Solving of linear Equations using SVD

- Solving a linear equation
- Gauss elimination and SVD
- HowTo
- Some tricks for SVD

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Problems in Linear Equations

- generally in solving **Ax=b** two error contributions possible:
  1. intrinsic errors in A and b
  2. numerical errors due to rounding
- Mathematical description and handling available

General Error Analysis

Matrix Norm necessary

- e.g. $$||A||_2 = \max\{|\lambda|^{1/2}, \lambda \text{ Eigenwert von } \bar{A}^T A\}$$

How does the final result (and its error) depend on the input value $\tilde{A} = A + \delta A$ and $\tilde{b} = b + \delta b$?

$$\frac{||\delta x||}{||x||} \leq \frac{\operatorname{cond}(A)}{1 - \operatorname{cond}(A)||\delta A||/||A||} \left\{ \frac{||\delta b||}{||b||} + \frac{||\delta A||}{||A||} \right\}$$

with

$$\operatorname{cond}_2(A) := ||A||_2 ||A^{-1}||_2 = \frac{|\lambda_{\max}|}{|\lambda_{\min}|}$$

Eigenvalues of matrix

Example for Condition of Matrix

$$\begin{bmatrix} 1.2969 & 0.8648 \\ 0.2161 & 0.1441 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0.8642 \\ 0.1440 \end{bmatrix}$$

Exact solution: **$(2,-2)^T$**

$\boxed{\mathrm{cond}(A)=1.513\times 10^8}$

Change input values:

0.8642 → 0.86419999
0.1440 → 0.14400001

approximate solution: **$(0.9911,-0.4870)^T$**

NOT ACCEPTABLE!

Gauss Elimination

- Error due to rounding:

n=m=30 $\rightarrow 10^{13}$

$$\frac{||\delta x||_\infty}{||x||_\infty} \leq \frac{\mathrm{cond}(A)}{1-\mathrm{cond}(A)||\delta A||_\infty/||A||_\infty}\{1.01\cdot 2^{n-1}(n^3+2n^2)\,\mathrm{eps}\}$$

with eps = machine accuracy (double: ~10^{-16})

However, this is the worst case scenario!

Singular Value Decomposition

- here for (nxn) case, valid also for (nxm)
- Solution of linear equations numerically difficult for matrices with bad condition:
  - regular matrices in numeric approximation can be singular
  - **SVD** helps finding and dealing with the sigular values

How does SVD work

Definition of singular value decomposition:

$$U^T A V = \Sigma = diag(\sigma_1, \ldots, \sigma_p)$$

with *U* and *V* orthogonal matrices

singular values

$$\begin{pmatrix} s_1 & & & \\ & s_2 & & \\ & & \ldots & \\ & & & s_n \end{pmatrix}$$

$$Av_i = \sigma_i u_i \,,$$
$$A^T u_i = \sigma_i v_i \,,$$

$$\Rightarrow \quad A^T A v_i = \sigma_i^2 v_i$$

σ_i are eigenvalues of $A^T A$

Solution

Solution of the Equation:

$$\boxed{\bar{x} = \sum_{i=1}^{r} \frac{u_i^T b}{\sigma_i} v_i}$$

numeric behaviour of SVD can be determined:

$$||\delta A||_2 \leq \text{eps}||A||_2$$

SVD after Golub and Reisch

Householder Matrix P,Q=1-ww^T
(numerically OK)

$$A = \begin{bmatrix} x & x & x & x \\ x & x & x & x \\ x & x & x & x \\ x & x & x & x \end{bmatrix} \to PA = A' = \begin{bmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \end{bmatrix}$$

$$A' = \begin{bmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \end{bmatrix} \to A'Q = A'' = \begin{bmatrix} x & * & 0 & 0 \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \end{bmatrix}$$

P,Q unitary

SVD after Golub and Reisch

- after a couple of iterations:

J_0 Bidiagonal form:

$$J_0 = \begin{bmatrix} q_1 & e_2 & 0 & 0 \\ 0 & q_2 & e_3 & * \\ 0 & 0 & q_3 & e_n \\ 0 & 0 & 0 & q_n \end{bmatrix}$$

with:

$$J_0 = P_n \ldots P_1 A Q_1 \ldots Q_{n-2}$$

Q and P Householder matirces

➢A and J_0 have the same singular values

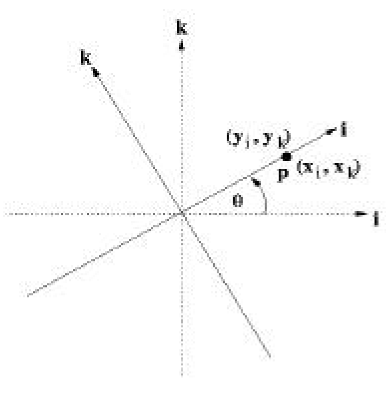
SVD after Golub and Reisch

use iterative procedure to transform bidiagonal J_0 matrix to diagonal form

- apply 'Givens reflections' to bidiagonal Matrix:

$$\bar{J}_0 = S_{n-1,n} \ldots S_{23} S_{12} J_0 T_{12} T_{23} \ldots T_{n-1,n}$$

'Givens Reflection':



~cubic convergence expected

Arithmetic Expenses

- **Gauss (normal) solution**
  - $\frac{1}{2}mn^2+1/6n^3$
- **SVD Golub-Reinsch**
  - $2mn^2+4n^3$

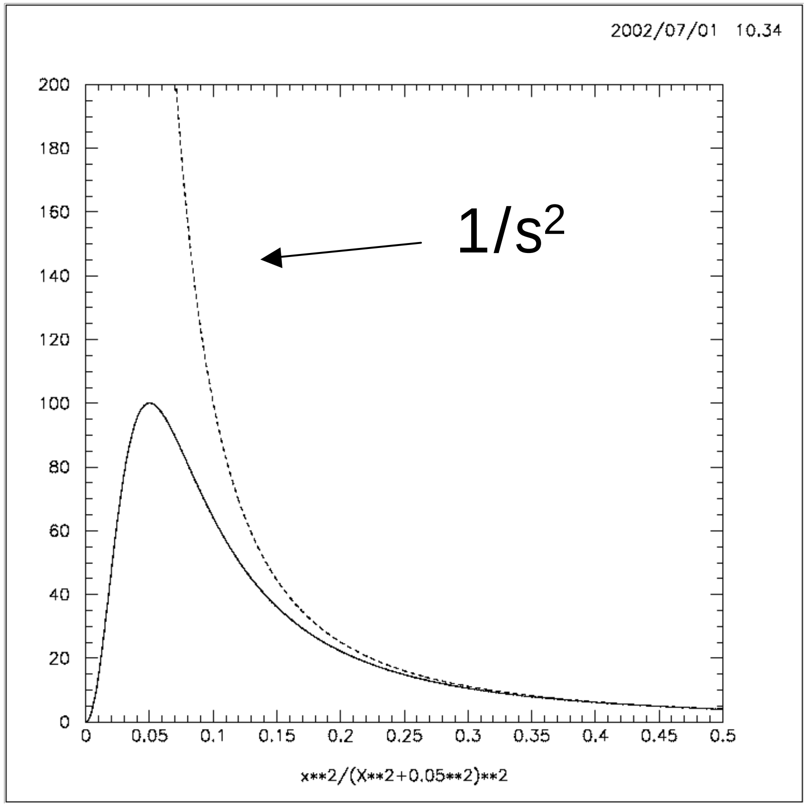
➢ **m=n :**

SVD is 9 times more expensive!

How to deal with Singularities

- singularities are determined with the SVD
- $1/\sigma_i$ is used for solution of linear equation
- relate $1/\sigma_i$ to machine accuracy and resolution τ
  - **usage of values s_i < t corrupts complet result!**
  - **careful handling necessary!**
- simple approach:
  - negelect all values in matrix with $\sigma_i < \tau$

Smooth cutoff

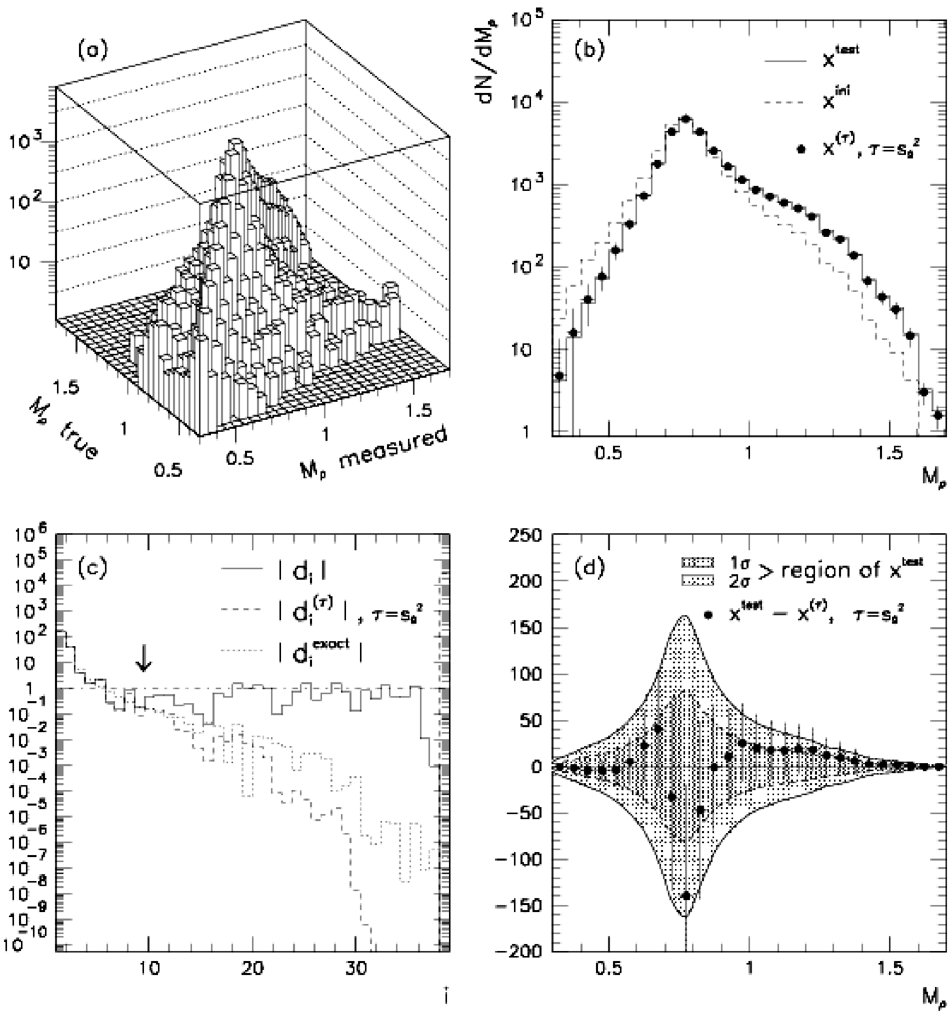


τ=0.05

- Regularisation of the singularities
- Replace the singular values with a function:

$$\frac{1}{s^2} \rightarrow \frac{s^2}{\left(s^2 + t^2\right)^2}$$

Example



Number-of-events response matrix

- $d_i \sim b_i/\Delta b_i$

  $d_i < 1 \rightarrow$

  statistically insignificant

- Problem for alignment:

  determination of τ

Conclusion

- Numeric Solution of a regular linear equation can be distorted by singular behaviour
- SVD returns singular values
- Singular values can be handeld with smooth cut off
- Mathemaical well described procedure