HW-3

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Homework 3 - Predictive Modeling in Finance and Insurance

1. Likelihood Function for mean of normal distribution

a. Joint Density Function

Note that $Y_1, Y_2,$ and Y_3 are independent. Therefore, their joint probability density function (p.d.f) is a product of their marginal probability density functions:

$$\begin{aligned}
f_{(Y_1,Y_2,Y_3)}(y_1,y_2,y_3) &= f_{Y_1}(y_1)f_{Y_2}(y_2)f_{Y_3}(y_3) \\
&= \frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{1}{2\sigma^2}(y_1-\mu_1)^2} * \frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{1}{2\sigma^2}(y_2-\mu_2)^2} * \frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{1}{2\sigma^2}(y_3-\mu_3)^2} \\
&= \frac{1}{(2\pi\sigma^2)^{\frac{3}{2}}} * e^{-\frac{1}{2\sigma^2}\left(\sum_{i=1}^3 (y_i-\mu_i)^2\right)}
\end{aligned}$$

b. Likelihood function and Log-Likelihood

The likelihood function is just the joint p.d.f, given parameter of interest $\overrightarrow{\mu} = (\mu_1, \mu_2, \mu_3)$:

$$L(\overrightarrow{\mu}) = f_{(Y_1,Y_2,Y_3)}(y_1,y_2,y_3;\mu) = \frac{1}{(2\pi\sigma^2)^{\frac{3}{2}}} * e^{-\frac{1}{2\sigma^2}\left(\sum_{i=1}^3 (y_i-\mu_i)^2\right)}$$

The log-likelihood is just the natural log of this function:

$$\begin{aligned}
\ell(\overrightarrow{\mu}) = \ln(L(\mu)) &= \ln\left(\frac{1}{(2\pi\sigma^2)^{\frac{3}{2}}}\right) + \ln\left(e^{-\frac{1}{2\sigma^2}\left(\sum_{i=1}^3 (y_i-\mu_i)^2\right)}\right) \\
&= -\frac{3}{2}\ln(2\pi\sigma^2) - \frac{1}{2\sigma^2}\left(\sum_{i=1}^3 (y_i-\mu_i)^2\right)
\end{aligned}$$

c. Score function, Observed Information, Expected Information

The score function is simply the derivative of the log likelihood with respect to the parameter of interest, $\overrightarrow{\mu}$. Note that the function is actually a matrix, as I takt the derivative with respect to μ_1, μ_2, and μ_3:

$$S(\overrightarrow{\mu}) = \frac{d}{d\mu}\ell(\mu) = \begin{bmatrix} \frac{d}{d\mu_1}\left(-\frac{3}{2}\ln(2\pi\sigma^2) - \frac{1}{2\sigma^2}\left(\sum_{i=1}^3 (y_i-\mu_i)^2\right)\right) \\ \frac{d}{d\mu_2}\left(-\frac{3}{2}\ln(2\pi\sigma^2) - \frac{1}{2\sigma^2}\left(\sum_{i=1}^3 (y_i-\mu_i)^2\right)\right) \\ \frac{d}{d\mu_3}\left(-\frac{3}{2}\ln(2\pi\sigma^2) - \frac{1}{2\sigma^2}\left(\sum_{i=1}^3 (y_i-\mu_i)^2\right)\right) \end{bmatrix} = \begin{bmatrix} \frac{y_1-\mu_1}{\sigma^2} \\ \frac{y_2-\mu_2}{\sigma^2} \\ \frac{y_3-\mu_3}{\sigma^2} \end{bmatrix}$$

Note that the observed information matrix is a matrix of second derivatives of the log-likelihood function.

Since we have 3 variables to differentiate with respect to, it is a 3 $\times$ 3 matrix, multiplied by -1:

$$j(\overrightarrow{\mu}; Y) = -1 * \begin{bmatrix} \frac{d^2\ell(\mu)}{d\mu_1^2} & \frac{d^2\ell(\mu)}{d\mu_2 d\mu_1} & \frac{d^2\ell(\mu)}{d\mu_3 d\mu_1} \\ \frac{d^2\ell(\mu)}{d\mu_1 d\mu_2} & \frac{d^2\ell(\mu)}{d\mu_2^2} & \frac{d^2\ell(\mu)}{d\mu_3 d\mu_2} \\ \frac{d^2\ell(\mu)}{d\mu_1 d\mu_3} & \frac{d^2\ell(\mu)}{d\mu_2 d\mu_3} & \frac{d^2\ell(\mu)}{d\mu_3^2} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sigma^2} & \frac{1}{\sigma^2} & \frac{1}{\sigma^2} \\ \frac{1}{\sigma^2} & \frac{1}{\sigma^2} & \frac{1}{\sigma^2} \\ \frac{1}{\sigma^2} & \frac{1}{\sigma^2} & \frac{1}{\sigma^2} \end{bmatrix} = \frac{1}{\sigma^2}\mathbf{1}_{3\times 3}$$

The expected information matrix is simply the expectation with respect to our observations of our observed information matrix:

$$i(\overrightarrow{\mu}) = \mathbb{E}[j(\overrightarrow{\mu}; Y)] = \frac{1}{\sigma^2}\mathbb{E}\left[\mathbf{1}_{3\times 3}\right] = \frac{1}{\sigma^2}\mathbf{1}_{3\times 3}$$

Given the observations, these matrices take on the values:

$$S(\overrightarrow{\mu}; Y) = \begin{bmatrix} \frac{4}{\sigma^2} & \frac{6.5}{\sigma^2} & \frac{5}{\sigma^2} \end{bmatrix}^T \text{ and } i(\overrightarrow{\mu}) = j(\overrightarrow{\mu}; Y) = \frac{1}{\sigma^2}\mathbf{1}_{3\times 3}$$

2. Fun with Distributions

a. Distribution of Y_1^2

Since $Y_1 \sim N(0,1)$, $Y_1^2 \sim \chi^2(1)$, or the chi-squared distribution with 1 degree of freedom.

b. Combination of Y_1 and Y_2

Note $\frac{Y_2-\mu_2}{\sigma_2} = \frac{Y_2-3}{2} \sim N(0,1)$; therefore:

$$\left(\frac{Y_2-3}{2}\right)^2 \sim \chi^2(1)$$

Using the independence of Y_1 and Y_2 and Cochran's Theorem:

$$y^T y = \begin{bmatrix} Y_1 & \frac{Y_2-3}{2} \end{bmatrix} * \begin{bmatrix} Y_1 \\ \frac{Y_2-3}{2} \end{bmatrix} = Y_1^2 + \left(\frac{Y_2-3}{2}\right)^2 = \chi^2(1+1) = \chi^2(2)$$

So, $y^T y$ has the chi-squared distribution with 2 degrees of freedom.

c. Multivariate Normal

Note that V in this case is the Variance-Covariance matrix. Since Y_1 and Y_2 are independent, the off-diagonal elements, which represent covariance, are 0. There diagonal elements are just $\sigma_1^2 = 1$ and $\sigma_2^2 = 4$, respectively, so:

$$V = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$$

I find the inverse of this 2 by 2 matrix:

$$V^{-1} = \frac{1}{1(4) - 0(0)} \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{4} \end{bmatrix}$$

Therefore:

$$\begin{aligned} y^T V^{-1} y &= \begin{bmatrix} Y_1 & Y_2 \end{bmatrix} * \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{4} \end{bmatrix} * \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} \\ &= \begin{bmatrix} Y_1 & \frac{Y_2}{4} \end{bmatrix} * \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} \\ &= Y_1^2 + \left(\frac{Y_2}{2}\right)^2 \end{aligned}$$

4. Linear Regression

a. Fitting model B

```
library(ggplot2)
library(readxl)
```

I first import the data:

```
carbData <- read_excel("Table 6.3 Carbohydrate diet-1.xls", skip = 2, sheet = "Sheet1")
```