A Comparison of the Binomial Expansion Technique and Gaussian Copula as a way to model default in CLO/CDOs

By: Dennis Goldenberg, Triet Vo, and Kezia Hana Setokusumo

Table of Contents

Introduction to CDO/CLOs

A Collateralized Loan Obligation, or CLO, is a security issued by a special purpose vehicle - usually a bank - that combines risky assets on its books and diversifies them to issue higher quality investment instruments with the pool of risky assets as the collateral. Unlike Collateralized Debt Obligations, or CDOs, which have a wide array of assets, CLOs are backed mostly by corporate credit (Dohnalek & Uytenhout, 2017).

To diversify risk, these corporate lines are divided into pieces, and then pieces from different loans are pooled together to become collateral for the CLO. Then, securities are issued in several “tranches”: these are claims to the interest revenue of a portion of the underlying Notional amount that is sold to investors. The investors are paid in order of seniority - that is, the most senior tranche, which usually is the biggest slice, receives the cash flows first. The equity - or most junior - tranche is the one that experiences losses as a result of default first. Thus, investments in more senior tranches are far safer but also offer a smaller yield. For the rest of this project, the loss function is calculated as the following (Witt, 2014):

$$L = min\left\{1, max\left\{0, \frac{Default\ Pct * Severity - Lower\ AP}{Upper\ AP - Lower\ AP}\right\}\right\} \text{ (*)}$$

Here, *Default Pct* equals the proportion of assets defaulted (the random variable), *Severity* is just $1 - R$, where *R* is the recovery rate, and *Upper/Lower AP* are the upper and lower attachment points of the tranche (percentages of the notional), respectively.

The degree of separation between the CLO and the underlying assets is vast: the CLO is essentially a pool of pieces of different loans. This complexity makes the loss distribution for a CLO (or, more precisely, a particular tranche) very difficult to model, for 2 reasons. Different bonds may have different notional amounts; thus, if there are m underlying assets, there are 2^m different default scenarios to account for. Also, the assets of companies selling bonds are likely correlated in their returns, which implies that company default is correlated as well. Determining the loss distribution entails dealing with both issues; this is an important task for rating agencies such as Moody's, whose ratings investors consider when investing. In this project, we evaluate two methods that attempt to model credit risk for a tranche of a CLO - Binomial Expansion Technique (BET) and the Gaussian copula. We describe the procedure, state their assumptions, and outline advantages and disadvantages. In so doing, we expound on a common misuse of the Gaussian copula - a period-by-period characterization that drops out lines of credit that default in a previous period - and why this is misaligned with the purpose of the copula. Then, we compare them by evaluating the validity of each of their assumptions and running both methods on a synthesized credit portfolio to illustrate.

The Binomial Expansion Technique

The Base Binomial Expansion Method

The Binomial Expansion technique looks to solve the first problem by representing the original credit portfolio as a hypothetical pool of independent, identically distributed (i.i.d) assets. The number of assets that is needed is called the **diversity score (D)**, which is chosen to match the variance of the default percentage of the underlying assets under simulation - essentially a Method of Moments estimate. If we let $a_1, a_2, ..., a_n$ be binary variables with $a_j = 1$ signifying default of asset j, and $z_1, z_2, ..., z_D$ representing the same thing for the hypothetical pool of i.i.d assets, then we get that $Z = \sum_{i=1}^{D} z_i \sim Binomial(D, p)$, as Z is the sum of D i.i.d Bernoulli random variables, each with a probability of default p. Thus, the default percentage would be $\frac{1}{D}\sum_{i=1}^{D} z_i$, and D would be calculated as a value that solves the equation (Witt, 2004):

$$Var\left(\frac{1}{D}\sum_{i=1}^{D} z_i\right) = \frac{p(1-p)}{D} = Var\left(\frac{1}{N}\sum_{i=1}^{N} a_i\right)$$

Meanwhile, p is calculated as the **weighted average probability of default** of the underlying pool. That is, the pool of assets underlying the CLO is simulated under different conditions of an underlying factor, the number of defaults is calculated, and the average is taken which is weighted by the probability measure for the underlying factor. Since each asset in the hypothetical pool is the same and independent, there are now only D + 1 scenarios: 0 defaults, 1 default, …., D defaults. The probability of each is given by:

$$P_j = P(Z = j) = \frac{D!}{j!(D-j)!} p^j (1 - p)^{D-j}$$

Then, represent E_j as the loss calculated in (*) with j defaults (note that this loss function assumes immediate asset loss). Therefore, the expected loss is calculated as:

$$E[L] = \sum_{i=0}^{D} P_j E_j$$

Moody's gives a rating by this expected loss and the maturity of the homogeneous assets.

This model assumes that the correlation of default is effectively captured by the variance of default percentage, and therefore never explicitly accounts for it. This works well when underlying assets are relatively uncorrelated and have a high diversity score. This assumption is called into question when the correlation between returns of different assets is particularly high, which would result in high variance and therefore a low diversity score, so you would be attempting to approximate a large portfolio with relatively few assets. It also assumes that the notional amount is roughly evenly distributed among assets so that a homogeneous pool is a good approximation and any one default has an equal impact on loss. Otherwise, the underlying asset pool would be dominated by a few lines of credit, whose default (or lack thereof) would be the only significant factor in determining loss; thus, representing this large portion of the notional amount in equal pieces whose default is independent of one another is inaccurate.

This method carries with it several key advantages over prior methods such as single event models or pure Monte Carlo simulation (Cifuentes, & O'Connor, 1996). It is far less computationally expensive than Monte Carlo and easier to describe, as there is a simple, discrete distribution (Binomial) that categorizes default. Further, each possible scenario is accounted for under the homogenous pool, and the effects of "tail events" - a large number of defaults - are accounted for by the stressing of the factor used to generate p. Finally, it simplifies calculations due to the reduction in the number of scenarios and the discrete nature of the binomial distribution.

However, this method's accuracy is highly dependent on the proximity of the underlying asset pool to the hypothetical, identically distributed pool. If the pool is significantly non-homogeneous - that is, a few lines of credit dominate the notional amount - then the assumptions that characterize the model fail, and the low diversity score contributes to a misrepresentation of the underlying portfolio. Further, by not explicitly accounting for correlation, it largely ignores the dependency structure between assets, which makes the weighted average probability of default estimate increasingly problematic with higher correlation.

Moody's Correlated Binomial Expansion Method

To account for some of the problems that the BET has, the correlated binomial expansion method described by Witt (2004) was implemented. It is similar to the original BET, except that it explicitly models correlation. The idealized portfolio of homogenous assets is still generated, and the number of said assets is calculated via variance matching - however, we drop the independence assumption for the homogenous assets. Let $x_1, x_2, ..., x_{D_\rho}$ be the indicator variable of whether a particular asset defaulted; two assumptions are made in the place of independence:

- Each pair of assets has a default correlation ρ between them
- (Specific to Moody's Correlated Binomial Expansion) If the assets are put in some arbitrary order, then $\rho = Corr(x_{j+1}, x_{j+2} | x_1 = 1, x_2 = 1, ..., x_j = 1)$. In other words, the correlation between any remaining assets remains constant, regardless of the default number.

Note the consequence of the second assumption: if we define $p_j = E[x_j | x_1 = 1, ... x_{j-1} = 1]$ as the probability of default of asset j given the previous $j - 1$ assets defaulted, and $p_1 = p$:

$$p_j = p_{j-1} + (1 - p_{j-1})\rho = 1 - (1 - p)(1 - \rho)^{j-1}$$

Thus, the probability of a jth default given j - 1 defaults is an increasing function with j, which accounts for the "tail events", or a large number of defaults, explicitly, as one default encourages others. Here, then, stress testing to account for large numbers of defaults is unnecessary.

The closed-form distribution of a correlated binomial is slightly more complicated. It is given by Witt (2004) as:

$$P(k\ defaults) = \frac{D_\rho!}{k!(D_\rho - k)!} E\left[\prod_{j=1}^{k} x_j \prod_{j=k+1}^{D_\rho} (1 - x_j)\right] = \frac{D_\rho!}{k!(D_\rho - k)!} \sum_{j=0}^{D_\rho - k}\left[(-1)^j * \frac{(D_\rho - k)!}{j!(D_\rho - k - j)!} \prod_{i=1}^{j+k} p_i\right]$$

This distribution[1] is similar to the regular binomial and reduces to it when $\rho = 0$. To generate this distribution, one must obtain estimates for ρ and D_ρ. Witt utilizes the underlying portfolio to calculate these estimates; given asset i has a notional amount N_i, default probability P_i, and survival probability $Q_i = 1 - P_i$, the estimate is given below:

$$\bar{\rho} = \frac{\sum_{i=1}^{m}\sum_{j=I+1}^{m} \rho_{ij} N_i N_j P_i P_j}{\sum_{i=1}^{m}\sum_{j=I+1}^{m} N_i N_j P_i P_j}$$

Then, if the diversity score for the uncorrelated BET, or D, is known, you can compute D_ρ by matching the variance of the default percentage to the uncorrelated BET. If not, then you match the second moment of the underlying portfolio; the results in each scenario are:

$$Var\left(\frac{1}{D_\rho}\sum_{i=1}^{D_\rho} x_i\right) = Var\left(\frac{1}{D}\sum_{i=1}^{D} z_i\right) \rightarrow D_\rho = \frac{(1-\bar{\rho})D}{1-\bar{\rho}D} \quad (**)$$

$$D_\rho = \frac{\left(\sum_{i=1}^{m} P_i N_i\right)\left(\sum_{i=1}^{m} Q_i N_i\right)(1-\bar{\rho})}{\sum_{i=1}^{m}\sum_{j=1}^{m} \rho_{ij}\sqrt{P_i Q_i P_j Q_j} N_i N_j - \bar{\rho}\left(\sum_{i=1}^{m} P_i N_i\right)} \quad (***)$$

This method carries several advantages to the original BET: first, as its correlation assumptions imply an increasing probability of default given more defaults, tail events are accounted for, making stress testing unnecessary. Second, the estimates of both the diversity score and correlation account for the different Notional amounts of the underlying portfolio; therefore, severe non-homogeneity of the underlying portfolio is not as problematic. This correlational BET can be thought of as an extension of those cases. However, Moody's correlated binomial specifically assumes that the underlying correlation structure remains constant regardless of the number of defaults, an assumption that may still systematically

[1] The proof of this can be found in Appendix II, or page 9, of Witt's (2004) report.

underestimate tail risk. Secondly, the more complex nature of the distribution of defaults reduces interpretability slightly. Therefore, this method's best use case is when the independence and homogeneity assumptions of the original BET fail.

The Gaussian Copula

Original Gaussian copula

The copula function was first introduced in 1959. The model links a joint distribution based on a given set of marginal distributions. By separating the modeling of joint distributions from marginal distributions, copula functions capture dependency structures, which are useful in risk management and finance.

The Gaussian copula is a type of copula function that utilizes a multivariate normal distribution to construct the dependence structure. Simplicity is often cited as the model's main benefit. The logic behind finding a joint distribution with the Gaussian copula is outlined below:

- Each random variable in the set $X_1, X_2, ..., X_n$ has its own marginal distribution functions $\{F_1(x_1), F_2(x_2), ..., F_n(x_n)\}$
- Under a Gaussian copula, the joint distribution of the variables can be written as $C\{F_1(x_1), F_2(x_2), ..., F_n(x_n)\} = \phi_N(\phi^{-1}(F_1(x_1)), \phi^{-1}(F_2(x_2)), ..., \phi^{-1}(F_n(x_n)))$

  Note that ϕ_N is the CDF and ϕ^{-1} is the inverse CDF.

Reiterating, Gaussian copulas are particularly known for their simplicity. The model is especially useful for studying the correlation structure between assets that have significant linear dependencies. As a result, the Gaussian copula is often used in risk management to depict the joint probabilities of extreme events. Its application extends to credit risk modeling, portfolio optimization, and evaluating systematic risks.

The Gaussian copula has several key assumptions and drawbacks. The model assumes that each asset follows a multivariate normal distribution, which is not always the case. Moreover, the Gaussian copula is known to underestimate tail dependence (i.e., the occurrence of extreme events at the same time), which is an important consideration in risk management practices.

Period-By-Period Gaussian copula

A Period-By-Period Gaussian model builds on the original copula function by introducing a model that projects the future values of certain variables over discretized time intervals. A Gaussian distribution is assumed. The Period-By-Period Gaussian model has the following components:

- The model considers a defined set of time intervals $t = 1, 2,..., N$ whereby N is the total number of steps.
- At each t, there's a state vector of random variables $X_t = (X_1^t, X_2^t,..., X_m^t)$, whereby m is the total number of assets.
- A transition model is used to capture the evolution of the state variables with each time interval, and this is done by assuming the variables follow a multivariate Gaussian distribution $X_{t+1} = AX_t + \epsilon_t$, whereby A is the transition matrix that depicts how the state vector changes with time and \epsilon_t is the Gaussian error.
- Each state vector is assumed to follow a normal distribution, such that $X_t \sim N(\mu_t, \Sigma_t)$. With this, we get the covariance matrix of the state vector.

Like the original Gaussian copula, the Period-By-Period variation can be used in portfolio optimization to optimize asset allocation based on returns and risk, risk management for investment portfolios and derivatives, and asset pricing by simulating future scenarios. The "fudged" period-by-period approach, using the Gaussian copula model, assumes that correlations remain constant over time and proceeds to maintain this same correlation despite names dropping out due to default. The “fudged” method oversimplifies the evolving underlying correlation and leads to inaccurate estimations of default probabilities and dependency structures, especially when applied to dynamic portfolios with evolving characteristics such as SIVs.

Key issues with the period-by-period fudged copula model:

1. The Gaussian copula model assumes a constant correlation between all variable pairs. However, in the financial market, correlations may change over time. With default occurring, and we begin to drop names, existing correlations should also change. Thus, without changing the correlation the accuracy of the model is impacted.
2. There is an evolving dependency structure of the portfolio. The “fudged” method neglects this dependency as it uses the same correlation structure at each period. In comparison, the original Gaussian copula model that is used for the whole survival time horizon would allow for a greater understanding of dependencies among variables.
3. Ultimately, the “fudged” method does not provide an accurate assessment of the portfolio risk. There is an underestimation of the true risk exposure the portfolio faces as the correlation structure is kept the same.

To put simply, the underestimation of defaults on the remaining names occurs because, at the beginning of each period, the period-by-period model assumes that nothing has happened. Furthermore, it is important to note that the Gaussian distribution may not hold in practice and that accuracy depends on how closely real-world data follows the normality assumption. Once again, tail risk is often underestimated in the Period-By-Period model, and this is an important factor to account for.

Comparing the 2 models

Discussing the drawbacks

The BET has several drawbacks. Moody's credit ratings of CLOs don't capture the same expected loss level as those of other fixed-income products. The BET also assumes that defaults aren't correlated across industries, which discounts the impact of market-wide shocks in the model. When considering a homogenous pool of assets, whether intra or inter-industry, with zero correlation, the BET is thought to compute the expected loss accurately. Nevertheless, the BET underestimates the expected loss when correlations exist among assets.

Similarly, the Gaussian copula presents several limitations. The correlations used in the model aren't very predictable, and its complex nature cannot be quantified with one value. Additionally, the copula method only has a single-period horizon, which restricts the extent of risk management strategies. The copula model doesn't use a stochastic process to describe the default correlation of the underlying, and doing so would better account for the time-varying nature of the parameter.

The period-by-period copula attempts to incorporate the changing nature of financial market dependencies. Instead of using a constant dependency variable over time, the period-by-period model tries to estimate how parameters might change in response to different factors, addressing the drawbacks outlined in the BET and Gaussian copula. Applying the period-by-period model would require estimating the copula variables for each time interval and using them to build joint distributions for risk management or portfolio optimization. The main impediment to this approach is the complexity and difficulty of gathering data to make accurate estimates.

Illustrative Example: BET vs. Copula

To demonstrate the differences in rating, we hypothesize a CLO with 100 underlying corporate credits, all from the insurance industry, and with a uniform maturity of 10 years. The first 25 assets are A rated, the next 25 BBB, the next 25 BB, and the final 25 B. The recovery rate was assumed to be 30%. Our hypothetical CLO is split into 4 tranches:

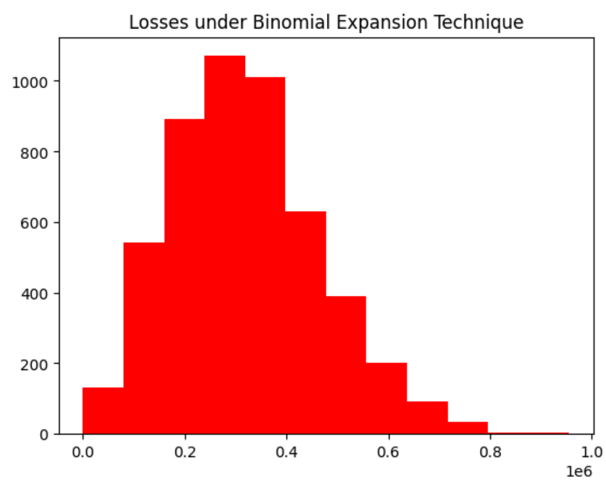
Name of Tranche	**Percentage of Notional**
Equity	[0%, 5%]
Mezzanine 1	[5%,20%]
Mezzanine 2	[20%,40%]
Senior	[40%,100%]

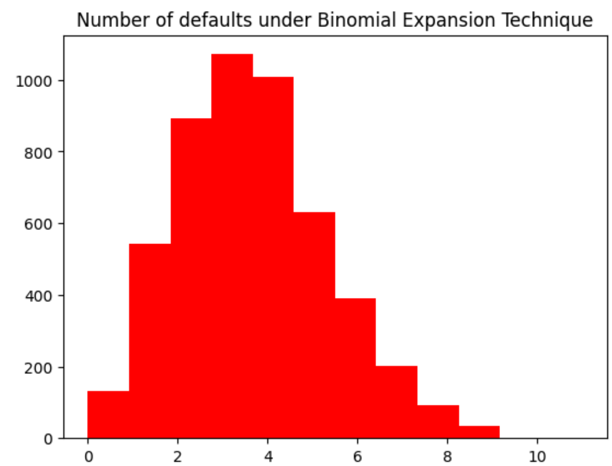
The Notionals for each credit are generated via uniform random variables, with the center of the distribution being $\frac{10000000}{100} = 100,000$. The sum of the notionals came out to be $10,298,085, right around this amount. The default probabilities over 10 years are obtained from Table 38 of S&P Ratings 2021 Annual Global Financial Services Default and rating transition study (Kraemer & Gunter, 2022). The diversity score was calculated using the technique outlined by Dohnalek and Uytenhout (2017, pp. 52-54), and it was estimated as $D = 83$. The weighted average probability was simply calculated as follows:

$$p = \frac{\sum_{i=1}^{100} p_i N_i}{\sum_{i=1}^{100} N_i} = 4.2177\%$$

Thus, defaults of the homogenous portfolio followed a $Binomial(83, .042177)$ distribution, with each asset being independent of the others and having a face value of $1/83$ of the portfolio.

5000 simulations were run, and the following graphs were generated for both the loss distribution and number of defaults (the loss function in (*)) was used:





Further, based on Moody's ratings for CLO tranches (Cifuentes & O'Connor, 1996, pp. 4), the following expected ratings were calculated based on expected loss in the tranche:

Tranche	**Expected Loss (%)**	**Rating**
Equity	57.245%	N/A
Mezzanine 1	0.601%	A2
Mezzanine 2	0%	Aaa
Senior	0%	Aaa

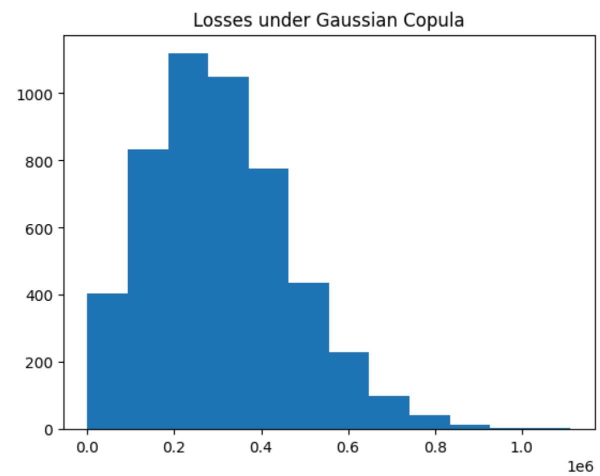
Then, the loss distribution was re-simulated using the gaussian copula. The q values in the marginal distributions, or $F^{-1}(x_i) = q$ were simply the 10-year default probabilities given the rating of the underlying credit, and the pairwise correlations were calculated using the technique described by Dohnalek and Uytenhout (2017, pp. 14-18), with Equation 2.6 in place of Equation 2.2 to calculate the joint probability of default of 2 names in the same rating class to avoid “spurious negative correlations”. This resulted in the following:

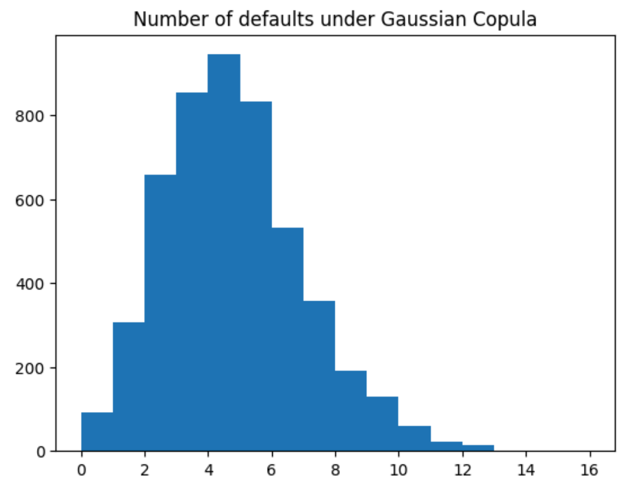
$JPD_{j,j}$	A	BBB	BB	B
A	.0273%	.0433%	.1503%	.1902%
BBB	.0433%	.0685%	.2379%	.3011%
BB	.1503%	.2379%	.8264%	1.0459%
B	.1902%	.3011%	1.0459%	1.3235%

From this, I calculated the the default correlations:

$\rho_{j,j}$	A	BBB	BB	B
A	.7858%	.6583%	1.4182%	1.2736%
BBB	.6523%	.3977%	.9711%	.674%
BB	1.4182%	.9711%	2.2641%	1.7492%
B	1.2736%	.674%	1.7492%	1.0428%

I generated the correlation matrix using these correlations (with 1 on the diagonal, as an asset is perfectly correlated with itself), simulated 5,000 multivariate normal distributions, calculated the number of defaults, and summed the loss using the notionals of said defaults in each simulation:





The expected Losses for each tranche were also given and shown below.

Tranche	**Expected Loss (%)**	**Rating**
Equity	57.378%	N/A
Mezzanine 1	0.626%	A2
Mezzanine 2	0%	Aaa
Senior	0%	Aaa

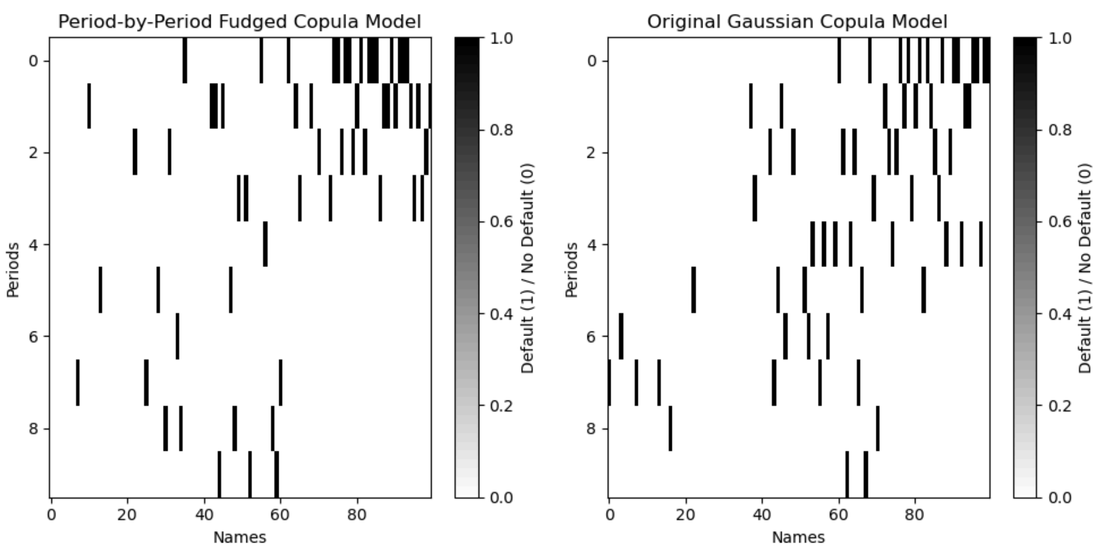
In comparing the two methods, the copula returned a fatter tail and a skewed distribution, which is more emblematic of the high severity and low-frequency nature of credit defaults. The expected losses were roughly equivalent, but the variance was far higher in the copula, for both number of defaults and losses:

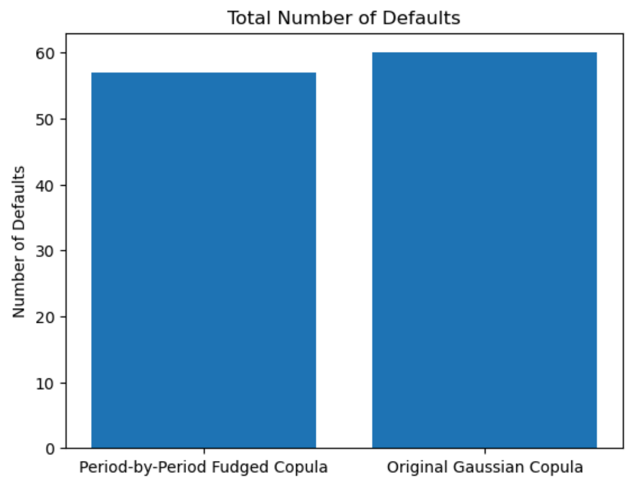
```
Variance of defaults, BET: 3.3902
Variance of defaults, Copula: 4.8184
Variance of Loss, BET: 25573026771.8725
Variance of, Loss, Copula: 26857607349.4339
```

This is due to the accounting for correlation in the copula. The difference in rating would have been more pronounced with more names and higher marginal default probabilities.

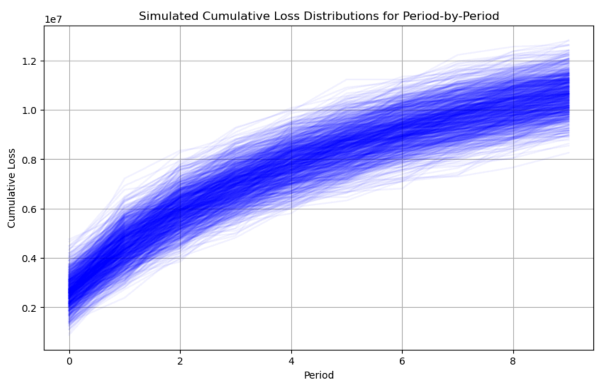
Illustrative Example: Gaussian Copula vs. "Fudged" Period-by-Period Copula

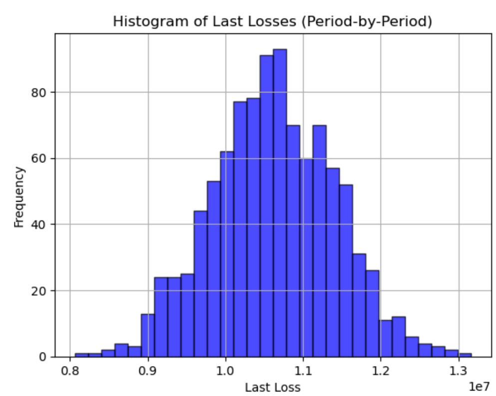
The following assumptions were made when comparing the two methods: a portfolio of 100 lines of corporate credit in the insurance industry, compiling of 25 BBB bonds, 25 BB bonds, 25 B bonds, and 25C bonds, with a 10-year horizon and an assumed correlation coefficient of 0.05. The "fudged" period-by-period copula model gives a smaller number of defaults in comparison to the original copula model. This is in alignment with prior discussion and criticism on the "fudged" period-by-period unchanging correlation structure. The default path of the portfolio for both models is included below along with the total number of defaults calculated.

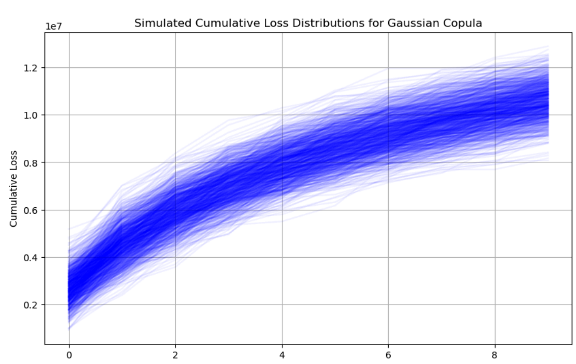


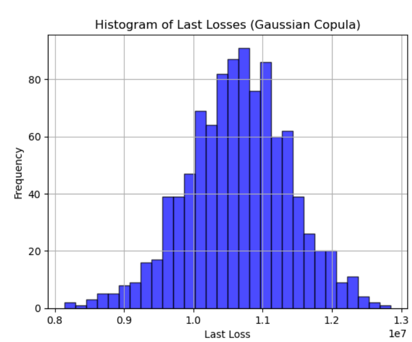


Below are the loss cumulative distributions/final loss distribution, generated with 1000 simulations, of the Period-by-Period model and the Original Gaussian Copula model. The Period-by-Period mean loss is lower than that of the Gaussian model, further supporting our criticism of the “fudged” model’s underprediction of loss.









Works Cited

Cifuentes, A., & O'Connor, G. (1996). (rep.). *The Binomial Expansion Method Applied CBO/CLO Analysis*. Moody's Investors Service, Inc. Retrieved from https://quantlabs.net/academy/download/free_quant_instituitional_books_/[Moody's]%20The%20Binomial%20Expansion%20Method%20Applied%20to%20CBO-CLO%20Analysis.pdf.

Dohnalek, J., & Uytenhout, J. M. (2017). *Are contemporary CLO rating models reliable?* (thesis).

Kraemer, N., & Gunter, E. (2022, December 12). *Default, transition, and recovery: 2021 Annual Global Financial Services Default and rating transition study*. S&P Global Ratings. https://www.spglobal.com/ratings/en/research/articles/221212-default-transition-and-recovery-2021-annual-global-financial-services-default-and-rating-transition-study-12573274

Witt, G. (2004). (rep.). *Moody's Correlated Binomial Default Distribution*. New York City, New York: Moody's Investors Service.