Basic vectors

$\vec{\nabla} = \hat{\varepsilon}_i \frac{\partial}{\partial x_i} \quad \vec{\nabla}\cdot\vec{A} = \frac{\partial A_i}{\partial x_i}$

$$\vec{\nabla}\times\vec{A} = \mathcal{E}_{ijk}\hat{\varepsilon}_i \frac{\partial A_k}{\partial x_j}$$

$\mathcal{E}_{iab}\mathcal{E}_{icd} = \delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc}$

$$\oint_S \vec{A}\cdot d\vec{S} = \int_V (\vec{\nabla}\cdot\vec{A})dV$$

$$\oint_l \vec{A}\cdot d\vec{l} = \int (\vec{\nabla}\times\vec{A})\cdot d\vec{S}$$

Dirac Delta

$\delta(x) = \lim_{m\to\infty} \frac{\sin mx}{\pi x} = \lim_{m\to\infty} \frac{m}{\sqrt{\pi}} \mathrm{e}^{-m^2x^2}$

$$\int \delta[g(x)]f(x)dx = \sum_{i\in g(x_i)=0} \frac{f(x_i)}{g'(x_i)}$$

$\int \delta^{(n)}(x)f(x)dx = (-1)^n f^{(n)}(0)$
$\nabla^2 \frac{1}{|\vec{r}|} = -4\pi\delta^3(\vec{r})$

$$\frac{\partial}{\partial x_p'}\left[\frac{1}{|\vec{x}-\vec{x}'|^n}\right] = \frac{n(x_p - x_p')}{|\vec{x}-\vec{x}'|^{n+2}}$$

$\delta(x-x') = \sum_n \frac{2}{L}\sin\frac{n\pi x'}{L}\sin\frac{n\pi x}{L}$

Electrostatics & Dipoles

$\vec{p} = Q(\vec{r}_+ - \vec{r}_-) \qquad \rho(\vec{r}) = -\vec{p}\cdot\vec{\nabla}\delta(\vec{r}-\vec{a})$
$\phi(\vec{r}) = k\frac{\vec{p}\cdot(\vec{x}-\vec{a})}{|\vec{x}-\vec{a}|^3}$

$$\vec{E} = \frac{k}{r^5}[3(\vec{r}\cdot\vec{p})\vec{r} - r^2\vec{p}]$$

$\vec{F}(\vec{x}) = (\vec{p}\cdot\vec{\nabla}')\vec{E}(\vec{x}')|_{\vec{x}'=\vec{x}}$
$\vec{E} = -\vec{\nabla}\phi \quad \oint \vec{E}\cdot d\vec{S} = \frac{Q}{\varepsilon_0} \quad \nabla^2\phi = -\frac{\rho}{\varepsilon_0}$
Work done in taking Q from a to b
$W = Q\int_{\vec{a}\ path}^{\vec{b}} \vec{E}(\vec{r}')\cdot d\vec{l}'$
On sheet $\sigma = -\varepsilon_0\frac{\partial V}{\partial n} \quad \frac{F}{A} = \frac{\sigma^2}{2\varepsilon_0}$
$U = \frac{k}{2}\int \frac{\rho\rho'}{|\vec{r}-\vec{r}'|}d^3rd^3r'$

Cylindrical and Spherical Coordinates

$d\vec{l} = ds\ \hat{s} + sd\phi\ \hat{\phi} + dz\ \hat{z}$
$d\vec{l} = dr\ \hat{r} + rd\theta\ \hat{\theta} + r sin\theta d\phi\ \hat{\phi}$
$dt = \vec{\nabla}t\cdot d\vec{l}$
$\vec{\nabla}t = \frac{\partial t}{\partial s}\hat{s} + \frac{1}{s}\frac{\partial t}{\partial\phi}\hat{\phi} + \frac{\partial t}{\partial z}\hat{z}$
$\vec{\nabla}t = \frac{\partial t}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial t}{\partial\theta}\hat{\theta} + \frac{1}{r sin\theta}\frac{\partial t}{\partial\phi}\hat{\phi}$
$\vec{\nabla}\cdot\vec{A} = \frac{1}{s}\frac{\partial}{\partial s}(sA_s) + \frac{1}{s}\frac{\partial A_\phi}{\partial\phi} + \frac{\partial A_z}{\partial z}$
$\vec{\nabla}\cdot\vec{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(A_\theta\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}$
$\vec{\nabla}\times\vec{A} = \left(\frac{1}{s}\frac{\partial A_z}{\partial\phi} - \frac{\partial A_\phi}{\partial z}\right)\hat{s} + \left(\frac{\partial A_s}{\partial z} - \frac{\partial A_z}{\partial s}\right)\hat{\phi}$
$+\frac{1}{s}\left(\frac{\partial}{\partial s}(sA_\phi) - \frac{\partial A_s}{\partial\phi}\right)\hat{z}$
$\nabla\times\vec{A} = \frac{1}{r\sin\theta}\left[\frac{\partial(A_\phi\sin\theta)}{\partial\theta} - \frac{\partial A_\theta}{\partial\phi}\right]\hat{r}$
$+\frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{\partial}{\partial r}(rA_\phi)\right]\hat{\theta} + \frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial\theta}\right]\hat{\phi}$

Some Maths

$\int_V(\phi\nabla^2\psi - \psi\nabla^2\phi)d^3r' = \int_S(\phi\vec{\nabla}\psi - \psi\vec{\nabla}\phi)\cdot d\vec{S}$
$\int_V \phi\nabla^2\psi d^3r' = \int_V \vec{\nabla}\cdot(\phi\vec{\nabla}\psi)d^3r' - \int_V(\vec{\nabla}\phi\cdot\vec{\nabla}\psi)d^3r'$

$$G(x,x') = \begin{cases} \frac{4\pi x'}{L}(L-x) & x > x' \\ \frac{4\pi x}{L}(L-x') & x < x' \end{cases}$$

$\phi(x) = \int_V \rho(x')G(x,x')d^3x'$
$\int_0^a \sin\frac{n\pi x}{a}\sin\frac{m\pi x}{a}dx = \delta_{nm}\frac{a}{2}$
$\sinh x = \frac{\mathrm{e}^x - \mathrm{e}^{-x}}{2} \quad \cosh x = \frac{\mathrm{e}^x + \mathrm{e}^{-x}}{2}$
$\cosh^2 x - \sinh^2 x = 1$
$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots$

Some Special Polynomials

$C_n(x) = \frac{1}{W(x)}\frac{d^n}{dx^n}[S^n(x)W(x)] \qquad C_1 : 1^{st} degree\ poly$
$S : poly\ of\ degree \le 2\ with\ real\ roots$
$W : real, integrable \quad C_n : poly\ degree \le n$
$\int_a^b W(x)C_n(x)C_m(x)dx = \delta_{mn}k_n$
Hermite $H_n : [-\infty,\infty], W(x) = \mathrm{e}^{-x^2}, S(x) = 1$
$\frac{d^2H_n(x)}{dx^2} - 2x\frac{dH_n(x)}{dx} + 2nH_n(x) = 0$
Legendre $P_n : [-1,1], W(x) = 1, S(x) = 1 - x^2$
$(1-x^2)\frac{d^2P_n(x)}{dx^2} - 2x\frac{dP_n(x)}{dx} + n(n+1)P_n(x) = 0$
$\int_{-1}^1 P_n(x)P_m(x)dx = \frac{2}{2n+1}\delta_{nm}$ Normalize $P_n(1) = 1$
$\frac{1}{|\vec{r}-\vec{r}'|} = \sum_{l=0}^\infty \frac{r_<^l}{r_>^{l+1}}P_l(\cos\theta)$
$P_0 = 1 \quad P_1 = x \quad P_2 = \frac{3x^2-1}{2}$

Capacitance & Inductance

$Q_i = C_{ij}\phi_j \quad W = \frac{1}{2}C_{ij}\phi_i\phi_j \quad \phi_j$: potential on j
$\Phi_i = M_{ij}I_j \quad M_{ij} = \frac{\mu_0}{4\pi}\oint\oint\frac{d\vec{l_i}\cdot d\vec{l_j}}{|\vec{r}_i - \vec{r}_j|}$
$\Phi = LI \quad \mathcal{E} = -L\frac{dI}{dt} \quad W = \frac{1}{2}LI^2$
$\Phi = \int\vec{B}\cdot d\vec{a} \quad \mathcal{E} = -\frac{d\Phi}{dt} = \oint\vec{E}\cdot d\vec{l} = V_+ - V_-$

Maxwell's Equations

$$\boldsymbol{\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0} \qquad \vec{\nabla}\cdot\vec{B} = 0 \qquad \vec{\nabla}\cdot\vec{J} = -\frac{\partial\rho}{\partial t}}$$

$$\boldsymbol{\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \qquad \vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t}}$$

$$\boldsymbol{\vec{F} = q\vec{E} + q\vec{v}\times\vec{B}}$$

Electrostatics In Media

$\vec{p}$: dipole/volume
$\rho_b(\vec{r}) = -\vec{\nabla}\cdot\vec{p}(\vec{r}) \quad \sigma_b(\vec{r}) = \vec{p}(\vec{r})\cdot\hat{n} \quad \rho = \rho_b + \rho_f$
$\vec{D} = \varepsilon_0\vec{E} + \vec{p} \quad \vec{\nabla}\cdot\vec{D} = \rho_f \quad \vec{D} = \varepsilon\vec{E} = \varepsilon_0(1+\chi_E)\vec{E}$
$\rho = k\int_S \frac{\sigma_b(\vec{r}')dS'}{|\vec{r}-\vec{r}'|} + k\int_V \frac{\rho_b(\vec{r}')dV'}{|\vec{r}-\vec{r}'|}$
$\vec{E} = \alpha\vec{P} \leftarrow dipole/atom \quad \alpha = 3\varepsilon_0 V \leftarrow volume$
if $\rho_f = 0 \quad D_\perp - D'_\perp = \sigma_f \quad E_\parallel = E'_\parallel$

Tensor in Electrostatics

$F_i = \int_V \rho(\vec{r}')\vec{E}(\vec{r};')dV' = \int_V \frac{\partial T_{ij}}{\partial x_j}d^3r = \int_S T_{ij}dS_j$
$T_{ij} = \varepsilon_0(E_iE_j - \frac{1}{2}\delta_{ij}|\vec{E}|^2)$

Magnetism

$\vec{B}(\vec{r}) = \frac{\mu_0 I}{4\pi}\oint \frac{d\vec{l}\times(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3} = \frac{\mu_0}{4\pi}\int \frac{dS'\vec{K}(\vec{r}')\times(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}$
$= \frac{\mu_0}{4\pi}\int \frac{dV'\vec{J}(\vec{r}')\times(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3} = \vec{\nabla}\times\vec{A} \quad \vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r}')}{|\vec{r}-\vec{r}'|}d^3r'$
$\vec{F}_{mag} = \int I d\vec{l}\times\vec{B} \quad \oint \vec{B}\cdot d\vec{l} = \mu_0 I_{enc} = \mu_0\int \vec{J}\cdot d\vec{S}$
$B_{ring} = \frac{\mu_0 I}{2R} \quad B_{sheet} = \frac{\mu_0 K}{2} \quad B_{tor} = \frac{\mu_0 I}{2\pi r}$
$\vec{B}_{above} - \vec{B}_{below} = \mu_0\vec{K}\times\hat{n} \quad \vec{A}_{above} = \vec{A}_{below}$
$\frac{\partial\vec{A}_{above}}{\partial n} - \frac{\partial\vec{A}_{below}}{\partial n} = -\mu_0\vec{K}$
$\vec{m} = I\int d\vec{a} \quad \vec{A}_{dip}(\vec{r}) = \frac{\mu_0}{4\pi}\frac{\vec{m}\times\vec{r}}{r^3}$
$Torque\ \vec{N} = \vec{m}\times\vec{B} \quad Force\ \vec{F}_{on\ \vec{m}} = \vec{\nabla}(\vec{m}\cdot\vec{B})$
$\vec{B}_{dip} = \frac{\mu_0}{4\pi r^3}\left(\frac{3(\vec{m}\cdot\vec{r})\vec{r}}{r^2} - \vec{m}\right) + \frac{\mu_0}{4\pi}\frac{8\pi}{3}\vec{m}\delta^3(\vec{r})$

Magnetism in Matter

$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{M}(\vec{r}')\times(\vec{r}-\vec{r}')d^3r'}{|\vec{r}-\vec{r}'|^3} \quad \vec{M} = \vec{m}/volume$
$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r}')}{|\vec{r}-\vec{r}'|}d^3r' + \frac{\mu_0}{4\pi}\int \frac{\vec{K}_b(\vec{r}')}{|\vec{r}-\vec{r}'|}dS'$
$\vec{J} = \vec{J}_b + \vec{J}_f \quad \vec{J}_b = \vec{\nabla}\times\vec{M} \quad \vec{K}_b = \vec{M}\times d\hat{S}$
Let $\vec{J}$ contained in volume
$\int[f\vec{J}\cdot\vec{\nabla}'g + g\vec{J}\cdot\vec{\nabla}'f + fg\vec{\nabla}\cdot\vec{J}]d^3r' = 0$
$\vec{A}_{dip} = -\frac{\mu_0}{8\pi r^3}\vec{r}\times\int \vec{r}'\times\vec{J}(\vec{r}')d^3r'$
$\vec{M} = \frac{1}{2}\vec{r}\times\vec{J} \quad \vec{m} = \int \vec{M}(\vec{r}')d^3r'$
Diamagnetism $\Delta\vec{m} = -\frac{1}{4m_e}e^2R^2\hat{\varepsilon}_3 \quad \Delta v = \frac{eRB}{2m_e}$
$\vec{H} = \frac{1}{\mu_0}\vec{B} - \vec{M} \quad \vec{\nabla}\times\vec{H} = \vec{J}_f$
$\vec{B} = \mu\vec{H} = \mu_0(1+\chi_m)H$
$B_\perp - B'_\perp = 0 \quad H_\parallel - H'_\parallel = \vec{K}_f\times\hat{n}$

EM Waves

$\vec{E}(\vec{r},t) = E_0\hat{\varepsilon}(\vec{k})\left(\mathrm{e}^{-i(\omega t-\vec{k}\cdot\vec{r})} + \mathrm{e}^{i(\omega t-\vec{k}\cdot\vec{r})}\right)$
$\vec{B}(\vec{r},t) = B_0\hat{\eta}(\vec{k})\left(\mathrm{e}^{-i(\omega t-\vec{k}\cdot\vec{r})} + \mathrm{e}^{i(\omega t-\vec{k}\cdot\vec{r})}\right)$
$\hat{\varepsilon}\times\hat{\eta} = \hat{k} \quad E_0 = cB_0$
$\vec{k} = |\vec{k}|\hat{\varepsilon}_3\ \vec{E} = \hat{\varepsilon}_1E_1\cos\omega t + \hat{\varepsilon}_2E_2\cos(\omega t+\Delta)$
$\Delta = 0 \implies Linear$
$\Delta = \frac{\pi}{2}, E_1 = E_2 \implies Right\ Circular(Left\ helicity)$
$\Delta = \frac{\pi}{2}, E_1 \neq E_2 \implies Elliptical$

Retarded Potential & Jefimenko's Equations

$t_r = t - \frac{|\vec{r}-\vec{r}'|}{c}$
$\phi(\vec{r},t) = k\int \frac{\rho(\vec{r}',t_r)}{|\vec{r}-\vec{r}'|}d^3r'$
$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r}',t_r)}{|\vec{r}-\vec{r}'|}d^3r'$
$\Box^2 := -\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \nabla^2$
$\Box^2\phi = -\frac{\rho}{\varepsilon_0}$
$\Box^2\vec{A} = -\mu_0\vec{J}$
$\vec{E}(\vec{r},t) = k\int\left[\frac{\rho(\vec{r}',t_r)(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3} + \frac{\dot{\rho}(\vec{r}',t_r)(\vec{r}-\vec{r}')}{c|\vec{r}-\vec{r}'|^2} - \frac{\dot{\vec{J}}(\vec{r}',t_r)}{c^2|\vec{r}-\vec{r}'|}\right]d^3r'$
$\vec{B}(\vec{r},t) = \frac{\mu_0}{4\pi}\int\left[\frac{\vec{J}(\vec{r}',t_r)}{|\vec{r}-\vec{r}'|^3} + \frac{\dot{\vec{J}}(\vec{r}',t_r)}{c|\vec{r}-\vec{r}'|^2}\right]\times(\vec{r}-\vec{r}')d^3r'$

Energy and Momentum

Displacement Current = $\varepsilon_0\frac{\partial\vec{E}}{\partial t}$

$$U_{em} = \frac{1}{2}\int_V\left(\varepsilon_0E^2 + \frac{1}{\mu_0}B^2\right)dV$$

Mech work done on particle $\frac{dW}{dt} = q\vec{E}\cdot\vec{v}$
$\frac{dW}{dt} = \int_V \vec{E}\cdot\vec{J}dV$
Poynting vector $\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B}$

$$\frac{dW_{mech}}{dt} + \int_V\left(\frac{\varepsilon_0}{2}\frac{\partial E^2}{\partial t} + \frac{1}{2\mu_0}\frac{\partial B^2}{\partial t}\right)dV + \int_a \vec{S}\cdot d\vec{a} = 0$$

$\frac{\partial}{\partial t}(u_m + u_{em}) + \vec{\nabla}\cdot\vec{S} = 0$
$\vec{p}_{em}/volume = \varepsilon_0\vec{E}\times\vec{B} = \varepsilon_0\mu_0\vec{S}$
$\vec{F}_{mech} = \int_V(\rho\vec{E} + \vec{J}\times\vec{B})dV$

$$T_{ij} = \varepsilon_0(E_iE_j - \frac{1}{2}\delta_{ij}E^2) + \frac{1}{\mu_0}(B_iB_j - \frac{1}{2}\delta_{ij}B^2)$$

$\partial_jT_{ij} = \varepsilon_0[E_i(\partial_jE_j) + E_j(\partial_jE_i) - \frac{1}{2}\partial_iE^2]$
$\quad + \frac{1}{\mu_0}[B_j(\partial_jB_i) - \frac{1}{2}\partial_iB^2]$

$$f_i = \partial_jT_{ij} - \mu_0\varepsilon_0\frac{\partial S_i}{\partial t} \qquad \vec{F}_{mech} = \int \vec{f}dV$$

$\int\frac{\partial}{\partial t}(p_{mech_i} + p_{em_i}) = \int_a T_{ij}da_j$

Maxwell's Equations in Matter

$$\vec{\nabla}\cdot\vec{D} = \rho_f \qquad \vec{\nabla}\cdot\vec{B} = 0$$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial B}{\partial t} \qquad \vec{\nabla}\times\vec{H} = \vec{J}_f + \frac{\partial\vec{D}}{\partial t}$$

Relativistic Equations

Field Tensor

$$F^{\mu\nu} = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{pmatrix}$$

Dual Tensor

$$G^{\mu\nu} = \begin{pmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & -E_z/c & E_y/c \\ -B_y & E_z/c & 0 & -E_x/c \\ -B_z & -E_y/c & E_x/c & 0 \end{pmatrix}$$

$J^\mu = \rho u^\mu = (c\rho, J_x, J_y, J_z)$

$$\boldsymbol{\partial_\mu J^\mu = 0 \qquad \partial_\nu F^{\mu\nu} = \mu_0 J^\mu \qquad \partial_\nu G^{\mu\nu} = 0}$$

$A^\mu = (\phi/c, A_x, A_y, A_z)$

$$\boldsymbol{F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu}$$

$\Box^2A^\mu = -\mu_0J^\mu$
Minkowski Force $K^\mu = \frac{dp^\mu}{d\tau} = qu_\nu F^{\mu\nu}$