2024 Spring, MATHGR5450 Credit Analytics: Assignment #2, Due Date: 10:00PM, March 26, 2024

1. (20 points) Suppose that we have a one-period default swap over time interval $[0, \Delta t]$. The premium P is paid at the beginning of the period, and the loss is paid at the time when default occurs, $1 - R$, where R is the recovery rate. We assume that the density function for the survival time is exponentially distributed, $f(t) = he^{-ht}$, and that we have a constant interest rate r over the time interval.
   (a) (5 points) Calculate the expected present value of loss leg for the one period default swap;
   (b) (10 points) If default occurs at time t, and $0 \le t \le \Delta t$, then default protection buyer would receive a refund of $P(1 - \frac{t}{\Delta t})$ at the time of default from the protection seller. Calculate the expected present value of this refund.
   (c) (5 points) What is the fair value of P for this one-period default swap?

2. (15 points) In credit portfolio analysis we usually assume that two firms from the same industry have a higher correlation (intra correlation) ρ_I than two firms from two different industries (inter correlation) ρ_O. We usually use the following model

$$X_i^k = \sqrt{\rho_I - \rho_O} \cdot X_k + \sqrt{\rho_O} \cdot X_M + \sqrt{1 - \rho_I} \cdot \epsilon_i. \tag{1}$$

   where X_k, X_M and ϵ_i are all standard normal distributions with mean zero and variance 1, and they are also independent from each other, and k represent a specific industry $(k = 1, 2, \ldots, K)$.
   (a) (5 points) Verify that intra and inter industry correlation are ρ_I and ρ_O.
   (b) (5 points) Suppose there are a total of 12 different industries, how many independent factors do we need to represent the correlation matrix?
   (c) (5 points) Suppose we have a total of 10 credit names, and they come from 5 different industries with exact two from the same industry, and $\rho_I = 0.50$, and $\rho_O = 0.10$. Let's form the correlation matrix and show them.

3. (20 points) A copula function is simply a multivariate uniform distribution over $[0, 1]$ hypecube. For bivariate function $C(u, v)$, $[0, 1]X[0, 1] \to [0, 1]$ we can call $C(u, v)$ is a bivariate copula function if it meets the following three conditions:
   a. $C(0, u) = C(u, 0) = 0$
   b. $C(1, u) = C(u, 1) = u$
   c. $C(u_2, v_2) - C(u_2, v_1) - C(u_1, v_2) + C(u_1, v_1) \ge 0$ for all $0 \le u_1 \le u_2 \le 1$ and $0 \le v_1 \le v_2 \le 1$,
   (a) (5 points) Suppose we have a continuous random variable X whose distribution is $F(x)$. Show that $F(X)$, as a function of the random variable X, is a uniform distribution.
   (b) (5 points) You take any univariate distribution functions $F_1(x_1)$ and $F_2(x_2)$, and then construct another function

$$F(x_1, x_2) = C(F_1(x_1), F_2(x_2))$$

Is this a bivariate distribution? what is the marginal distribution of this bivariate distribution?

(c) (5 points) Show that $C(u, v) = \min(u, v)$ is a bivariate copula function if u and v are two uniform distributions over $(0, 1]$.

(d) (5 points) Show that for any bivariate copula function, it satisfies

$$\max(u + v - 1, 0) \leq C(u, v) \leq \min(u, v)$$

(hint: use the Bonferroni inequalities $Pr(A)+Pr(B)-1 \leq Pr(A \cap B) \leq \min(Pr(A), Pr(B))$).

4. (20 points) (a) (5 points) We have a non-negative random random variable X whose distribution function is $F(x)$ and whose density function is $f(x)$. Prove that

$$E(X) = \int_0^\infty (1 - F(x))dx$$

(b) (5 points) We have a securitized tranche whose subordination levels are L_l and L_u, and the underlying portfolio loss distribution is described as a random variable X as in above. Express the loss of the tranche in terms of the portfolio loss X.

(c) (5 points) Show that the loss for the tranche can be expressed as a call spread on the underlying loss distribution.

(d) (5 points) Show that the expected loss for the tranche can be expressed as

$$\int_{L_l}^{L_u} (1 - F(x))dx$$

5. (25 points) In a one-factor Gaussian copula credit model, the correlation structure for standardized asset returns is as follow

$$X_i = \sqrt{\rho}\, X_M + \sqrt{1 - \rho}\, \epsilon_i$$

where X_i, X_M are standardized asset returns, and the market asset return with mean zero and variance 1, ϵ_i's are non correlated noises with mean zero and variance 1, $Cov(X_i, X_j) = \rho$ for $i \neq j$.

The survival time for each credit is linked to its asset return as follows

$$\tau_i = F_i^{-1}(N(X_i))$$

where $F(t)$ is the survival time distribution, and $N()$ is the standard normal distribution function.

Conditional on the common market factor X_M, all credit survival times are independent to each other.

(a) (5 points) Prove that the conditional default probability

$$q_i(t|X_M) = Pr[\tau_i < t|X_M]$$

can be linked to the non-conditional default probability

$$q_i(t) = Pr[\tau_i < t]$$

as follows:

$$q_i(t|X_M) = N\left[\frac{N^{-1}(q_i(t)) - \sqrt{\rho}X_M}{\sqrt{1-\rho}}\right]$$

where $N()$ is the standard normal distribution function, and $q_i(t) = Pr[\tau_i < t]$.

(b) (5 points) We consider a credit portfolio consisting of n underlying credits whose notional amounts are N_i and fixed recovery rates are R_i $(i = 1, \cdots n)$. We consider the aggregate loss from today to time t as a fixed sum of random variables:

$$L_n(t) = \sum_{i=1}^{n} X_i = \sum_{i=1}^{n} (1 - R_i) \cdot N_i \cdot I_{[\tau_i < t]} = \sum_{i=1}^{n} B_i \cdot I_{[\tau_i < t]} \tag{2}$$

where τ_i is the survival time for the ith credit in the credit portfolio. The distribution function of survival time τ_i is denoted as $F_i(t) = \Pr[\tau_i \leq t]$.

Conditional on the market factor $X_M = v$, the portfolio loss is simply a sum of independent, individual losses. Provide the mean M_v and variance σ_v^2 for the conditional portfolio loss.

(c) (10 points) The conditional Normal approximation approach uses a Normal distributions to approximate the total loss distribution by matching the mean and variance as computed above. Prove, under this conditional normal assumption, that the conditional expected loss for the tranche with attachment K_L^T, and detachment K_U^T are given as follows:

$$\begin{aligned} E[L^T(t)|v] &= (M_v - K_L^T) N\left(\tfrac{M_v - K_L^T}{\sigma_v}\right) + \sigma_v \cdot n\left(\tfrac{M_v - K_L^T}{\sigma_v}\right) \\ &\quad - \left(M_v - K_U^T\right) N\left(\tfrac{M_v - K_U^T}{\sigma_v}\right) - \sigma_v \cdot n\left(\tfrac{M_v - K_U^T}{\sigma_v}\right) \end{aligned}$$

where $n()$ is one-dimensional normal density function.

(d) (5 points) Provide an expression for the unconditional expected loss of tranche with attachment and detachment points K_L^T and K_U^T, and convert the expression so that an numerical integration could be readily applied.

6. (60 points) Suppose we have a credit portfolio of 125 names, each credit has a notional amount of \$80000, a credit spread 120 bps, and a recovery value of 40%.

(a) (5 points) Based on the Duffie-Singleton one period model, what is the annual default probability for each of the credits?

(b) (5 points) What is the expected loss of the above credit portfolio in the next 5-years? Does this portfolio expected loss depend on the correlation?

(c) (30 points) Implement three portfolio loss calculation methods: homogeneous portfolio based on the conditional binomial distribution, conditional normal approximation, and recursive method for a given pairwise asset return correlation ρ.

(d) (5 points) Compare the loss distributions with $rho = 0, 20\%, 50\%75\%, 95\%$. In this calculation you need to use Gaussian quadrature for the numerical integration. How many quadrature points should be used for each correlation?

(e) (5 points) Calculate the loss distribution for CDO tranches of $[0, 3\%]$, $[3\%, 7\%]$, $[7\%, 10\%]$, $[10\%, 15\%]$, $[15\%, 30\%]$, $[30\%, 100\%]$ for each of the above correlations.

(f) (5 points) Based on the above calculation, for an equity tranche $[0, 3\%]$ holder(this means he is a protection buyer for loss range of $[0, 3\%]$), how does he like the asset correlation, how about the senior tranche $[30\%, 100\%]$ holder? how about the mezzanine tranche $[7\%, 10\%]$ holder?

(g) (5 points) When we use conditional normal approximation, the loss could take negative value as a normal variable always has a positive probability to become a negative value. But for equity tranche $[0, 3\%]$ we integrate the loss distribution from 0 to 3%, which will make the total loss from adding all tranche losses not equal to the expected loss of the credit portfolio. One possible solution is to integrate from $-\infty$ to 3%. Does this help?