Project Report on:

Measure of Entanglement

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Chapter 1

Introduction

This project was done as part of the course Quantum Mechanics 1, offered at CMI during August-November 2025.

This project aims understanding of Hidden Variable Theories, Entangled states and Measure of Entanglement. Chapter 2 discuss Hidden Variable Theories and different inequalities they predict. Chapter 3 discuss entangled pure and mixed bipartite states. Chapter 4 discuss methods to quantify entanglement of pure and mixed bipartite states.

Chapter 2

Hidden Variable Theories & BCHSH Inequalities

2.1 EPR Paradox

In 1935, Einstein, Podolsky, and Rosen published a paper[1] (known as the EPR paper) arguing that just because quantum mechanics prevent us from knowing simultaneous values of some variables, does not imply that they do not exist before the measurements.

Consider 2 systems A & B which were initially interacting, then moved far away, with the following combined wavefunction:

$$\psi(x_1, x_2) = \delta(x_1 - x_2 + x_0) = \int_{-\infty}^{\infty} U_p(x_1) U_{(-p)}(x_2 - x_0) dp$$

$$U_p(x_1) = \frac{1}{\sqrt{2\pi\hbar}} e^{i\frac{x_1 p}{\hbar}} \qquad U_{(-p)}(x_2 - x_0) = \frac{1}{\sqrt{2\pi\hbar}} e^{-i\frac{(x_2 - x_0)p}{\hbar}}$$

$(\hat{P}U_p(x_1) = pU_p(x_1) \ \hat{P}U_{(-p)}(x_2 - x_0) = -pU_{(-p)}(x_2 - x_0))$

Suppose B makes measurement on U_{-p} and gets $-p_0$, then U_p collapses to U_{p_0}.

The same wavefunction can be written as:

$$\psi(x_1, x_2) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f_{x_1}(x) g_{x_2}(x) dx$$

$$f_{x_1}(x) = \delta(x - x_1) \quad g_{x_2}(x) = \delta(x - x_2 + x_0)$$

Now A makes a measurement on f_{x_1} and gets a, then g_{x_2} collapses to g_{a+x_0}

A & B can share this information, and now they know the position and momentum of the particles with high accuracy(in the sense of violating the uncertainty relation).

To prevent this, there should be some *"Spooky Action at a distance"*. And hence they argued that the wave function does not give a complete description of the physical state.

A similar example was given by Bohm[2] in 1951. Consider a system of 2 spin $\frac{1}{2}$ particles in the singlet state:

$$|\psi\rangle_{AB} = \frac{1}{\sqrt{2}}(|+z\rangle_A \otimes |-z\rangle_B - |-z\rangle_A \otimes |+z\rangle_B) = -\frac{1}{\sqrt{2}}(|+x\rangle_A \otimes |-x\rangle_B - |-x\rangle_A \otimes |+x\rangle_B)$$

S_z, S_x do not commute, hence cannot have simultaneous eigen functions (similar to P, X).

2.2 Hidden Variable Theory (HVT)

Hidden Variable Theories were constructed in trying to overcome the above problems. HVT says that every states have some *Hidden Variables* that fixes values for all observables. Because our present methods of preparation of states cannot distinguish the hidden variables, what we call identical states are not really identical, as they differ in their hidden variables.

Properties of Hidden Variables

Let $\hat{A}$ be an observable with eigenvalues a_1, a_2,... Collectively denoted by $V(A)$. (i.e. Any statement with $V(A)$ should hold if $V(A)$ is replaced by any a_j)

If $\hat{A}$, $\hat{B}$, $\hat{C}$,... satisfies

$$f(\hat{A}, \hat{B}, \hat{C}, ...) = 0$$

then

$$f(V(A), V(B), V(C), ...) = 0$$

2.3 Bell's Construction of HVT in 2D Hilbert Space

Bell successfully constructed a hidden variable theory in 2-dimensional Hilbert space that is consistent with quantum mechanics[3]. Consider a 2D Hilbert space $\mathcal{H}^2$. The following are the Pauli Matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

We denote $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$.

Let $\hat{A}$ be an observable in $\mathcal{H}^2$. $\hat{A}$ can be written in terms of the Pauli Matrices and Identity matrix:

$$\hat{A} = a_o\hat{I} + \vec{a} \cdot \vec{\sigma} = \begin{pmatrix} a_0 + a_3 & a_1 - ia_2 \\ a_1 + ia_2 & a_0 - a_3 \end{pmatrix}$$

$(\vec{a} = (a_1, a_2, a_3) \qquad a_0, a_1, a_2, a_3 \in \mathbb{R}\ )$

For a unit vector $\hat{n} \in \mathbb{R}^3$, define $|\hat{n}\rangle$ as $\vec{\sigma} \cdot \hat{n}|\hat{n}\rangle = |\hat{n}\rangle$

$$\langle\hat{n}|\sigma_i|\hat{n}\rangle = n_i \qquad \langle\hat{n}|\hat{A}|\hat{n}\rangle = a_o + \vec{a} \cdot \hat{n}$$

Eigenvalues of $\hat{A}$ are $a_o \pm a$.

$\langle\hat{n}|\sigma_i|\hat{n}\rangle = \frac{1}{2}\left(\langle\hat{n}|\vec{\sigma} \cdot \hat{n}\sigma_i|\hat{n}\rangle + \langle\hat{n}|\sigma_i\vec{\sigma} \cdot \hat{n}|\hat{n}\rangle\right) = \frac{1}{2}\left(\langle\hat{n}|n_j\left(\sigma_j\sigma_i + \sigma_i\sigma_j\right)|\hat{n}\rangle\right)$
$= \langle\hat{n}|n_j\delta_{ij}|\hat{n}\rangle = n_i$
$\langle\hat{n}|\hat{A}|\hat{n}\rangle = a_o + a_i\langle\hat{n}|\sigma_i|\hat{n}\rangle = a_o + \vec{a} \cdot \hat{n}$
$\det(\hat{A} - \lambda\hat{I}) = 0 \implies \lambda^2 - 2\lambda a_o + a_o^2 - a^2 = 0 \implies \lambda = a_o \pm a$
where $a := |\vec{a}|$.

2.3.1 HVT Construction and Consistency with QM

The construction of HVT is as follows.

Every state has a hidden variable $\hat{m}$ (a unit vector in $\mathbb{R}^3$)

$$\langle \hat{n}, \hat{m}|\hat{A}|\hat{n}, \hat{m}\rangle := \begin{cases} a_0 + a & \text{if } (\hat{n}+\hat{m})\cdot\vec{a} > 0 \\ a_0 - a & \text{if } (\hat{n}-\hat{m})\cdot\vec{a} \leq 0 \end{cases}$$

Now, lets check the consistency with quantum mechanics.

As the preparation does not have control over the hidden variables, the expectation value of $\hat{A}$ for the state $|\hat{n}\rangle$ is obtained by averaging over all $\hat{m}$.

$$\langle \hat{n}|\hat{A}|\hat{n}\rangle_{HVT} = \int \frac{d\Omega_m}{4\pi}\langle \hat{n}, \hat{m}|\hat{A}|\hat{n}, \hat{m}\rangle$$

($d\Omega_m$ is the infinitesimal solid angle associated with $\hat{m}$.)

For evaluating the integral, choose coordinates s.t. $\vec{a} = a\hat{z}, a \geq 0$.

$$\langle \hat{n}|\hat{A}|\hat{n}\rangle_{HVT} = \frac{1}{4\pi}\int_0^{2\pi} d\phi_m \int_{-1}^{1} d(\cos\theta_m)\langle \hat{n}, \hat{m}|\hat{A}|\hat{n}, \hat{m}\rangle \qquad (m_3 = \cos\theta_m)$$

$$= \frac{1}{2}\int_{-1}^{-n_3} d\cos\theta_m(a_0 - a) + \frac{1}{2}\int_{-n_3}^{1} d\cos\theta_m(a_0 + a)$$

$$= \frac{1}{2}[(1-n_3)(a_0 - a) + (1+n_3)(a_0+a)] = a_0 + an_3 = a_0 + \vec{a}\cdot\hat{n} = \langle \hat{n}|\hat{A}|\hat{n}\rangle$$

$$\boldsymbol{\langle \hat{n}|\hat{A}|\hat{n}\rangle_{HVT} = \langle \hat{n}|\hat{A}|\hat{n}\rangle}$$

This can be extended to show that $\langle \hat{k}|\hat{A}|\hat{n}\rangle_{HVT} = \langle \hat{k}|\hat{A}|\hat{n}\rangle$

Hence this construction of hidden variable theory in $\mathcal{H}^2$ is consistent with quantum mechanics.

2.4 Higher Dimensions

Bell successfully constructed HVT in 2D. But HVT is inconsistent with QM in higher dimensions. We will look at an example.[4, 5]

2.4.1 Mermin Example in 4D

Consider two sets of Pauli matrices($\alpha\&\beta$) forming 9 observables:

$$\begin{array}{ccc} a_{11} = \sigma_1^\alpha & a_{12} = \sigma_1^\beta & a_{13} = \sigma_1^\alpha\sigma_1^\beta \\ a_{21} = \sigma_2^\beta & a_{22} = \sigma_2^\alpha & a_{23} = \sigma_2^\alpha\sigma_2^\beta \\ a_{31} = \sigma_1^\alpha\sigma_2^\beta & a_{32} = \sigma_2^\alpha\sigma_1^\beta & a_{33} = \sigma_3^\alpha\sigma_3^\beta \end{array}$$

These have the following properties:

- Observables in each row commutes with each other. So does in each column.
- Row Product: $a_{i1}a_{i2}a_{i3} = I \quad$ for $i = 1, 2, 3$

- Column Product: $a_{1i}a_{2i}a_{3i} = I \quad$ for $i = 1, 2$
- Column Product: $a_{13}a_{23}a_{33} = -I$

(Simultaneous measurement of commuting observables can be done.)
Simultaneous measurement of observables in a row is done.

$$\prod_{i=1}^{3} a_{i1}a_{i2}a_{i3} = I \implies \prod_{i,j} V(a_{ij}) = 1$$

Similarly, simultaneous measurement of observables in a column is done.

$$\prod_{i=1}^{3} a_{1i}a_{2i}a_{3i} = -I \implies \prod_{i,j} V(a_{ij}) = -1$$

Which is a contradiction as $V(a_{ij})$ are just numbers, and hence commute.
As H^4 is a subspace of $H^{n \geq 4}$, this example works for higher dimensions.

2.4.2 Non-Contextuality and Locality

In the above, **non-contextuality** is assumed.

Non-Contextuality

Value obtained in measurement of an observable does not depend upon whether it is measured along with a commuting set of Observables $(\hat{O}_2, \hat{O}_3, ...)$ or another commuting set of observables $(\hat{O}'_2, \hat{O}'_3, ...)$.

Weaker Assumption: Locality

We can rather make a weaker assumption of **locality**.

Locality

Value obtained in measurement of an observable does not depend upon whether it is measured along with a commuting set of Observables $(\hat{O}_2, \hat{O}_3, ...)$ or another commuting set of observables $(\hat{O}'_2, \hat{O}'_3, ...)$ when $(\hat{O}_2, \hat{O}_3, ...)$ and $(\hat{O}'_2, \hat{O}'_3, ...)$ are physically far away.
(i.e. "no action at a distance")

A similar example in 8D Hilbert space (with 3 set of Pauli matrices) reach in a similar inconsistency with assuming locality instead of non-contextuality.

A simple example in 8D is **GHZ States** (there are 8 of them).

One being $\frac{1}{\sqrt{2}}(|+++\rangle - |---\rangle)$. (By considering 4 commuting observables: $O_1 = \sigma_1^A \sigma_2^B \sigma_2^C$, $O_2 = \sigma_2^A \sigma_1^B \sigma_2^C$, $O_3 = \sigma_2^A \sigma_2^B \sigma_1^C$, $O_4 = \sigma_1^A \sigma_1^B \sigma_1^C$ and using that $O_1 O_2 O_3 = -O_4$)

There exists examples in 3 dimensions but are more complicated.

A better method to study inconsistencies is discussed in the following section.

2.5 BCHSH Inequality

Local Hidden Variable Theories(LHVT) predicts different inequalities that are violated in QM and observed to violate in experiments.

2.5.1 Bell's Inequality

Let's consider a inequality derived by Bell[6].

Consider a system of 2 spin $\frac{1}{2}$ particles in the singlet state(Bell's State).

(A&B are far away from each other)

$$|\psi\rangle_{AB} = \frac{1}{\sqrt{2}}\left(|+\hat{n}\rangle_A \otimes |-\hat{n}\rangle_B - |-\hat{n}\rangle_A \otimes |+\hat{n}\rangle_B\right)$$

The total spin operator, $\vec{S}_{AB} = \vec{S}_A \otimes \hat{I}_B + \hat{I}_A \otimes \vec{S}_B$

This is a spin 0 state, $\vec{S}_{AB}|\psi\rangle_{AB} = 0 \implies \vec{S}_A \otimes \hat{I}_B|\psi\rangle_{AB} = -\hat{I}_A \otimes \vec{S}_B|\psi\rangle_{AB}$

Let $\hat{\alpha}, \hat{\beta} \in \mathbb{R}^3$ unit vectors.

$${}_{AB}\langle\psi|[(\vec{\hat{S}}_A \cdot \hat{\alpha} \otimes \hat{I}_B)(\hat{I}_A \otimes \vec{\hat{S}}_B \cdot \hat{\beta})]|\psi\rangle_{AB} = -\frac{\hbar^2}{4}\hat{\alpha}\cdot\hat{\beta}$$

${}_{AB}\langle\psi|[(\vec{\hat{S}}_A \cdot \hat{\alpha} \otimes \hat{I}_B)(\hat{I}_A \otimes \vec{\hat{S}}_B \cdot \hat{\beta})]|\psi\rangle_{AB} = -{}_{AB}\langle\psi|[(\vec{\hat{S}}_A \cdot \hat{\alpha} \otimes \hat{I}_B)(\vec{\hat{S}}_A \cdot \hat{\beta} \otimes \hat{I}_B)]|\psi\rangle_{AB}$
$= -{}_{AB}\langle\psi|[\vec{\hat{S}}_A \cdot \hat{\alpha}\vec{\hat{S}}_A \cdot \hat{\beta} \otimes \hat{I}_B]|\psi\rangle_{AB}$
$= -{}_{AB}\langle\psi|[(\frac{\hbar^2}{4}\hat{\alpha}\cdot\hat{\beta}\hat{I}_A + i\frac{\hbar}{2}\epsilon_{ijk}\alpha_i\beta_j(\hat{S}_A)_k) \otimes \hat{I}_B]|\psi\rangle_{AB}$
(Using the identity $\vec{\hat{S}} \cdot \hat{\alpha}\vec{\hat{S}} \cdot \hat{\beta} = \frac{\hbar^2}{4}\hat{\alpha}\cdot\hat{\beta} + i\frac{\hbar}{2}\epsilon_{ijk}\alpha_i\beta_j\hat{S}_k$)
Claim: ${}_{AB}\langle\psi|[(\hat{S}_A)_k \otimes \hat{I}_B]|\psi\rangle_{AB} = 0$
Proof: As the operator in B-space is Identity, only ${}_B\langle+\hat{n}|\hat{I}_B|+\hat{n}\rangle_B$ & ${}_B\langle-\hat{n}|\hat{I}_B|-\hat{n}\rangle_B$ are non-zero.
Now using the result $\langle\hat{n}|\sigma_i|\hat{n}\rangle = n_i$, the equality follows.
Hence ${}_{AB}\langle\psi|[(\vec{\hat{S}}_A \cdot \hat{\alpha} \otimes \hat{I}_B)(\hat{I}_A \otimes \vec{\hat{S}}_B \cdot \hat{\beta})]|\psi\rangle_{AB} = -\frac{\hbar^2}{4}\hat{\alpha}\cdot\hat{\beta}$

Now we assume LHVT with hidden variable μ.

For fixed μ, the measurement outcome $O(\vec{S}_A\cdot\hat{\alpha}, \mu), O(\vec{S}_B\cdot\hat{\beta}, \mu)$ have definite value (each takes one among $\pm\frac{\hbar}{2}$).(Here $O(\vec{S}_A\cdot\hat{\alpha}, \mu)$ means the fixed value of observable $\vec{S}_A\cdot\hat{\alpha}$ given the hidden variable is μ)

By Locality, these values are independent.

Hence the correlation function $E(\hat{\alpha}, \hat{\beta}) = \langle\vec{S}_A \cdot \hat{\alpha}\vec{S}_B \cdot \hat{\beta}\rangle$ is

$$E(\hat{\alpha}, \hat{\beta})_{LHVT} = \int d\mu K(\mu) O(\vec{S}_A\cdot\hat{\alpha}, \mu) O(\vec{S}_B\cdot\hat{\beta}, \mu)$$

where $K(\mu)$ is the probability density of μ, with $\int d\mu K(\mu) = 1$

We need $E(\hat{\alpha}, \hat{\beta})_{LHVT} = E(\hat{\alpha}, \hat{\beta})_{QM}$. Especially when $\hat{\alpha} = \hat{\beta}$ we need:

$$\int d\mu K(\mu) O(\vec{S}_A\cdot\hat{\alpha}, \mu) O(\vec{S}_B\cdot\hat{\alpha}, \mu) = -\frac{\hbar^2}{4}$$

This is only possible if $O(\vec{S}_A\cdot\hat{\alpha}, \mu) = -O(\vec{S}_B\cdot\hat{\alpha}, \mu)$.

(As their absolute value is $\frac{\hbar}{2}$)

Let $\hat{\gamma} \in \mathbb{R}^3$ unit vector. We get,

Bell's Inequality

$$\left|\boldsymbol{E}(\hat{\boldsymbol{\alpha}}, \hat{\boldsymbol{\beta}})_{\mathrm{LHVT}} - \boldsymbol{E}(\hat{\boldsymbol{\alpha}}, \hat{\boldsymbol{\gamma}})_{\mathrm{LHVT}}\right| \leq \frac{\hbar^2}{4} + \boldsymbol{E}(\hat{\boldsymbol{\beta}}, \hat{\boldsymbol{\gamma}})_{\mathrm{LHVT}}$$

$$\begin{aligned}
&\left|E(\hat{\alpha}, \hat{\beta})_{\mathrm{LHVT}} - E(\hat{\alpha}, \hat{\gamma})_{\mathrm{LHVT}}\right| \\
&\leq \int d\mu K(\mu) \left|O(\vec{S}_A \cdot \hat{\alpha}, \mu) O(\vec{S}_B \cdot \hat{\beta}, \mu) - O(\vec{S}_A \cdot \hat{\alpha}, \mu) O(\vec{S}_B \cdot \hat{\gamma}, \mu)\right| \\
&= \int d\mu K(\mu) \left|O(\vec{S}_A \cdot \hat{\alpha}, \mu) O(\vec{S}_B \cdot \hat{\beta}, \mu)\right| \left|1 - \tfrac{4}{\hbar^2} O(\vec{S}_B \cdot \hat{\beta}, \mu) O(\vec{S}_B \cdot \hat{\gamma}, \mu)\right| \\
&= \tfrac{\hbar^2}{4} \int d\mu K(\mu) \left(1 - \tfrac{4}{\hbar^2} O(\vec{S}_B \cdot \hat{\beta}, \mu) O(\vec{S}_B \cdot \hat{\gamma}, \mu)\right) \\
&= \tfrac{\hbar^2}{4} \int d\mu K(\mu) \left(1 + \tfrac{4}{\hbar^2} O(\vec{S}_A \cdot \hat{\beta}, \mu) O(\vec{S}_B \cdot \hat{\gamma}, \mu)\right) = \tfrac{\hbar^2}{4} + E(\hat{\beta}, \hat{\gamma})_{\mathrm{LHVT}}
\end{aligned}$$

Violation of Bell's Inequality

Take $\hat{\alpha} = \hat{\epsilon}_3$, $\hat{\beta} = \frac{\sqrt{3}}{2}\hat{\epsilon}_1 + \frac{1}{2}\hat{\epsilon}_3$, $\hat{\gamma} = \frac{\sqrt{3}}{2}\hat{\epsilon}_1 - \frac{1}{2}\hat{\epsilon}_3$

$|E(\hat{\alpha}, \hat{\beta})_{QM} - E(\hat{\alpha}, \hat{\gamma})_{QM}| = \frac{\hbar^2}{4} \geq \frac{\hbar^2}{8} = \frac{\hbar^2}{4} + E(\hat{\beta}, \hat{\gamma})_{QM}$

Violates Bell's inequality.

2.5.2 CHSH Inequality

A more general inequality is the CHSH inequality.[7]

(Bell's inequality is a special case of CHSH inequality)

Consider 2 systems A & B moving away from each other.

A has two observables $(\hat{x}_1, \hat{x}_2)$ (outcome a_1, a_2). B has two observables $(\hat{y}_1, \hat{y}_2)$ (outcome b_1, b_2). $a_i, b_i \in \{\pm 1\}$

Correlation $\langle a_i b_j \rangle = \sum_{a'_i, b'_j} a'_i b'_j P(a'_i b'_j | \hat{x}_i, \hat{y}_j)$

(Where $P(a'_i b'_j | \hat{x}_i, \hat{y}_j)$ is the probability of getting a'_i for $\hat{x}_i$ and b'_j for $\hat{y}_j$.)

Assuming LHVT, we get,

CHSH Inequality

$$|\boldsymbol{S}| := |\langle \boldsymbol{a_1 b_1} \rangle + \langle \boldsymbol{a_1 b_2} \rangle + |\langle \boldsymbol{a_2 b_1} \rangle - \langle \boldsymbol{a_2 b_2} \rangle|| \leq \boldsymbol{2}$$

Violation of CHSH Inequality

Take $|\psi\rangle_{AB} = \frac{1}{\sqrt{2}}(|+z\rangle_A |-z\rangle_B - |-z\rangle_A |+z\rangle_B)$

$\hat{x}_1 = \hat{\sigma}_1^A$, $\hat{x}_2 = \hat{\sigma}_2^A$, $\hat{y}_1 = -\frac{\hat{\sigma}_1^B + \hat{\sigma}_2^B}{\sqrt{2}}$, $\hat{y}_2 = \frac{-\hat{\sigma}_1^B + \hat{\sigma}_2^B}{\sqrt{2}}$

$\langle a_1 b_1 \rangle = \langle a_1 b_2 \rangle = \langle a_2 b_1 \rangle = \frac{1}{\sqrt{2}}$, $\langle a_2 b_2 \rangle = -\frac{1}{\sqrt{2}}$

$|S| = \left|\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - (-\frac{1}{\sqrt{2}})\right| = 2\sqrt{2} > 2$

Violates CHSH inequality.

Several experiments have been conducted, especially using photon polarization and the inequality is observed to be violated.

2.6 Legget-Garg Inequality

Given below is an example of Legget-Garg inequality[8], which is an inequality for time correlation.

Consider an observable $\hat{O}(t)$ in Heisenberg representation (with outcomes ± 1).

For $t_i \neq t_j$, define the time correlation $C_{ij} := \langle \hat{O}(t_i)\hat{O}(t_j)\rangle$.

Under the 3 assumptions of macroscopic realism, non invasive measurement, and causality, we get,

Legget-Garg Inequality

$$\boldsymbol{-3 \leq C_{12} + C_{32} - C_{31} \leq 1}$$

Macroscopic realism

The value of observables preexists before measurement.

Non invasive measurement

The value of an observable can be obtained without disturbing it.

Violation of Legget-Garg Inequality

Take $\hat{O} = \sigma_3$ and Hamiltonian $H = \frac{\hbar\omega}{2}\sigma_1$. Take $t_3 - t_2 = t_2 - t_1 = \tau$, and $\omega\tau = \frac{\pi}{3}$.

$C_{12} + C_{32} - C_{31} = \frac{3}{2} > 1$

Violates Legget-Garg inequality.

Violation of Legget-Garg inequality has also been observed.

The states that violate the above discussed inequalities exhibit interesting properties, and they open a new window towards understanding the nature of quantum mechanics. The following chapters study these states.

Chapter 3

Entanglement

This chapter study the definition and basic properties of both pure and mixed bipartite entangled states.

3.1 Pure States

Consider 2 Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$ of dimension N.
(They can have different dimensions, we assume same for simplicity)

Definition of Entangled Pure State

A pure bipartite state $|\psi\rangle_{AB} \in \mathcal{H}_A \otimes \mathcal{H}_B$ is said to be **separable** if

$$\exists\ |\phi\rangle_A \in \mathcal{H}_A, |\phi\rangle_B \in \mathcal{H}_B \quad s.t. \quad |\psi\rangle_{AB} = |\phi\rangle_A \otimes |\phi\rangle_B$$

Or in terms of Schmidt Decomposition, if the **Schmidt rank $\boldsymbol{r = 1}$**.
If a state is not separable, then it is said to be **entangled**.

The Schmidt Decomposition Theorem

For any pure state $|\psi\rangle_{AB} \in \mathcal{H}_A \otimes \mathcal{H}_B$, there exists orthonormal bases $\{|\phi_i\rangle_A\}$ and $\{|\phi_i\rangle_B\}$ of $\mathcal{H}_A$ and $\mathcal{H}_B$ respectively, such that:

$$|\psi\rangle_{AB} = \sum_{i=1}^{r} \lambda_i |\phi_i\rangle_A \otimes |\phi_i\rangle_B$$

where $\lambda_i > 0$ are the **Schmidt coefficients** and $r(\leq N)$ is the **Schmidt rank**. ($\sum_{i=1}^{r} \lambda_i^2 = 1$)

It is easy to show that separable states satisfy the BCHSH inequality. It can also be shown[9] that for any pure bipartite entangled states, there exists a setting of observables that violate the BCHSH inequality. All the examples we saw of violation of the inequalities were entangled states.

3.2 Mixed States

Consider again 2 Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$ of dimension N. ($\mathcal{H} := \mathcal{H}_A \otimes \mathcal{H}_B$)

A mixed state in $\mathcal{H}$ is represented by the density matrix $\rho_{AB} = \sum_{i=1}^{R} p_i |\psi_i\rangle\langle\psi_i|$ where $p_i \geq 0, \sum_{i=1}^{R} p_i = 1$, $|\psi_i\rangle \in \mathcal{H}$.

Definition of Entangled Mixed State

A mixed state ρ_{AB} in $\mathcal{H}$ is said to be **separable** if it can be written as

$$\rho_{AB} = \sum_{i=1}^{R} p_i \rho_{iA} \otimes \rho_{iB}$$

where $p_i \geq 0, \sum_i p_i = 1$. ρ_{iA} and ρ_{iB} are mixed states in $\mathcal{H}_A$ and $\mathcal{H}_B$ respectively.

If a state is not separable, then it is said to be **entangled**.

It is easy to see that separable states satisfy the BCHSH inequality. But unlike pure entangled states, it is not necessary that we can find a set of observables for a mixed entangled state that violates the inequality. This makes the study of mixed entangled states much harder and we need new methods to study entanglement.

3.3 The Peres-Horodecki method of Positive Partial Transpose

The density matrix ρ of a separable state in $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ can be written in component form as

$$\rho_{a\mu,b\nu} = \sum_i p_i \rho^A_{iab} \rho^B_{i\mu\nu}$$

Define

$$\sigma_{b\mu,a\nu} := \rho_{a\mu,b\nu}$$

i.e. ρ^A_i has been transposed, hence called partial transpose.

$$\sigma = \sum_{i=1}^{R} p_i {\rho^A_i}^T \otimes \rho^B_i$$

ρ is Hermitian, hence σ is Hermitian.
Eigenvalues does not change when the operator is transposed.
Hence ${\rho^A_i}^T$ is also non-negative with unit trace.
$\therefore$ ${\rho^A_i}^T$ are also legitimate density matrices.
Hence, σ should have only non-negative eigenvalues.

Positive Partial Transpose(PPT)

ρ is said to be PPT if σ has only non-negative eigenvalues.

Hence, PPT is a necessary condition for separability. [10]

Separability $\implies$ PPT

Sufficient Condition for $2 \times 2, 2 \times 3$ dimensional Hilbert spaces[11]

In $H^2 \times H^2, H^2 \times H^3$,

$$\textbf{Separable} \iff \textbf{PPT}$$

But this is not true for higher dimensions, there are PPT states that are entangled.[12]

The concept of PPT is also helpful in quantifying entanglement and will be used in next chapter.

3.4 Faster than light communication and No Cloning Theorem

Here we discuss the major reason why HVT were introduced, a thought that "Spooky Action at a Distance" violates causality.

3.4.1 Impossibility of Super Luminal Communication

An entangled state, such as the singlet state $|\psi\rangle_{AB} = \frac{1}{\sqrt{2}}(|+z\rangle_A|-z\rangle_B - |-z\rangle_A|+z\rangle_B)$, exhibits instantaneous collapse upon measurement. If Observer A measures σ_z^A and obtains $|+z\rangle_A$, the state collapses instantly to $|-z\rangle_B$.

Can this be used for super luminal communication?

(for example, communicate whether A did σ_z^A or σ_x^A)

However, A's measurement outcome is fundamentally probabilistic (50% chance for $|+z\rangle_A$ and 50% for $|-z\rangle_A$). Suppose the state collapsed to $|+z\rangle_A|-z\rangle_B$ on A measuring σ_z^A, then B will get $|-z\rangle_B$ on measurement of σ_z^B. But B could also get $|-z\rangle_B$ if A had measured σ_x^A. Hence B doesn't know the observable A used.

No super luminal communication is possible.

3.4.2 No Cloning Theorem

In the above example, super luminal communication would have been possible if B was able to make multiple copies(clones) of the state after A does the measurement. But this is also prohibited in quantum mechanics.

No Cloning Theorem

There exists no unitary operation U that takes $|\psi\rangle_A \otimes |\phi\rangle_B$ to $|\psi\rangle_A \otimes |\psi\rangle_B$ for all states $|\psi\rangle$.

Proof: Assume such U exists for states $|\psi\rangle$ and $|\psi'\rangle$, such that. Let $|\phi\rangle_B$ be a state s.t.

$$U|\psi\rangle_A|\phi\rangle_B = |\psi\rangle_A|\psi\rangle_B, \quad U|\psi'\rangle_A|\phi\rangle_B = |\psi'\rangle_A|\psi'\rangle_B$$

$$_A\langle\psi|\ \psi'\rangle_A = (\langle\psi|_A\langle\phi|_B)\,(|\psi'\rangle_A|\phi\rangle_B) = (\langle\psi|_A\langle\phi|_B)\,U^\dagger U\,(|\psi'\rangle_A|\phi\rangle_B)$$

$$= (\langle\psi|_A\langle\psi|_B)\,(|\psi'\rangle_A|\psi'\rangle_B) = \ _A\langle\psi|\ \psi'\rangle_A\ _B\langle\psi|\ \psi'\rangle_B$$

Hence, $\langle\psi|\ \psi'\rangle \in \{0,1\}$. This contradicts the assumption that $|\psi\rangle$ and $|\psi'\rangle$ are arbitrary.

3.5 Quantum Teleportation

Quantum teleportation is a process to transfer an unknown quantum state $|\psi\rangle_{A'}$ from Observer A to Observer B (without physically carrying the state from A to B) using a shared entangled state, local operations and classical communication.

A wants to teleport an unknown state $|\psi\rangle_{A'} = a|+z\rangle_{A'} + b|-z\rangle_{A'}$ to B. They share the Bell state $|\Psi^-\rangle_{AB} = \frac{1}{\sqrt{2}}(|+z\rangle_A|-z\rangle_B - |-z\rangle_A|+z\rangle_B)$.

Initial Three-Qubit State of the system is:$|\Psi\rangle_{\text{total}} = |\psi\rangle_{A'} \otimes |\Psi^-\rangle_{AB}$.

$$|\Psi\rangle_{\text{total}} = \frac{1}{2}\left[|\Psi^-\rangle_{A'A}(-a|+z\rangle_B - b|-z\rangle_B) + |\Psi^+\rangle_{A'A}(-a|+z\rangle_B + b|-z\rangle_B)\right]$$

$$+\frac{1}{2}\left[|\Phi^+\rangle_{A'A}(a|-z\rangle_B - b|+z\rangle_B) + |\Phi^-\rangle_{A'A}(a|-z\rangle_B + b|+z\rangle_B)\right]$$

where $\{|\Phi^\pm\rangle_{A'A}, |\Psi^\pm\rangle_{A'A}\}$ are the four Bell states.

Now, A makes a simultaneous measurement of the commuting observables $M_1 = \sigma_1^{A'} \otimes \sigma_1^A$ and $M_3 = \sigma_3^{A'} \otimes \sigma_3^A$.

This collapse the state to one of the Bell states.

Each of them corresponds to different values of $[M_1, M_3]$.

This value is shared to B using classical communication and B does an appropriate local operation.

$M_1,\ M_3$	Operation B does
$1, -1$	$-\sigma_3$
$-1, -1$	$-I$
$1, 1$	$i\sigma_2$
$-1, 1$	σ_1

After this, B will have the state $|\psi\rangle$.

The original state $|\psi\rangle_{A'}$ is altered during the measurement, ensuring consistency with the No Cloning Theorem.

Chapter 4

Measures of Entanglement

In this chapter, we will study some methods to quantify entanglement of bipartite states. We will work with 2 Hilbert spaces $\mathcal{H}_A \& \mathcal{H}_B$ of dimension N. $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$

4.1 Pure States

Consider the pure state in Schmidt Decomposition $|\psi\rangle = \sum_{i=1}^{r} \lambda_i |\phi_i\rangle_A \otimes |\phi_i\rangle_B \in \mathcal{H}$ where $\lambda_i > 0, \sum_{i=1}^{r} \lambda_i^2 = 1$

The density matrix of this pure state is $\rho = |\psi\rangle\langle\psi|$

The reduced density matrix of A is obtained by taking partial trace of ρ w.r.t B.

$$\rho_A = \mathrm{Tr}_B(\rho)$$

(The partial trace is obtained by taking sum in between states of B)

It follows that

$$\rho_A = \sum_{i=1}^{r} \lambda_i^2 |\phi_i\rangle_{AA}\langle\phi_i|$$

A measure for pure state entanglement is the von Neumann entropy of ρ_A.

$$E(|\psi\rangle) = S(\rho_A) = -\mathrm{Tr}(\rho_A \ln \rho_A)$$

Taking trace using $\rho_A = \sum_{i=1}^{r} \lambda_i^2 |\phi_i\rangle_{AA}\langle\phi_i|$, we get

Measure of Entanglement of Pure States

$$\boldsymbol{E(|\psi\rangle) = S(\rho_A) = -\sum_{i=1}^{r} \lambda_i^2 \ln(\lambda_i^2)}$$

Note that $S(\rho_A) = S(\rho_B) = E(|\psi\rangle) = -\sum_{i=1}^{r} \lambda_i^2 \ln(\lambda_i^2)$.

Note also that for separable states, $r = 1 \implies \lambda_1 = 1 \implies E(|\psi\rangle) = 0$ $(\lim_{x\to 0} x^2 \ln(x^2) = 0)$

Example

Consider the Bell's State $|\psi\rangle = \frac{1}{\sqrt{2}}(|+z\rangle_A|-z\rangle_B - |-z\rangle_A|+z\rangle_B)$

$E(|\psi\rangle) = \ln 2$

It is standard to take this value as 1 ebit. $\log_2$ is used instead of ln when $N = 2$.

Maximally Entangled

$E(|\psi\rangle) = -\sum_{i=1}^{r} \lambda_i^2 \ln(\lambda_i^2)$ is maximized when $\lambda_i^2 = \frac{1}{N}\ \forall i$.

Hence, $E_{max} = \ln N$

When a state attains this value of E, it is called **Maximally Entangled**.

The Bell's state is maximally entangled.

4.2 Mixed States

As we need the measure of mixed states to agree with measure of pure states, we require the following properties:

- $E(\rho) = 0 \iff \rho$ is separable.
- (A&B far away) Any processes involving only local operations and classical communication cannot increase $E(\rho)$
- $E(\rho)$ is continuous.(Under a continuous change of ρ)
- $E(\underbrace{\rho \otimes \cdots \otimes \rho}_{n}) = nE(\rho)$
- $0 \leq E(\rho) \leq \ln N$

4.2.1 Entanglement of Formation & Concurrence

A density matrix in $\mathcal{H}$ is of the form

$$\rho = \sum_{i=1}^{R} p_i \rho_i$$

where $\rho_i := |\psi_i\rangle\langle\psi_i|$ is density matrix of a pure state in $\mathcal{H}$. ($p_i \geq 0, \sum_{i=1}^{R} p_i = 1$)

Note that this decomposition is not unique.

We define the entanglement of formation as

Entanglement of Formation

$$\boldsymbol{E(\rho) = \min\{\sum_{i=1}^{R} p_i E(\rho_i)\}}$$

where the minimum is among all pure state decompositions. $E(\rho_i) = E(|\psi_i\rangle)$ is the (von Neumann)measure of entanglement for pure states.

$\underline{N = 2}$

As the entanglement of formation involves a minimum over all pure state decompositions, in general the calculation is not analytical and usually solved numerically in higher dimensions. But for $N = 2$, there is a analytical formula called the Wootter's formula.[13]

Define

$$|\tilde{\psi}\rangle := (\sigma_2 \otimes \sigma_2)|\psi\rangle^*$$

$$\boldsymbol{\tilde{\rho} := (\sigma_2 \otimes \sigma_2)\rho^*(\sigma_2 \otimes \sigma_2)}$$

where * denotes complex conjugate.

$$\rho = \begin{pmatrix} a_{11} & a_{12} & b_{11} & b_{12} \\ a_{12}^* & a_{22} & b_{21} & b_{22} \\ b_{11}^* & b_{21}^* & d_{11} & d_{12} \\ b_{12}^* & b_{22}^* & d_{12}^* & d_{22} \end{pmatrix} \implies \tilde{\rho} = \begin{pmatrix} d_{22} & -d_{12} & -b_{22} & b_{12} \\ -d_{12}^* & d_{11} & b_{21} & -b_{11} \\ -b_{22}^* & b_{21}^* & a_{22} & -a_{12} \\ b_{12}^* & -b_{11}^* & -a_{12}^* & a_{11} \end{pmatrix}$$

It can be shown that the operator $\rho\tilde{\rho}$ has only non-negative eigenvalues.

Let the eigenvalues of $\boldsymbol{\rho\tilde{\rho}}$ be $\lambda_1^2, \lambda_2^2, \lambda_3^2$ and λ_4^2 in the order $\lambda_1^2 \geq \lambda_2^2 \geq \lambda_3^2 \geq \lambda_4^2$. $(\lambda_i \geq 0)$

$\lambda_i s$ are also the eigenvalues of the Hermitian Matrix $\mathcal{R} = \sqrt{\sqrt{\rho}\tilde{\rho}\sqrt{\rho}}$.

Define **Concurrence** of ρ as

$$\boldsymbol{C(\rho) = \max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\}}$$

Wootter's Formula

$$\boldsymbol{E(\rho) = E(C(\rho)) = H_2\left(\frac{1 + \sqrt{1 - C^2(\rho)}}{2}\right)}$$

where $H_2(x) = -x\log_2 x - (1-x)\log_2(1-x)$ is the Shannon Entropy.

$E(C(\rho)) = 0 \iff C(\rho) = 0, \qquad E(C(\rho)) = 1 \iff C(\rho) = 1.$

Also E is a convex function of C.

Example: Werner State

Consider the Werner state

$$\hat{\rho}_W = x|\psi\rangle\langle\psi| + \frac{1-x}{4} I_A \otimes I_B, \qquad \text{for } -\frac{1}{3} \leq x \leq 1$$

where $|\psi\rangle = \frac{1}{\sqrt{2}}(|+z\rangle_A|-z\rangle_B - |-z\rangle_A|+z\rangle_B)$ is a Bell's state.

$$\rho_W = \begin{pmatrix} \frac{1-x}{4} & 0 & 0 & 0 \\ 0 & \frac{1+x}{4} & -\frac{x}{2} & 0 \\ 0 & -\frac{x}{2} & \frac{1+x}{4} & 0 \\ 0 & 0 & 0 & \frac{1-x}{4} \end{pmatrix} = \tilde{\rho}_W$$

When $x \geq 0$, $\lambda_1 = \frac{1+3x}{4}$ and $\lambda_2 = \lambda_3 = \lambda_4 = \frac{1-x}{4}$. $C(\rho_W) = \max\{0, \frac{3x-1}{2}\}$.

When $-\frac{1}{3} \leq x < 0$, $\lambda_1 = \lambda_2 = \lambda_3 = \frac{1-x}{4}$ and $\lambda_4 = \frac{1+3x}{4}$. $C(\rho_W) = 0$.

Hence Werner state is separable if $-\frac{1}{3} \leq x \leq \frac{1}{3}$, and entangled if $\frac{1}{3} < x \leq 1$.

It is maximally entangled for $x = 1$.

4.2.2 Monogamy of Entanglement

Consider 3 different systems α, β and γ.

Let $C_{\alpha\beta}$ and $C_{\alpha\gamma}$ denote the concurrence of the systems $\alpha - \beta$ and $\alpha - \gamma$ respectively. Then the following inequality holds,

$$\boldsymbol{C^2_{\alpha\beta} + C^2_{\alpha\gamma} \leq 1}$$

Especially, if α and β are maximally entangled, then $C_{\alpha\beta} = 1 \implies C_{\alpha\gamma} = 0$, i.e. α and γ are not entangled.

This property of entanglement is called monogamy of entanglement.

Chapter 5

Summary

There exists hidden variable theories that are consistent with quantum mechanics in 2D Hilbert Space. Hidden variable theories are inconsistent with quantum mechanics in higher dimensions. BCHSH inequalities provide methods to confirm and study properties of quantum mechanics. The method of positive partial transpose gives a necessary condition for separable bipartite mixed states and sufficient condition if the Hilbert space dimension is 2×2 or 2×3. Wootter's formula provides an analytical way to compute entanglement of formation for 2×2 dimensional bipartite states in terms of concurrence.

Entanglement has opened new ways for differentiating Quantum Mechanics from Classical Mechanics. Entanglement, teleportation and measure of entanglement are used in multiple fields such as quantum computation, quantum communication, particle physics, condensed matter physics, etc.

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