeric(names(which.min(mseVecL)))
optLMSE <- mseVecL[as.character(optL)]
sprintf("Optimal shrinkage: %.3f, Test MSE: %.5f", optL, optLMSE)
```

```
## [1] "Optimal shrinkage: 0.164, Test MSE: 16.60171"
```

iv. Comparing test MSEs

I summarize the results of the above 3 procedures:

```
mseSum <- as.data.frame(cbind(c("Base Boosted Model", "Boosted + Optimal depth",
                                "Boosted + Optimal depth and shrinkage"),
                              unname(c(MSEgB, optDMSE, optLMSE))))
colnames(mseSum) <- c("Model Specifications", "Test MSE")
mseSum$`Test MSE` <- as.numeric(mseSum$`Test MSE`)
mseSum
```

```
##                     Model Specifications Test MSE
## 1                   Base Boosted Model 18.47792
## 2               Boosted + Optimal depth 17.60284
## 3 Boosted + Optimal depth and shrinkage 16.60171
```

With each optimization of parameters, the test MSE was reduced, improving the overall accuracy of the model.

3. Gradient Boosted Tree for Q1 data

a. Building the First tree

The first tree is just one leaf, with no splits. Therefore, the prediction at that tree is simply the average of all weights, or $\hat{y} = \bar{y} = \frac{88+82+60+73+77+55+80}{7} = 73.57$. I calculate the residuals by generating a data frame of table 1 data, and then subtracting $\bar{y}$:

```
Height <- c(1.6,1.7,1.5,1.8,1.5,1.4,1.7)
Gender <- c(0,1,1,0,0,1,0)
Weight <- c(88,82,60,73,77,55,80)
table1 <- as.data.frame(cbind(Height, Gender, Weight))
table1$Gender <- factor(table1$Gender)
residual <- table1$Weight - mean(table1$Weight)
resid <- as.data.frame(cbind(1:7, Height, Gender, residual))
colnames(resid) <- c("Observation", "Height", "Gender","Residual")
resid
```

```
##   Observation Height Gender    Residual
## 1           1    1.6      0  14.4285714
## 2           2    1.7      1   8.4285714
## 3           3    1.5      1 -13.5714286
## 4           4    1.8      0  -0.5714286
## 5           5    1.5      0   3.4285714
## 6           6    1.4      1 -18.5714286
## 7           7    1.7      0   6.4285714
```

b. Creating the 2nd tree

i. Determining root node for 2nd tree

The split at the root node has to put at least one data point into each child node. Since the heights have the possible values $\{1.4, 1.5, 1.6, 1.7, 1.8\}$, there are 4 possible splits on height, and since gender has the possible values $\{M, F\}$, there is 1 possible split on gender. I examine all splits below:

- Splitting on Height ≤ 1.4: In this case, the left child has one data point, data point 6; the mean response is just $\bar{y}_l = y_6 = -18.57$, so $SSE_l = 0$. The other side contains the other 6 points; here the mean response is $\bar{y}_r = \frac{14.43+8.43-13.57-0.57+3.43+6.43}{6} = 3.10$. Thus, $SSE_r = \sum_{i \in R}(y_i - \bar{y}_r)^2 = 459.33$. Therefore, $SSE = 0 + 459.33 = 459.33$.
- Splitting on Height ≤ 1.5: In this case, the left child has 3 data points: 3, 5, and 6. Therefore, the mean response is $\bar{y}_l = \frac{-13.57+3.43-18.57}{3} = -9.57$. Therefore, $SSE_l = \sum_{i \in L}(y_i - \bar{y}_l)^2 = 266$. The other side contains the other 4 data points, and the mean response is $\bar{y}_r = \frac{14.43+8.43-0.57+6.43}{4} = 7.18$. Here, $SSE_R = \sum_{i \in R}(y_i - \bar{y}_r)^2 = 114.75$. So, $SSE = 266 + 114.75 = 380.75$.
- Splitting on Height ≤ 1.6: In this case, the left child has 4 data points: 1,3,5, and 6. The mean response is $\bar{y}_l = \frac{14.43-13.57+3.43-18.57}{4} = -3.57$. Consequently, $SSE_l = \sum_{i \in L}(y_i - \bar{y}_l)^2 = 698$. Then, the right child has 3 data points: 2, 4, and 7. The mean response is $\bar{y}_r = \frac{8.43-0.57+6.43}{3} = 4.76$. So, $SSE_r = \sum_{i \in R}(y_i - \bar{y}_r)^2 = 44.67$. Finally, $SSE = 698 + 44.67 = 742.67$.
- Splitting on Height ≤ 1.7: In this case, the left child has 6 data points: 1,2,3,5,6,7. The mean response is $\bar{y}_l = \frac{14.43+8.43-13.57+3.43-18.57+6.43}{6} = 0.10$. Then, $SSE_l = \sum_{i \in L}(y_i - \bar{y}_l)^2 = 861.33$. The right child only has one data point: point 4. So, the mean response is $\bar{y}_r = y_4 = -0.57$, and $SSE_r = 0$. Finally, $SSE = 861.33 + 0 = 861.33$.
- Splitting on Gender < 0.5: Encode $\{M : 0, F : 1\}$. Thus, all males (points 1,4,5, 7) are in the left child, and all females (points 2,3, 6) are in the right child. Thus, $\bar{y}_l = \frac{14.43-0.57+3.43+6.43}{4} = 5.93$, and $SSE_l =$

$\sum_{i \in L}(y_i - \bar{y}_l)^2 = 121$. Similarly, $\bar{y}_r = \frac{8.43-13.57-18.57}{3} = -7.90$ and $SSE_r = \sum_{i \in R}(y_i - \bar{y}_r)^2 = 412.67$. Thus, $SSE = 121 + 412.67 = 533.67$.

I summarize in a table:

Split	Points in L	Points in R	SSE_l	SSE_r	SSE
Height ≤ 1.4	6	1,2,3,4,5,7	0	459.33	459.33
Height ≤ 1.5	3,5,6	1,2,4,7	266	114.75	380.75
Height ≤ 1.6	1,3,5,6	2,4,7	698	44.67	742.67
Height ≤ 1.7	1,2,3,5,6,7	4	861.33	0	861.33
Gender < 0.5	1,4,5,7	2,3,6	121	412.67	533.67

Therefore, the root node is split on Height $\leq \mathbf{1.5}$, the left leaf contains $3, 5, 6$, the right leaf contains $1, 2, 4, 7$, and their SSE's are 266 and 114.75 respectively, with an association SSE of 380.75. Note that the SSE values are the same as in the original creation of the tree. This is because the residuals are simply the original values with the mean subtracted, and $Var[Y - \bar{Y}] = Var[Y]$.

ii. The second internal node

As described earlier, the SSE calculations are fundamentally the same as in Question 1. Therefore, the second internal node would also be the same, with the second split being on the left leaf, and the variable split on would be Gender $< \mathbf{0.5}$. Thus, leaves L_1 and L_2 are created, with L_1 containing data point 5 and L_2 containing data points 3 and 6. The respective SSE values are $SSE_{L_1} = 0$ and $SSE_{L_2} = 12.5$, as before. The predictions for the residuals would be $\bar{y}_{L_1} = y_5 = 3.43$ and $\bar{y}_{L_2} = \frac{-13.57+-18.57}{2} = -16.07$ respectively.

iii. Leaves, predictions for residuals

After these two splits, there are 3 leaves; I summarize the leaves, the data points in the leaves, the predictions for the residual in each leaf, and the updated weight prediction for the data points $\bar{y} + 0.1(\text{pred(resid)})$:

Leaf Name	Data points in leaf	Prediction for Residual	Updated Weight Prediction
L_1	5	3.43	$73.57 + 0.1(3.43) = \mathbf{73.873}$
L_2	3,6	-16.07	$73.57 + 0.1(-16.07) = \mathbf{71.963}$
R	1,2,4,7	7.18	$73.57 + 0.1(7.18) = \mathbf{74.288}$

c. Creating the 3rd tree

I first need to find the new residuals:

```
resid2 <- resid
resid2$Residual <- Weight - c(74.288,74.288,71.963,74.288,73.873,71.963,74.288)
resid2
```

```
##   Observation Height Gender Residual
## 1           1    1.6      0   13.712
## 2           2    1.7      1    7.712
## 3           3    1.5      1  -11.963
## 4           4    1.8      0   -1.288
## 5           5    1.5      0    3.127
## 6           6    1.4      1  -16.963
## 7           7    1.7      0    5.712
```

Now, I fit the tree again, fitting on the residuals of tree number 2. I examine each potential split:

- Splitting on Height ≤ 1.4: In this case, the left child has one data point, data point 6; the mean response is just $\bar{y}_l = y_6 = -16.963$, so $SSE_l = 0$. The other side contains the other 6 points; here the mean response is $\bar{y}_r = \frac{13.712+7.712-11.963-1.288+3.127+5.712}{6} = 2.835$. Thus, $SSE_r = \sum_{i \in R}(y_i - \bar{y}_r)^2 = 386.44$. Therefore, $SSE = 0 + 386.44 = 386.44$.

- Splitting on Height ≤ 1.5: In this case, the left child has 3 data points: 3, 5, and 6. Therefore, the mean response is $\bar{y}_l = \frac{-11.963+3.127-16.963}{3} = -8.6$. Therefore, $SSE_l = \sum_{i \in L}(y_i - \bar{y}_l)^2 = 218.77$. The other side contains the other 4 data points, and the mean response is $\bar{y}_r = \frac{13.712+7.712-1.288+5.712}{4} = 6.462$. Here, $SSE_R = \sum_{i \in R}(y_i - \bar{y}_r)^2 = 114.75$. So, $SSE = 218.77 + 114.75 = 333.52$.
- Splitting on Height ≤ 1.6: In this case, the left child has 4 data points: 1,3,5, and 6. The mean response is $\bar{y}_l = \frac{13.712-11.963+3.127-16.963}{4} = -3.022$. Consequently, $SSE_l = \sum_{i \in L}(y_i - \bar{y}_l)^2 = 592.13$. Then, the right child has 3 data points: 2, 4, and 7. The mean response is $\bar{y}_r = \frac{7.712-1.288+5.712}{3} = 4.045$. So, $SSE_r = \sum_{i \in R}(y_i - \bar{y}_r)^2 = 44.67$. Finally, $SSE = 592.13 + 44.67 = 636.80$.
- Splitting on Height ≤ 1.7: In this case, the left child has 6 data points: 1,2,3,5,6,7. The mean response is $\bar{y}_l = \frac{13.712+7.712-11.963+3.127-16.963+5.712}{6} = 0.223$. Then, $SSE_l = \sum_{i \in L}(y_i - \bar{y}_l)^2 = 720.46$. The right child only has one data point: point 4. So, the mean response is $\bar{y}_r = y_4 = -0.57$, and $SSE_r = 0$. Finally, $SSE = 720.46 + 0 = 720.46$.
- Splitting on Gender < 0.5: Encode $\{M : 0, F : 1\}$. Thus, all males (points 1,4,5, 7) are in the left child, and all females (points 2,3, 6) are in the right child. Thus, $\bar{y}_l = \frac{13.712-1.288+3.127+5.712}{4} = 5.316$, and $SSE_l = \sum_{i \in L}(y_i - \bar{y}_l)^2 = 119.05$. Similarly, $\bar{y}_r = \frac{7.712-11.963-16.963}{3} = -7.071$ and $SSE_r = \sum_{i \in R}(y_i - \bar{y}_r)^2 = 340.32$. Thus, $SSE = 119.05 + 340.32 = 459.37$.

I summarize in the table below:

Split	Points in L	Points in R	SSE_l	SSE_r	SSE
Height ≤ 1.4	6	1,2,3,4,5,7	0	386.44	386.44
Height ≤ 1.5	3,5,6	1,2,4,7	218.77	114.75	333.52
Height ≤ 1.6	1,3,5,6	2,4,7	592.13	44.67	636.80
Height ≤ 1.7	1,2,3,5,6,7	4	720.46	0	720.46
Gender < 0.5	1,4,5,7	2,3,6	119.05	340.32	459.37

Note that the lowest remaining SSE after split belongs to the split corresponding with Height $\leq$ **1.5**. So, leaf L contains $3, 5, 6$ and leaf R contains $1, 2, 4, 7$ after the first split. Note that $SSE_L = 218.77 > 114.75 = SSE_R$. Therefore, if I were to find a split among the three data points in leaf L whose decrease in SSE was greater than SSE_R, I know that this is the optimal second split. Data points $3, 5, 6$ have heights $\{1.4, 1.5\}$ and genders $\{M, F\}$. Therefore, there is one potential split on height and one potential split on gender; I examine both (calling the resulting leaves L_1 and L_2):

- Splitting on Height $\leq$ 1.4: Only observation 6 fits this criteria; therefore, $\bar{y}_{L_1} = y_6 = -16.963$ and $SSE_{L_1} = 0$. Then, observations 3 and 5 are in L_2, so $\bar{y}_{L_2} = \frac{-11.963+3.127}{2} = -4.418$. Thus, $SSE_{L_2} = \sum_{i \in L_2}(y_i - \bar{y}_{L_2})^2 = 113.85$. So, $SSE_{\text{tot}} = 0 + 113.85 = 113.85$.
- Splitting on Gender < 0.5: Only observation 5 is male of the 3 observations in L; therefore, $\bar{y}_{L_1} = y_5 = 3.127$ and $SSE_{L_1} = 0$. Then, observations 3 and 6 are female and in L_2. Thus, $\bar{y}_{L_2} = \frac{-11.963-16.963}{2} = -14.463$ and $SSE_{L_2} = \sum_{i \in L_2}(y_i - \bar{y}_{L_2})^2 = 12.5$. Thus $SSE_{\text{tot}} = 0 + 12.5 = 12.5$.

Thus, splitting on Gender < 0.5 is the best split among data in Leaf L and $218.77 - 12.5 = 206.27 > 114.75$ so the reduction in SSE is greater than the entirety of the SSE in leaf R; thus, the split on leaf L on Gender $<$ **0.5** is the optimal second split. After this split, there are 3 leaves, so the third tree is completed - note that the leaves are grouped exactly the same as they were in the second tree. Therefore, I summarize the leaves, the observations in each leaf, and the updated weight prediction for the leaves ($\hat{y}_{\text{Tree}_3} = \hat{y}_{\text{Tree}_2} + 0.1 * \text{pred}(\text{Resid}_2)$):

Leaf Name	Data points in leaf	Prediction for Resid_2	Updated Weight Prediction
L_1	5	3.127	$73.873 + 0.1(3.127) =$ **74.1857**
L_2	3,6	-14.463	$71.963 + 0.1(-14.463) =$ **70.5167**
R	1,2,4,7	6.462	$74.288 + 0.1(6.462) =$ **74.9342**

I find the residuals of the third tree:

```
pred3 <- c(74.9342, 74.9342, 70.5167, 74.9342, 74.1857, 70.5167, 74.9342)
resid3 <- resid2
```

```
resid3$Residual <- Weight - pred3
resid3
```

```
##   Observation Height Gender Residual
## 1           1    1.6      0  13.0658
## 2           2    1.7      1   7.0658
## 3           3    1.5      1 -10.5167
## 4           4    1.8      0  -1.9342
## 5           5    1.5      0   2.8143
## 6           6    1.4      1 -15.5167
## 7           7    1.7      0   5.0658
```

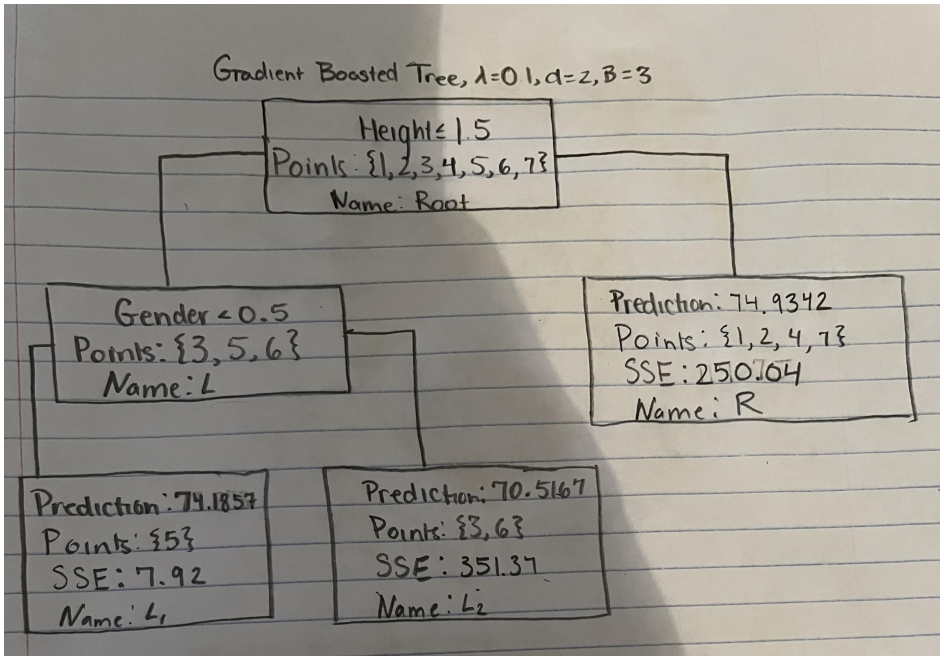
d. Drawing whole gradient boosted tree

To get the Sum of Squared Errors for each leaf, I take the sum of squared residuals:

- $SSE_{L_1} = \sum_{i \in L_1} e^2_{\text{tree } 3,i} = 2.8143^2 = 7.92$
- $SSE_{L_2} = \sum_{i \in L_2} e^2_{\text{tree } 3,i} = (-10.5167)^2 + (-15.5167)^2 = 351.37$
- $SSE_R = \sum_{i \in R} e^2_{\text{tree } 3,i} = 13.0658^2 + 7.0658^2 + (-1.9342)^2 + 5.0658^2 = 250.04$

Using this information, I plot the final, boosted tree:

```
knitr::include_graphics("tree3.jpg")
```



e. Predicting weight for male with height 1.45m

In this case, I would predict that $\hat{y}(\text{M}, 1.45) = 74.1857$. This is exactly what I found in part d, because both trees 2 and 3 had the exact same splits, which means tree 3 and the total, gradient boosted tree are exactly the same.