5 Representation of Tensor Functions

5.1 The Basic Idea of the Representation of Tensor Functions

1. Modeling a physical process often means to find a function relating two tensors. From the context we can assume that such a function exists, but it is not known and it can often be determined only from experiments. Typical examples of such functions are the stress–displacement relation $\underline{\underline{T}} = \underline{\underline{f}}(\underline{\underline{D}})$ for an elastic solid, where the (symmetric) stress tensor $\underline{\underline{T}}$ depends on an (also symmetric) displacement tensor $\underline{\underline{D}}$; or the yield-stress condition $\sigma = f(\underline{\underline{T}})$ in plasticity theory, which relates the experimentally determined yield-stress σ to the three-dimensional stress state described by the stress tensor $\underline{\underline{T}}$.

The functions f or $\underline{\underline{f}}$ in the examples are called, according to the order of the tensors, scalar-valued or tensor-valued tensor functions. Contrary to what one may expect, these functions cannot have an arbitrary form, because the transformation properties of a tensor impose certain restrictions. For example, if in a given Cartesian coordinate system $\sigma = f(T_{ij})$ and $T_{ij} = f_{ij}(D_{kl})$, then, under a change of the coordinate system, substituting the transformed coordinates $\tilde{T}_{ij}$ into the function f must result in the same scalar $\sigma = f(\tilde{T}_{ij})$, and similarly, substituting the transformed coordinates $\tilde{D}_{kl}$ into the function $\underline{\underline{f}}$ must give the transformed coordinates $\tilde{T}_{ij} = f_{ij}(\widetilde{D}_{kl})$. If we assume that the tensors are polar, then, according to the transformation equations (2.17), a scalar-valued function f must satisfy

$$f(T_{mn}) = f(\alpha_{im}\,\alpha_{jn}\,T_{ij}), \tag{a}$$

and a tensor-valued function $\underline{\underline{f}}$ is restricted by the condition

$$\alpha_{mi}\,\alpha_{nj}\,f_{mn}(D_{kl}) = f_{ij}(\alpha_{pk}\,\alpha_{ql}\,D_{pq}). \tag{b}$$

2. How to evaluate conditions of the type (a) or (b) and how to use them to construct functions relating tensors is the subject of a theory called the theory of the representation of tensor functions. The basic idea is that a scalar-valued function cannot depend on single tensor coordinates, but only on scalar combinations of tensor coordinates, which are clearly invariant under a coordinate transformation. This explains why, instead of the theory of the representation of tensor functions, we also speak of the theory of invariants. We will show that for any symmetric tensor, such as the stress tensor $\underline{\underline{T}}$, the three basic invariants $\operatorname{tr}\underline{\underline{T}}$, $\operatorname{tr}\underline{\underline{T}}^2$, and $\operatorname{tr}\underline{\underline{T}}^3$ form a complete set of invariants, so we can write the scalar-valued function f in our example as a function of the three basic invariants (or, according to Section 3.15, of another equivalent set of invariants, such as the eigenvalues or the main invariants), so we have

$$\sigma = f(\operatorname{tr}\underline{\underline{T}}, \operatorname{tr}\underline{\underline{T}}^2, \operatorname{tr}\underline{\underline{T}}^3).$$

https://doi.org/10.1515/9783110404265-005

These considerations also apply to tensor-valued functions, if we make them scalar-valued by introducing an auxiliary tensor. In our example, then $\underline{\underline{T}} = \underline{f}(\underline{\underline{D}})$ gives temporarily

$$\underline{\underline{T}} \cdot\cdot\, \underline{\underline{H}} = f(\underline{\underline{D}}, \underline{\underline{H}}),$$

where the auxiliary tensor $\underline{\underline{H}}$ must appear in such a way that it can be canceled out in the end.

Before we turn to the details, we have to clarify how many tensor invariants are needed for a (yet to be defined) representation to be complete. For this we first need to generalize the Cayley–Hamilton theorem, which in its basic version relates only powers of the same tensor, while in the theory of the representation of tensor functions, products of different tensors often appear as well. Since the tensors in physical applications are often polar, we restrict ourselves to polar tensors for the rest of this chapter (this includes scalars and vectors as tensors of order zero and one, respectively), without mentioning this every time. In the few cases where we consider axial tensors, or if the distinction between polar and axial tensors is important for the argument, we will mention this explicitly.

3. In physics, representation of tensor functions is as important as dimensional analysis. Physical quantities are the product of a numerical value and a unit, but the quantity itself is independent of the chosen unit, i. e. if we change the unit, the numerical value changes correspondingly. This property carries over to equations between physical quantities, where we say that an equation must be dimensionally homogeneous. Specifically, it follows that physical quantities cannot be related by an arbitrary function, but only by functions which are invariant under a change of units. This condition of invariance under a change of units has its analog for tensors in the relations (a) and (b), which result from the requirement that an equation must be invariant under a change of the coordinate system.

In dimensional analysis we use the condition of invariance under a change of units to replace relations between quantities by relations between dimensionless combinations of quantities (which do not need units). In many cases, this leads to restrictions on the permissible functions. Analogously, in the theory of the representation of tensor functions, we change to scalar combinations of tensor coordinates (i. e. tensor invariants), which again leads to restrictions on the permissible functions.

5.2 Generalized Cayley–Hamilton Theorem

1. According to (1.20), $\delta^{ijkl}_{pqrs} = 0$, and then (1.19) gives

$$\begin{vmatrix} \delta_{ip} & \delta_{iq} & \delta_{ir} & \delta_{is} \\ \delta_{jp} & \delta_{jq} & \delta_{jr} & \delta_{js} \\ \delta_{kp} & \delta_{kq} & \delta_{kr} & \delta_{ks} \\ \delta_{lp} & \delta_{lq} & \delta_{lr} & \delta_{ls} \end{vmatrix} = 0.$$

Contraction with $a_{pi}\, b_{qj}\, c_{rk}$ gives, if we multiply the first row by a_{pi}, the second row by b_{qj}, and the third row by c_{rk},

$$\begin{vmatrix} a_{pp} & a_{pq} & a_{pr} & a_{ps} \\ b_{qp} & b_{qq} & b_{qr} & b_{qs} \\ c_{rp} & c_{rq} & c_{rr} & c_{rs} \\ \delta_{lp} & \delta_{lq} & \delta_{lr} & \delta_{ls} \end{vmatrix} = 0.$$

We expand this determinant along the last row, so we obtain

$$-\delta_{lp} \begin{vmatrix} a_{pq} & a_{pr} & a_{ps} \\ b_{qq} & b_{qr} & b_{qs} \\ c_{rq} & c_{rr} & c_{rs} \end{vmatrix} + \delta_{lq} \begin{vmatrix} a_{pp} & a_{pr} & a_{ps} \\ b_{qp} & b_{qr} & b_{qs} \\ c_{rp} & c_{rr} & c_{rs} \end{vmatrix} - \delta_{lr} \begin{vmatrix} a_{pp} & a_{pq} & a_{ps} \\ b_{qp} & b_{qq} & b_{qs} \\ c_{rp} & c_{rq} & c_{rs} \end{vmatrix} + \delta_{ls} \begin{vmatrix} a_{pp} & a_{pq} & a_{pr} \\ b_{qp} & b_{qq} & b_{qr} \\ c_{rp} & c_{rq} & c_{rr} \end{vmatrix} = 0.$$

Multiplying δ_{lp}, δ_{lq}, and δ_{lr} into the first, second, and third row of the first three determinants, respectively, gives

$$-\begin{vmatrix} a_{lq} & a_{lr} & a_{ls} \\ b_{qq} & b_{qr} & b_{qs} \\ c_{rq} & c_{rr} & c_{rs} \end{vmatrix} + \begin{vmatrix} a_{pp} & a_{pr} & a_{ps} \\ b_{lp} & b_{lr} & b_{ls} \\ c_{rp} & c_{rr} & c_{rs} \end{vmatrix} - \begin{vmatrix} a_{pp} & a_{pq} & a_{ps} \\ b_{qp} & b_{qq} & b_{qs} \\ c_{lp} & c_{lq} & c_{ls} \end{vmatrix} + \delta_{ls} \begin{vmatrix} a_{pp} & a_{pq} & a_{pr} \\ b_{qp} & b_{qq} & b_{qr} \\ c_{rp} & c_{rq} & c_{rr} \end{vmatrix} = 0.$$

We expand each of the determinants and obtain

$$\begin{aligned} &- a_{lq}\, b_{qr}\, c_{rs} - a_{lr}\, b_{qs}\, c_{rq} - a_{ls}\, b_{qq}\, c_{rr} + a_{ls}\, b_{qr}\, c_{rq} + a_{lr}\, b_{qq}\, c_{rs} + a_{lq}\, b_{qs}\, c_{rr} \\ &+ a_{pp}\, b_{lr}\, c_{rs} + a_{pr}\, b_{ls}\, c_{rp} + a_{ps}\, b_{lp}\, c_{rr} - a_{ps}\, b_{lr}\, c_{rp} - a_{pr}\, b_{lp}\, c_{rs} - a_{pp}\, b_{ls}\, c_{rr} \\ &- a_{pp}\, b_{qq}\, c_{ls} - a_{pq}\, b_{qs}\, c_{lp} - a_{ps}\, b_{qp}\, c_{lq} + a_{ps}\, b_{qq}\, c_{lp} + a_{pq}\, b_{qp}\, c_{ls} + a_{pp}\, b_{qs}\, c_{lq} \\ &+ \delta_{ls}\,(a_{pp}\, b_{qq}\, c_{rr} + a_{pq}\, b_{qr}\, c_{rp} + a_{pr}\, b_{qp}\, c_{rq} - a_{pr}\, b_{qq}\, c_{rp} - a_{pq}\, b_{qp}\, c_{rr} \\ &\qquad - a_{pp}\, b_{qr}\, c_{rq}) = 0. \end{aligned}$$

Translating this expression into symbolic notation gives

$$\begin{aligned}
&-\underline{\underline{a}}\cdot\underline{\underline{b}}\cdot\underline{\underline{c}}-\underline{\underline{a}}\cdot\underline{\underline{c}}\cdot\underline{\underline{b}}-\underline{\underline{a}}\,\mathrm{tr}\,\underline{\underline{b}}\,\mathrm{tr}\,\underline{\underline{c}}+\underline{\underline{a}}\,\mathrm{tr}(\underline{\underline{b}}\cdot\underline{\underline{c}})+\underline{\underline{a}}\cdot\underline{\underline{c}}\,\mathrm{tr}\,\underline{\underline{b}}+\underline{\underline{a}}\cdot\underline{\underline{b}}\,\mathrm{tr}\,\underline{\underline{c}}\\
&+\underline{\underline{b}}\cdot\underline{\underline{c}}\,\mathrm{tr}\,\underline{\underline{a}}+\underline{\underline{b}}\,\mathrm{tr}(\underline{\underline{a}}\cdot\underline{\underline{c}})+\underline{\underline{b}}\cdot\underline{\underline{a}}\,\mathrm{tr}\,\underline{\underline{c}}-\underline{\underline{b}}\cdot\underline{\underline{c}}\cdot\underline{\underline{a}}-\underline{\underline{b}}\cdot\underline{\underline{a}}\cdot\underline{\underline{c}}-\underline{\underline{b}}\,\mathrm{tr}\,\underline{\underline{a}}\,\mathrm{tr}\,\underline{\underline{c}}\\
&-\underline{\underline{c}}\,\mathrm{tr}\,\underline{\underline{a}}\,\mathrm{tr}\,\underline{\underline{b}}-\underline{\underline{c}}\cdot\underline{\underline{a}}\cdot\underline{\underline{b}}-\underline{\underline{c}}\cdot\underline{\underline{b}}\cdot\underline{\underline{a}}+\underline{\underline{c}}\cdot\underline{\underline{a}}\,\mathrm{tr}\,\underline{\underline{b}}+\underline{\underline{c}}\,\mathrm{tr}(\underline{\underline{a}}\cdot\underline{\underline{b}})+\underline{\underline{c}}\cdot\underline{\underline{b}}\,\mathrm{tr}\,\underline{\underline{a}}\\
&+\underline{\underline{\delta}}\Big(\mathrm{tr}\,\underline{\underline{a}}\,\mathrm{tr}\,\underline{\underline{b}}\,\mathrm{tr}\,\underline{\underline{c}}+\mathrm{tr}(\underline{\underline{a}}\cdot\underline{\underline{b}}\cdot\underline{\underline{c}})+\mathrm{tr}(\underline{\underline{a}}\cdot\underline{\underline{c}}\cdot\underline{\underline{b}})-\mathrm{tr}(\underline{\underline{a}}\cdot\underline{\underline{c}})\,\mathrm{tr}\,\underline{\underline{b}}-\mathrm{tr}(\underline{\underline{a}}\cdot\underline{\underline{b}})\,\mathrm{tr}\,\underline{\underline{c}}\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad-\mathrm{tr}(\underline{\underline{b}}\cdot\underline{\underline{c}})\,\mathrm{tr}\,\underline{\underline{a}}\Big)=\underline{\underline{0}}.
\end{aligned}$$

Finally, after sorting these terms and reversing the signs, we obtain

$$\begin{aligned}
&\underline{\underline{a}}\cdot\left(\underline{\underline{b}}\cdot\underline{\underline{c}}+\underline{\underline{c}}\cdot\underline{\underline{b}}\right)+\underline{\underline{b}}\cdot\left(\underline{\underline{a}}\cdot\underline{\underline{c}}+\underline{\underline{c}}\cdot\underline{\underline{a}}\right)+\underline{\underline{c}}\cdot\left(\underline{\underline{a}}\cdot\underline{\underline{b}}+\underline{\underline{b}}\cdot\underline{\underline{a}}\right)\\
&-\left(\underline{\underline{b}}\cdot\underline{\underline{c}}+\underline{\underline{c}}\cdot\underline{\underline{b}}\right)\mathrm{tr}\,\underline{\underline{a}}-\left(\underline{\underline{a}}\cdot\underline{\underline{c}}+\underline{\underline{c}}\cdot\underline{\underline{a}}\right)\mathrm{tr}\,\underline{\underline{b}}-\left(\underline{\underline{a}}\cdot\underline{\underline{b}}+\underline{\underline{b}}\cdot\underline{\underline{a}}\right)\mathrm{tr}\,\underline{\underline{c}}\\
&+\left[\mathrm{tr}\,\underline{\underline{b}}\,\mathrm{tr}\,\underline{\underline{c}}-\mathrm{tr}(\underline{\underline{b}}\cdot\underline{\underline{c}})\right]\underline{\underline{a}}+\left[\mathrm{tr}\,\underline{\underline{a}}\,\mathrm{tr}\,\underline{\underline{c}}-\mathrm{tr}(\underline{\underline{a}}\cdot\underline{\underline{c}})\right]\underline{\underline{b}}\\
&\qquad\qquad\qquad\qquad\qquad+\left[\mathrm{tr}\,\underline{\underline{a}}\,\mathrm{tr}\,\underline{\underline{b}}-\mathrm{tr}(\underline{\underline{a}}\cdot\underline{\underline{b}})\right]\underline{\underline{c}}\\
&-\Big[\mathrm{tr}\,\underline{\underline{a}}\,\mathrm{tr}\,\underline{\underline{b}}\,\mathrm{tr}\,\underline{\underline{c}}+\mathrm{tr}(\underline{\underline{a}}\cdot\underline{\underline{b}}\cdot\underline{\underline{c}})+\mathrm{tr}(\underline{\underline{c}}\cdot\underline{\underline{b}}\cdot\underline{\underline{a}})\\
&\qquad-\mathrm{tr}(\underline{\underline{b}}\cdot\underline{\underline{c}})\,\mathrm{tr}\,\underline{\underline{a}}-\mathrm{tr}(\underline{\underline{a}}\cdot\underline{\underline{c}})\,\mathrm{tr}\,\underline{\underline{b}}-\mathrm{tr}(\underline{\underline{a}}\cdot\underline{\underline{b}})\,\mathrm{tr}\,\underline{\underline{c}}\Big]\underline{\underline{\delta}}=\underline{\underline{0}}.
\end{aligned}\tag{5.1}$$

2. If we set $\underline{\underline{a}}=\underline{\underline{b}}=\underline{\underline{c}}$, then (5.1) simplifies to

$$6\,\underline{\underline{a}}^3-6\,\mathrm{tr}\,\underline{\underline{a}}\,\underline{\underline{a}}^2+3\left[\mathrm{tr}^2\,\underline{\underline{a}}-\mathrm{tr}\,\underline{\underline{a}}^2\right]\underline{\underline{a}}-\left[\mathrm{tr}^3\,\underline{\underline{a}}-3\,\mathrm{tr}\,\underline{\underline{a}}\,\mathrm{tr}\,\underline{\underline{a}}^2+2\,\mathrm{tr}\,\underline{\underline{a}}^3\right]\underline{\underline{\delta}}=0.$$

After dividing by −6 and using (3.98) we see that this is the Cayley–Hamilton theorem (3.94), so we call (5.1) the generalized Cayley–Hamilton theorem.

3. For the representation of tensor functions, the first row of (5.1) is of particular importance, so we introduce the following symbol:

$$\underline{\underline{\Sigma}}:=\underline{\underline{a}}\cdot\left(\underline{\underline{b}}\cdot\underline{\underline{c}}+\underline{\underline{c}}\cdot\underline{\underline{b}}\right)+\underline{\underline{b}}\cdot\left(\underline{\underline{a}}\cdot\underline{\underline{c}}+\underline{\underline{c}}\cdot\underline{\underline{a}}\right)+\underline{\underline{c}}\cdot\left(\underline{\underline{a}}\cdot\underline{\underline{b}}+\underline{\underline{b}}\cdot\underline{\underline{a}}\right).\tag{5.2}$$

Then we obtain from (5.1), after solving for the first row, i. e.

$$\begin{aligned}
\underline{\underline{\Sigma}}=&\left(\underline{\underline{b}}\cdot\underline{\underline{c}}+\underline{\underline{c}}\cdot\underline{\underline{b}}\right)\mathrm{tr}\,\underline{\underline{a}}+\left(\underline{\underline{a}}\cdot\underline{\underline{c}}+\underline{\underline{c}}\cdot\underline{\underline{a}}\right)\mathrm{tr}\,\underline{\underline{b}}+\left(\underline{\underline{a}}\cdot\underline{\underline{b}}+\underline{\underline{b}}\cdot\underline{\underline{a}}\right)\mathrm{tr}\,\underline{\underline{c}}\\
&-\left[\mathrm{tr}\,\underline{\underline{b}}\,\mathrm{tr}\,\underline{\underline{c}}-\mathrm{tr}(\underline{\underline{b}}\cdot\underline{\underline{c}})\right]\underline{\underline{a}}-\left[\mathrm{tr}\,\underline{\underline{a}}\,\mathrm{tr}\,\underline{\underline{c}}-\mathrm{tr}(\underline{\underline{a}}\cdot\underline{\underline{c}})\right]\underline{\underline{b}}\\
&\qquad\qquad\qquad\qquad\qquad-\left[\mathrm{tr}\,\underline{\underline{a}}\,\mathrm{tr}\,\underline{\underline{b}}-\mathrm{tr}(\underline{\underline{a}}\cdot\underline{\underline{b}})\right]\underline{\underline{c}}\\
&+\Big[\mathrm{tr}\,\underline{\underline{a}}\,\mathrm{tr}\,\underline{\underline{b}}\,\mathrm{tr}\,\underline{\underline{c}}+\mathrm{tr}(\underline{\underline{a}}\cdot\underline{\underline{b}}\cdot\underline{\underline{c}})+\mathrm{tr}(\underline{\underline{c}}\cdot\underline{\underline{b}}\cdot\underline{\underline{a}})\\
&\qquad-\mathrm{tr}(\underline{\underline{b}}\cdot\underline{\underline{c}})\,\mathrm{tr}\,\underline{\underline{a}}-\mathrm{tr}(\underline{\underline{a}}\cdot\underline{\underline{c}})\,\mathrm{tr}\,\underline{\underline{b}}-\mathrm{tr}(\underline{\underline{a}}\cdot\underline{\underline{b}})\,\mathrm{tr}\,\underline{\underline{c}}\Big]\underline{\underline{\delta}}.
\end{aligned}\tag{5.3}$$

5.3 Invariants of Vectors and Second-Order Tensors

Let us recall our definition of an invariant. An invariant is a scalar combination of tensor coordinates which (for polar scalars) does not change under a change of the coordinate system, or (for axial scalars) at most its sign changes. Following our definition from Section 5.1, however, we use the term invariant in this chapter only for polar scalars.

Problem 5.1.
Show, using the transformation equations (2.17) or (2.18), that tr $\underline{\underline{T}}$ and tr $\underline{\underline{T}}^2$ are invariants of a (polar or axial) second-order tensor.

In Section 3.15, we saw that the eigenvalues, the main invariants, and the basic invariants are three different sets of three invariants of a second-order tensor. The elements of each of these three sets are independent of each other, i. e. no element of a set can be computed from the other two elements of the same set; but the three sets are not independent of each other, i. e. if we know the invariants of one set, then we can compute the invariants of the other sets. However, it is not clear if the three invariants of a set form a complete set, i. e. if further invariants exist, which cannot be computed from these invariants. A complete set of invariants is called a basis (of invariants).

The easiest way to compute invariants is from appropriate contractions of tensor coordinates. An example are the main invariants of a second-order tensor. Such invariants are polynomials of tensor coordinates and we call them polynomial invariants. In the theory of the representation of tensor functions we usually consider only polynomial invariants and we use the following terms. A polynomial invariant which can be written as a polynomial of other polynomial invariants is called reducible (with respect to these invariants); otherwise it is called irreducible. A complete set of irreducible invariants, i. e. a set such that all other polynomial invariants can be written in terms of these irreducible invariants, is called an integrity basis. However, it is also possible that polynomial invariants are related by a polynomial and yet none of them can be written *as a polynomial* of the other invariants. Such polynomials are called syzygies;[1] an example is equation (2.46) for the scalar products of the six vectors $\underline{a}$, $\underline{b}$, $\underline{c}$, $\underline{d}$, $\underline{e}$, and $\underline{f}$.

An example of nonpolynomial invariants are the eigenvalues of a second-order tensor. We computed them in Section 3.11.2 from a cubic equation whose coefficients are polynomial invariants, but an eigenvalue cannot be written as a polynomial of tensor coordinates.

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1 Greek: pair (from *syn* [together] and *zygon* [yoke]), i. e. literally, a pair coupled together by a yoke, e. g. in astronomy a generic term for conjunction and opposition of two planets, and in poetry the juxtaposition of two equal metrical feet.

The polynomial invariants are a subset of all the invariants of a tensor. Since reducible invariants are polynomial combinations of tensor coordinates, it may still happen that there are more irreducible invariants than independent invariants. In other words, an integrity basis can contain more elements than a basis.

5.3.1 Invariants of Vectors

A single vector $\underline{u}$ has only one irreducible invariant, i. e. its square $u_i\, u_i$.

From two vectors $\underline{u}$ and $\underline{v}$ we can form a total of three irreducible invariants:
- the squares $u_i\, u_i$ and $v_i\, v_i$ of each of the vectors;
- the scalar product $u_i\, v_i$.

The scalar product is an example of a so-called simultaneous invariant, i. e. an invariant which consists of coordinates of different tensors.

As the number of vectors increases, the number of irreducible invariants quickly increases. For three vectors $\underline{u}$, $\underline{v}$, $\underline{w}$, there are already seven independent scalar combinations:
- the squares $u_i u_i$, $v_i v_i$, $w_i\, w_i$, of each of the vectors;
- the scalar products $u_i v_i$, $u_i w_i$, $v_i w_i$, of each of the vector pairs;
- the triple product $\varepsilon_{ijk}\, u_i\, v_j\, w_k$ of all three vectors.

The triple product plays a particular role. If the vectors $\underline{u}$, $\underline{v}$, $\underline{w}$ are polar, then the first six invariants are also polar scalars, whereas the triple product is, due to the ε-tensor, an axial scalar. However, according to our definition, an axial scalar is no invariant, i. e. from three (polar) vectors $\underline{u}$, $\underline{v}$, $\underline{w}$, we can form six irreducible invariants.

5.3.2 Independent Invariants of a Second-Order Tensor

1. Let us first determine the number of independent invariants of an arbitrary second-order tensor $\underline{\underline{T}}$. To this end, we decompose the tensor into its symmetric part $\underline{\underline{S}}$ and its antimetric part $\underline{\underline{A}}$. We can write the antimetric part $\underline{\underline{A}}$, according to Section 3.3, in terms of its corresponding vector $\underline{b}$, so we have

$$T_{ij} = S_{ij} + A_{ij} = S_{ij} + \varepsilon_{ijk}\, b_k .$$

If we assume that $\underline{\underline{T}}$ is polar, then $\underline{\underline{S}}$ and $\underline{\underline{A}}$ are also polar and $\underline{b}$ is axial.

The decomposition into a symmetric part and an antimetric part is, according to Section 2.6, No. 7, independent of the coordinate system. So we can also use the principal axis system of $\underline{\underline{S}}$ to represent the coordinates of the full tensor $\underline{\underline{T}}$, and thus we have for its coordinate matrix

$$\widetilde{T}_{ij} = \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{pmatrix} + \begin{pmatrix} 0 & \beta_3 & -\beta_2 \\ -\beta_3 & 0 & \beta_1 \\ \beta_2 & -\beta_1 & 0 \end{pmatrix},$$

$$\widetilde{T}_{ij} = \begin{pmatrix} \sigma_1 & \beta_3 & -\beta_2 \\ -\beta_3 & \sigma_2 & \beta_1 \\ \beta_2 & -\beta_1 & \sigma_3 \end{pmatrix}. \tag{5.4}$$

Here the σ_i are the (not necessarily different) eigenvalues of $\underline{\underline{S}}$ and the β_i are the coordinates of $\underline{b}$ in the principal axis system of $\underline{\underline{S}}$. We already know from Section 3.11.2 that the eigenvalues σ_i are invariants. The coordinates of $\underline{b}$ can be interpreted as the projections of the vector $\underline{b}$ onto the principal axes of $\underline{\underline{S}}$; hence the β_i are also invariants, because the direction of the principal axes of $\underline{\underline{S}}$ and the direction of $\underline{b}$ are independent of the coordinate system.

Thus the principal axis system of the symmetric part $\underline{\underline{S}}$ is a distinguished coordinate system of the tensor $\underline{\underline{T}}$; the coordinate matrix of $\underline{\underline{T}}$ is then (and only then) antimetric, except on the main diagonal.

We conclude that, in general, a second-order tensor has six independent invariants; more than six independent invariants are not possible, because the tensor is uniquely determined by the σ_i and β_i. However, these invariants are not polynomial invariants, i. e. they cannot be written as polynomials of tensor coordinates in an arbitrary Cartesian coordinate system.

2. If the symmetric part $\underline{\underline{S}}$ has multiple eigenvalues, the number of independent invariants is reduced.

For an eigenvalue of multiplicity two, we choose the x, y-plane as the eigenplane of the principal axis system and we put the x-axis such that $\underline{b}$ lies in the x, z-plane. Then the tensor $\underline{\underline{T}}$ has the coordinate matrix

$$\widetilde{T}_{ij} = \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_1 & 0 \\ 0 & 0 & \sigma_3 \end{pmatrix} + \begin{pmatrix} 0 & \beta_3 & 0 \\ -\beta_3 & 0 & \beta_1 \\ 0 & -\beta_1 & 0 \end{pmatrix} = \begin{pmatrix} \sigma_1 & \beta_3 & 0 \\ -\beta_3 & \sigma_1 & \beta_1 \\ 0 & -\beta_1 & \sigma_3 \end{pmatrix},$$

i. e. only four independent invariants exist.

For an eigenvalue of multiplicity three, every coordinate system is also a principal axis system of $\underline{\underline{S}}$. So we can put the x-axis in the direction of $\underline{b}$ and we get for the coordinate matrix of $\underline{\underline{T}}$

$$\widetilde{T}_{ij} = \begin{pmatrix} \sigma & 0 & 0 \\ 0 & \sigma & 0 \\ 0 & 0 & \sigma \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \beta \\ 0 & -\beta & 0 \end{pmatrix} = \begin{pmatrix} \sigma & 0 & 0 \\ 0 & \sigma & \beta \\ 0 & -\beta & \sigma \end{pmatrix},$$

i. e. $\underline{\underline{T}}$ has only two independent invariants.

3. If the tensor is symmetric, we have $\underline{\underline{A}} = \underline{\underline{0}}$. A symmetric tensor has at most three independent invariants, namely if all its eigenvalues are different. If the tensor has an eigenvalue of multiplicity two, it has only two invariants; and if it has an eigenvalue of multiplicity three, only one invariant exists.

If the tensor is antimetric, we have $\underline{\underline{S}} = \underline{\underline{0}}$. An antimetric tensor has only one independent invariant, i. e. the square of the corresponding vector.

In addition, we consider orthogonal tensors. In a coordinate system in which the z-axis is the axis of rotation or the axis of rotary reflection, an orthogonal tensor has, according to (3.75) or (3.76), the coordinate matrix

$$\tilde{R}_{ij} = \begin{pmatrix} \cos\vartheta & -\sin\vartheta & 0 \\ \sin\vartheta & \cos\vartheta & 0 \\ 0 & 0 & \pm 1 \end{pmatrix}.$$

This is a representation of a tensor with respect to the principal axis system of its symmetric part, so we note that with the angle ϑ only one independent invariant exists.

5.3.3 Irreducible Invariants of Second-Order Tensors

1. Irreducible invariants are by definition polynomial invariants, which are obtained from appropriate contractions of tensor coordinates. For second-order tensors, we can write the polynomial invariants also as the trace of a sequence of scalar products between these tensors. We begin with a few statements, which we need later:

I. The trace of a sequence of scalar products of second-order tensors does not change if we cyclically permute the order of the factors in the scalar products, i. e.

$$\mathrm{tr}(\underline{\underline{a}} \cdot \underline{\underline{b}} \cdot \ldots \cdot \underline{\underline{c}} \cdot \underline{\underline{d}}) = a_{ij}\, b_{jk} \cdots c_{mn}\, d_{ni} = b_{jk} \cdots c_{mn}\, d_{ni}\, a_{ij} = \mathrm{tr}(\underline{\underline{b}} \cdot \ldots \cdot \underline{\underline{c}} \cdot \underline{\underline{d}} \cdot \underline{\underline{a}}),$$

so we have

$$\mathrm{tr}(\underline{\underline{a}} \cdot \underline{\underline{b}} \cdot \ldots \cdot \underline{\underline{c}} \cdot \underline{\underline{d}}) = \mathrm{tr}(\underline{\underline{b}} \cdot \ldots \cdot \underline{\underline{c}} \cdot \underline{\underline{d}} \cdot \underline{\underline{a}}) = \cdots = \mathrm{tr}(\underline{\underline{d}} \cdot \underline{\underline{a}} \cdot \underline{\underline{b}} \cdot \ldots \cdot \underline{\underline{c}}). \tag{5.5}$$

II. The trace of a sequence of scalar products of second-order tensors does not change if we transpose each factor, while reversing the order of the factors, i. e.

$$\begin{aligned} &\mathrm{tr}(\underline{\underline{a}} \cdot \underline{\underline{b}} \cdot \ldots \cdot \underline{\underline{c}} \cdot \underline{\underline{d}}) = a_{ij}\, b_{jk} \cdots c_{mn}\, d_{ni} \\ &= d^{\mathrm{T}}_{in}\, c^{\mathrm{T}}_{nm} \cdots b^{\mathrm{T}}_{kj}\, a^{\mathrm{T}}_{ji} = \mathrm{tr}(\underline{\underline{d}}^{\mathrm{T}} \cdot \underline{\underline{c}}^{\mathrm{T}} \cdot \ldots \cdot \underline{\underline{b}}^{\mathrm{T}} \cdot \underline{\underline{a}}^{\mathrm{T}}), \end{aligned}$$

so we have

$$\mathrm{tr}(\underline{\underline{a}} \cdot \underline{\underline{b}} \cdot \ldots \cdot \underline{\underline{c}} \cdot \underline{\underline{d}}) = \mathrm{tr}(\underline{\underline{d}}^{\mathrm{T}} \cdot \underline{\underline{c}}^{\mathrm{T}} \cdot \ldots \cdot \underline{\underline{b}}^{\mathrm{T}} \cdot \underline{\underline{a}}^{\mathrm{T}}). \tag{5.6}$$

In particular, if all factors are equal, we have

$$\operatorname{tr} \underline{\underline{a}}^n = \operatorname{tr} (\underline{\underline{a}}^{\mathrm{T}})^n = \operatorname{tr} (\underline{\underline{a}}^n)^{\mathrm{T}}, \tag{5.7}$$

where n is a natural number.

III. The trace of a scalar product of a symmetric tensor $\underline{\underline{s}} = \underline{\underline{s}}^{\mathrm{T}}$ and an antimetric tensor $\underline{\underline{a}} = -\underline{\underline{a}}^{\mathrm{T}}$ is zero.

Since

$$(\underline{\underline{s}} \cdot \underline{\underline{a}})^{\mathrm{T}} \overset{(2.35)}{=} \underline{\underline{a}}^{\mathrm{T}} \cdot \underline{\underline{s}}^{\mathrm{T}} = -\underline{\underline{a}} \cdot \underline{\underline{s}} \tag{a}$$

and also

$$\operatorname{tr}(\underline{\underline{s}} \cdot \underline{\underline{a}}) \overset{(5.7)}{=} \operatorname{tr} (\underline{\underline{s}} \cdot \underline{\underline{a}})^{\mathrm{T}} \overset{(a)}{=} -\operatorname{tr}(\underline{\underline{a}} \cdot \underline{\underline{s}}) \overset{(5.5)}{=} -\operatorname{tr}(\underline{\underline{s}} \cdot \underline{\underline{a}}),$$

we obtain

$$\operatorname{tr}(\underline{\underline{s}} \cdot \underline{\underline{a}}) = 0. \tag{5.8}$$

2. In order to write the irreducible invariants of a second-order tensor as the trace of a sequence of scalar products, the main question is how many of these invariants form an integrity basis. Answering this question here in all generality would be beyond the scope of this book. We restrict ourselves to the case that the factors in the scalar product depend only on two different tensors $\underline{\underline{a}}$ and $\underline{\underline{b}}$. We will follow a systematical approach and increase the number of the factors in the scalar product step-by-step; the number of these factors is also called the degree of the invariant.

With only one factor, there are only two ways to form invariants in the described manner, namely the trace of each of the two tensors themselves, i. e.

$$I_{11} := \operatorname{tr} \underline{\underline{a}}, \qquad I_{12} := \operatorname{tr} \underline{\underline{b}}. \tag{5.9}$$

Scalar products of two factors can be formed in four different ways: $\underline{\underline{a}}^2$, $\underline{\underline{b}}^2$, $\underline{\underline{a}} \cdot \underline{\underline{b}}$, $\underline{\underline{b}} \cdot \underline{\underline{a}}$; after taking cyclical permutations (5.5) into account, only three invariants of degree two remain:

$$I_{21} := \operatorname{tr} \underline{\underline{a}}^2, \qquad I_{22} := \operatorname{tr} \underline{\underline{b}}^2, \qquad I_{23} := \operatorname{tr}(\underline{\underline{a}} \cdot \underline{\underline{b}}). \tag{5.10}$$

From three factors, there are altogether eight possible scalar products: $\underline{\underline{a}}^3$, $\underline{\underline{b}}^3$, $\underline{\underline{a}}^2 \cdot \underline{\underline{b}}$, $\underline{\underline{a}} \cdot \underline{\underline{b}} \cdot \underline{\underline{a}}$, $\underline{\underline{b}} \cdot \underline{\underline{a}}^2$, $\underline{\underline{b}}^2 \cdot \underline{\underline{a}}$, $\underline{\underline{b}} \cdot \underline{\underline{a}} \cdot \underline{\underline{b}}$, $\underline{\underline{a}} \cdot \underline{\underline{b}}^2$; after computing the traces and taking cyclical permutations according to (5.5) into account, only four invariants of degree three remain:

$$\begin{aligned} I_{31} &:= \operatorname{tr} \underline{\underline{a}}^3, & I_{32} &:= \operatorname{tr} \underline{\underline{b}}^3, \\ I_{33} &:= \operatorname{tr}(\underline{\underline{a}}^2 \cdot \underline{\underline{b}}), & I_{34} &:= \operatorname{tr}(\underline{\underline{a}} \cdot \underline{\underline{b}}^2). \end{aligned} \tag{5.11}$$

The invariants of degree one, two, and three, which we have considered so far, are all irreducible, because none can be written in terms of the others. However, for invariants of degree four, a fundamental restriction appears, since invariants such as $\mathrm{tr}(\underline{\underline{a}}^3 \cdot \underline{\underline{b}})$ and $\mathrm{tr}(\underline{\underline{b}}^3 \cdot \underline{\underline{a}})$ are reducible. To see this, we solve the Cayley–Hamilton theorem (3.94) for $\underline{\underline{a}}^3$ and scalar multiply it from the right by $\underline{\underline{b}}$, so we have

$$\underline{\underline{a}}^3 \cdot \underline{\underline{b}} = A'' \, \underline{\underline{a}}^2 \cdot \underline{\underline{b}} - A' \, \underline{\underline{a}} \cdot \underline{\underline{b}} + A \, \underline{\underline{b}}.$$

Computing the trace gives

$$\mathrm{tr}(\underline{\underline{a}}^3 \cdot \underline{\underline{b}}) = A'' \, \mathrm{tr}(\underline{\underline{a}}^2 \cdot \underline{\underline{b}}) - A' \, \mathrm{tr}(\underline{\underline{a}} \cdot \underline{\underline{b}}) + A \, \mathrm{tr}\, \underline{\underline{b}},$$

i. e. $\mathrm{tr}(\underline{\underline{a}}^3 \cdot \underline{\underline{b}})$ can be written in terms of invariants of degree less than or equal to three, since A'', A', and A are, according to (3.99), also functions of $\mathrm{tr}\, \underline{\underline{a}}$, $\mathrm{tr}\, \underline{\underline{a}}^2$, and $\mathrm{tr}\, \underline{\underline{a}}^3$. The same is true for $\mathrm{tr}(\underline{\underline{b}}^3 \cdot \underline{\underline{a}})$, because $\underline{\underline{a}}$ and $\underline{\underline{b}}$ can be interchanged in the above derivation.

We can extend these considerations to invariants of the form $\mathrm{tr}(\underline{\underline{c}} \cdot \underline{\underline{a}}^3 \cdot \underline{\underline{d}})$ and $\mathrm{tr}(\underline{\underline{c}} \cdot \underline{\underline{b}}^3 \cdot \underline{\underline{d}})$, where $\underline{\underline{c}}$ and $\underline{\underline{d}}$ are arbitrary second-order tensors; in particular they can be scalar products of the factors $\underline{\underline{a}}$ and $\underline{\underline{b}}$. Such invariants can be reduced, using the Cayley–Hamilton theorem (3.94), to invariants of a smaller degree. In other words, invariants with a term of power three (or higher) are reducible. Hence we only need to investigate sequences of scalar products with the elements $\underline{\underline{a}}^2,\ \underline{\underline{b}}^2,\ \underline{\underline{a}},\ \underline{\underline{b}}$.

For scalar products with at least four factors, we can derive another relation, using the generalized Cayley–Hamilton theorem, which will narrow down the search for irreducible invariants. For this we set in (5.1) $\underline{\underline{a}} = \underline{\underline{b}} = \underline{\underline{n}}$ and scalar multiply from the right by an arbitrary tensor $\underline{\underline{d}}$. For the first row we get with (5.2)

$$\begin{aligned}
\underline{\underline{\Sigma}} \cdot \underline{\underline{d}} &= \left[\underline{\underline{n}} \cdot \left(\underline{\underline{n}} \cdot \underline{\underline{c}} + \underline{\underline{c}} \cdot \underline{\underline{n}}\right) + \underline{\underline{n}} \cdot \left(\underline{\underline{n}} \cdot \underline{\underline{c}} + \underline{\underline{c}} \cdot \underline{\underline{n}}\right) + \underline{\underline{c}} \cdot \left(\underline{\underline{n}} \cdot \underline{\underline{n}} + \underline{\underline{n}} \cdot \underline{\underline{n}}\right)\right] \cdot \underline{\underline{d}} \\
&= 2\left[\underline{\underline{n}} \cdot \underline{\underline{c}} \cdot \underline{\underline{n}} + \underline{\underline{n}}^2 \cdot \underline{\underline{c}} + \underline{\underline{c}} \cdot \underline{\underline{n}}^2\right] \cdot \underline{\underline{d}} = 2\left[\underline{\underline{n}} \cdot \underline{\underline{c}} \cdot \underline{\underline{n}} \cdot \underline{\underline{d}} + \underline{\underline{n}}^2 \cdot \underline{\underline{c}} \cdot \underline{\underline{d}} + \underline{\underline{c}} \cdot \underline{\underline{n}}^2 \cdot \underline{\underline{d}}\right].
\end{aligned}$$

Computing the trace, taking cyclic permutations into account, yields

$$\begin{aligned}
&\mathrm{tr}(\underline{\underline{\Sigma}} \cdot \underline{\underline{d}}) = 2\left[\mathrm{tr}(\underline{\underline{n}} \cdot \underline{\underline{c}} \cdot \underline{\underline{n}} \cdot \underline{\underline{d}}) + \mathrm{tr}(\underline{\underline{n}}^2 \cdot \underline{\underline{c}} \cdot \underline{\underline{d}}) + \mathrm{tr}(\underline{\underline{n}}^2 \cdot \underline{\underline{d}} \cdot \underline{\underline{c}})\right], \\
&\mathrm{tr}(\underline{\underline{n}} \cdot \underline{\underline{c}} \cdot \underline{\underline{n}} \cdot \underline{\underline{d}}) = -\left(\mathrm{tr}(\underline{\underline{n}}^2 \cdot \underline{\underline{c}} \cdot \underline{\underline{d}}) + \mathrm{tr}(\underline{\underline{n}}^2 \cdot \underline{\underline{d}} \cdot \underline{\underline{c}})\right) + \frac{1}{2}\, \mathrm{tr}(\underline{\underline{\Sigma}} \cdot \underline{\underline{d}}).
\end{aligned}$$

Using (5.3), we can write $\mathrm{tr}(\underline{\underline{\Sigma}} \cdot \underline{\underline{d}})$ in terms of invariants of at least one degree less than the degree of $\mathrm{tr}(\underline{\underline{n}} \cdot \underline{\underline{c}} \cdot \underline{\underline{n}} \cdot \underline{\underline{d}})$, $\mathrm{tr}(\underline{\underline{n}}^2 \cdot \underline{\underline{c}} \cdot \underline{\underline{d}})$, and $\mathrm{tr}(\underline{\underline{n}}^2 \cdot \underline{\underline{d}} \cdot \underline{\underline{c}})$. In the theory of the representation of tensor functions we also say that $\mathrm{tr}(\underline{\underline{n}} \cdot \underline{\underline{c}} \cdot \underline{\underline{n}} \cdot \underline{\underline{d}})$ and $-\left(\mathrm{tr}(\underline{\underline{n}}^2 \cdot \underline{\underline{c}} \cdot \underline{\underline{d}}) + \mathrm{tr}(\underline{\underline{n}}^2 \cdot \underline{\underline{d}} \cdot \underline{\underline{c}})\right)$ are equivalent and we write

$$\mathrm{tr}(\underline{\underline{n}} \cdot \underline{\underline{c}} \cdot \underline{\underline{n}} \cdot \underline{\underline{d}}) \equiv -\left(\mathrm{tr}(\underline{\underline{n}}^2 \cdot \underline{\underline{c}} \cdot \underline{\underline{d}}) + \mathrm{tr}(\underline{\underline{n}}^2 \cdot \underline{\underline{d}} \cdot \underline{\underline{c}})\right). \tag{5.12}$$

Equivalent invariants have the same degree and differ only in their (reducible) invariants of lower degrees; in particular, we can replace an irreducible invariant in an integrity basis by an equivalent invariant. If a computation shows that an invariant is equivalent to zero, then this invariant is reducible. The important implication of the equivalence relation (5.12) is that invariants with two factors which are equal but appear separated from each other can be written as the sum of two other invariants in which this factor appears squared.

With these preparations we can now reduce our search for fourth-order irreducible invariants to the following scalar products: $\underline{\underline{a}}^2 \cdot \underline{\underline{b}} \cdot \underline{\underline{a}}$, $\underline{\underline{a}} \cdot \underline{\underline{b}} \cdot \underline{\underline{a}}^2$, $\underline{\underline{b}}^2 \cdot \underline{\underline{a}} \cdot \underline{\underline{b}}$, $\underline{\underline{b}} \cdot \underline{\underline{a}} \cdot \underline{\underline{b}}^2$, $\underline{\underline{a}} \cdot \underline{\underline{b}} \cdot \underline{\underline{a}} \cdot \underline{\underline{b}}$, $\underline{\underline{b}} \cdot \underline{\underline{a}} \cdot \underline{\underline{b}} \cdot \underline{\underline{a}}$, $\underline{\underline{a}} \cdot \underline{\underline{b}}^2 \cdot \underline{\underline{a}}$, $\underline{\underline{b}} \cdot \underline{\underline{a}}^2 \cdot \underline{\underline{b}}$, $\underline{\underline{a}}^2 \cdot \underline{\underline{b}}^2$, $\underline{\underline{b}}^2 \cdot \underline{\underline{a}}^2$. After computing the trace and using cyclic permutation we have $\operatorname{tr}(\underline{\underline{a}}^2 \cdot \underline{\underline{b}} \cdot \underline{\underline{a}}) = \operatorname{tr}(\underline{\underline{a}} \cdot \underline{\underline{b}} \cdot \underline{\underline{a}}^2) = \operatorname{tr}(\underline{\underline{a}}^3 \cdot \underline{\underline{b}})$ and $\operatorname{tr}(\underline{\underline{b}}^2 \cdot \underline{\underline{a}} \cdot \underline{\underline{b}}) = \operatorname{tr}(\underline{\underline{b}} \cdot \underline{\underline{a}} \cdot \underline{\underline{b}}^2) = \operatorname{tr}(\underline{\underline{b}}^3 \cdot \underline{\underline{a}})$, so these invariants are reducible. With the same reasoning we have $\operatorname{tr}(\underline{\underline{a}} \cdot \underline{\underline{b}} \cdot \underline{\underline{a}} \cdot \underline{\underline{b}}) = \operatorname{tr}(\underline{\underline{b}} \cdot \underline{\underline{a}} \cdot \underline{\underline{b}} \cdot \underline{\underline{a}})$ and $\operatorname{tr}(\underline{\underline{a}}^2 \cdot \underline{\underline{b}}^2) = \operatorname{tr}(\underline{\underline{b}}^2 \cdot \underline{\underline{a}}^2) = \operatorname{tr}(\underline{\underline{a}} \cdot \underline{\underline{b}}^2 \cdot \underline{\underline{a}}) = \operatorname{tr}(\underline{\underline{b}} \cdot \underline{\underline{a}}^2 \cdot \underline{\underline{b}})$, so only two invariants remain, i. e. $\operatorname{tr}(\underline{\underline{a}} \cdot \underline{\underline{b}} \cdot \underline{\underline{a}} \cdot \underline{\underline{b}})$ and $\operatorname{tr}(\underline{\underline{a}}^2 \cdot \underline{\underline{b}}^2)$, and we have to check if they are irreducible. Setting in (5.12) $\underline{\underline{n}} = \underline{\underline{a}}$, $\underline{\underline{c}} = \underline{\underline{d}} = \underline{\underline{b}}$ we obtain

$$\operatorname{tr}(\underline{\underline{a}} \cdot \underline{\underline{b}} \cdot \underline{\underline{a}} \cdot \underline{\underline{b}}) \equiv -2 \operatorname{tr}(\underline{\underline{a}}^2 \cdot \underline{\underline{b}}^2),$$

i. e. both invariants are equivalent. Hence only one irreducible invariant of degree four exists and we choose

$$I_{41} := \operatorname{tr}(\underline{\underline{a}}^2 \cdot \underline{\underline{b}}^2). \tag{5.13}$$

When investigating scalar products with five factors, we consider cyclic permutations from the outset and then we see that there are only two ways of computing invariants of degree five, which are composed of the elements $\underline{\underline{a}}^2$, $\underline{\underline{b}}^2$, $\underline{\underline{a}}$, $\underline{\underline{b}}$, namely $\operatorname{tr}(\underline{\underline{a}} \cdot \underline{\underline{b}} \cdot \underline{\underline{a}} \cdot \underline{\underline{b}}^2)$ and $\operatorname{tr}(\underline{\underline{b}} \cdot \underline{\underline{a}} \cdot \underline{\underline{b}} \cdot \underline{\underline{a}}^2)$. From (5.12) it follows that

$$\begin{aligned} &\operatorname{tr}(\underline{\underline{a}} \cdot \underline{\underline{b}} \cdot \underline{\underline{a}} \cdot \underline{\underline{b}}^2) \equiv -2 \operatorname{tr}(\underline{\underline{a}}^2 \cdot \underline{\underline{b}}^3), \\ &\operatorname{tr}(\underline{\underline{b}} \cdot \underline{\underline{a}} \cdot \underline{\underline{b}} \cdot \underline{\underline{a}}^2) \equiv -2 \operatorname{tr}(\underline{\underline{b}}^2 \cdot \underline{\underline{a}}^3); \end{aligned}$$

however, because of $\underline{\underline{a}}^3$ and $\underline{\underline{b}}^3$ both invariants are reducible. In other words, no irreducible invariants of degree five exist.

In order to find the irreducible invariants of degree six, which can be formed from the elements $\underline{\underline{a}}^2$, $\underline{\underline{b}}^2$, $\underline{\underline{a}}$, $\underline{\underline{b}}$, we start, after taking cyclic permutations into account, from the invariants $\operatorname{tr}(\underline{\underline{a}} \cdot \underline{\underline{b}}^2 \cdot \underline{\underline{a}} \cdot \underline{\underline{b}}^2)$, $\operatorname{tr}(\underline{\underline{b}} \cdot \underline{\underline{a}}^2 \cdot \underline{\underline{b}} \cdot \underline{\underline{a}}^2)$, $\operatorname{tr}(\underline{\underline{a}} \cdot \underline{\underline{b}} \cdot \underline{\underline{a}} \cdot \underline{\underline{b}} \cdot \underline{\underline{a}} \cdot \underline{\underline{b}})$, $\operatorname{tr}(\underline{\underline{a}}^2 \cdot \underline{\underline{b}}^2 \cdot \underline{\underline{a}} \cdot \underline{\underline{b}})$, $\operatorname{tr}(\underline{\underline{a}}^2 \cdot \underline{\underline{b}} \cdot \underline{\underline{a}} \cdot \underline{\underline{b}}^2)$. We then investigate, using (5.12), which of them are irreducible. For the first two invariants we have

$$\mathrm{tr}(\underline{\underline{a}} \cdot \underline{\underline{b}}^2 \cdot \underline{\underline{a}} \cdot \underline{\underline{b}}^2) \equiv -2\,\mathrm{tr}(\underline{\underline{a}}^2 \cdot \underline{\underline{b}}^4),$$
$$\mathrm{tr}(\underline{\underline{b}} \cdot \underline{\underline{a}}^2 \cdot \underline{\underline{b}} \cdot \underline{\underline{a}}^2) \equiv -2\,\mathrm{tr}(\underline{\underline{b}}^2 \cdot \underline{\underline{a}}^4);$$

but because of $\underline{\underline{a}}^4$ and $\underline{\underline{b}}^4$ these invariants are reducible. For the third invariant we have

$$\mathrm{tr}(\underline{\underline{a}} \cdot (\underline{\underline{b}} \cdot \underline{\underline{a}} \cdot \underline{\underline{b}}) \cdot \underline{\underline{a}} \cdot \underline{\underline{b}}) \equiv -\mathrm{tr}(\underline{\underline{a}}^2 \cdot \underline{\underline{b}} \cdot \underline{\underline{a}} \cdot \underline{\underline{b}}^2) - \mathrm{tr}(\underline{\underline{a}}^2 \cdot \underline{\underline{b}}^2 \cdot \underline{\underline{a}} \cdot \underline{\underline{b}}),$$

i. e. the third invariant is equivalent to the (negative) sum of the last two invariants. For the last two invariants we obtain

$$\mathrm{tr}(\underline{\underline{a}}^2 \cdot \underline{\underline{b}}^2 \cdot \underline{\underline{a}} \cdot \underline{\underline{b}}) = \mathrm{tr}(\underline{\underline{a}} \cdot (\underline{\underline{a}} \cdot \underline{\underline{b}}^2) \cdot \underline{\underline{a}} \cdot \underline{\underline{b}}) \equiv -\underbrace{\mathrm{tr}(\underline{\underline{a}}^3 \cdot \underline{\underline{b}}^3)}_{\equiv 0} - \mathrm{tr}(\underline{\underline{a}}^2 \cdot \underline{\underline{b}} \cdot \underline{\underline{a}} \cdot \underline{\underline{b}}^2),$$

i. e. they are equivalent and can only be reduced as a sum, but not individually, to invariants of lower degree. So we can regard one of them as irreducible and we choose

$$I_{61} := \mathrm{tr}(\underline{\underline{a}}^2 \cdot \underline{\underline{b}} \cdot \underline{\underline{a}} \cdot \underline{\underline{b}}^2). \tag{5.14}$$

It turns out that we do not have to look further for invariants of degree seven or higher, because they will always be reducible. This follows from two facts. First, a factor can appear in irreducible invariants only as itself or squared. Second, the equivalence relation (5.12) states that invariants appearing twice can be replaced by invariants with quadratic factors. From the elements $\underline{\underline{a}}^2$, $\underline{\underline{b}}^2$, $\underline{\underline{a}}$, $\underline{\underline{b}}$ we can form, without repetition, at most invariants of degree six, because for higher degrees at least one of the elements appears twice. If the element appearing twice is $\underline{\underline{a}}$ or $\underline{\underline{b}}$, we can replace the invariant under consideration, using (5.12), with equivalent invariants with $\underline{\underline{a}}^2$ or $\underline{\underline{b}}^2$; however, the elements $\underline{\underline{a}}^2$ and $\underline{\underline{b}}^2$ already appear, so that applying (5.12) again leads to invariants with $\underline{\underline{a}}^4$ or $\underline{\underline{b}}^4$, which are reducible. If the element appearing twice is $\underline{\underline{a}}^2$ or $\underline{\underline{b}}^2$, then applying (5.12) once is already sufficient to reduce the invariant under consideration to invariants of lower degree.

3. We summarize the results of our investigations.

We computed invariants as the trace of a sequence of scalar products of second-order tensors, and we restricted our search for invariants to the case that the factors in the sequence of scalar products are two different tensors $\underline{\underline{a}}$ and $\underline{\underline{b}}$. Using the Cayley–Hamilton theorem (3.94) and its generalization (5.1), we could show that invariants with more than six factors are always reducible. Finally, we were able to find with (5.9), (5.10), (5.11), (5.13), and (5.14) a total of 11 irreducible invariants (some of which are simultaneous invariants, since they are built from the coordinates of different tensors):

$$
\begin{aligned}
&I_{11} = \operatorname{tr} \underline{\underline{a}}, && I_{12} = \operatorname{tr} \underline{\underline{b}}, \\
&I_{21} = \operatorname{tr} \underline{\underline{a}}^2, && I_{22} = \operatorname{tr} \underline{\underline{b}}^2, && I_{23} = \operatorname{tr}(\underline{\underline{a}} \cdot \underline{\underline{b}}), \\
&I_{31} = \operatorname{tr} \underline{\underline{a}}^3, && I_{32} = \operatorname{tr} \underline{\underline{b}}^3, && I_{33} = \operatorname{tr}(\underline{\underline{a}}^2 \cdot \underline{\underline{b}}), && I_{34} = \operatorname{tr}(\underline{\underline{a}} \cdot \underline{\underline{b}}^2), \\
&I_{41} = \operatorname{tr}(\underline{\underline{a}}^2 \cdot \underline{\underline{b}}^2), \\
&I_{61} = \operatorname{tr}(\underline{\underline{a}}^2 \cdot \underline{\underline{b}} \cdot \underline{\underline{a}} \cdot \underline{\underline{b}}^2).
\end{aligned}
\tag{5.15}
$$

It remains to answer the question if these irreducible invariants form a complete set. We will do this by means of examples.

4. As a first example, we consider a single tensor $\underline{\underline{T}}$ and set in (5.15) $\underline{\underline{a}} = \underline{\underline{T}}$, $\underline{\underline{b}} = \underline{\underline{T}}^{\mathrm{T}}$. Since the trace does not change if we transpose its argument and since, according to (5.7), transposition and exponentiation are interchangeable within a trace, we see immediately that some of the invariants in (5.15) are the same. We have

$$I_{11} = I_{12} = \operatorname{tr} \underline{\underline{T}}, \quad I_{21} = I_{22} = \operatorname{tr} \underline{\underline{T}}^2, \quad I_{31} = I_{32} = \operatorname{tr} \underline{\underline{T}}^3,$$

and further investigation of I_{33} and I_{34} shows that

$$I_{34} = \operatorname{tr}(\underline{\underline{T}} \cdot (\underline{\underline{T}}^{\mathrm{T}})^2) \overset{(5.6)}{=} \operatorname{tr}(\underline{\underline{T}}^2 \cdot \underline{\underline{T}}^{\mathrm{T}}) = I_{33}.$$

Thus a single second-order tensor has, in general, seven irreducible invariants:

$$
\begin{aligned}
I_1 &:= T_{ii} &&= \operatorname{tr} \underline{\underline{T}}, \\
I_2 &:= T_{ij}\, T_{ji} &&= \operatorname{tr} \underline{\underline{T}}^2, \\
I_3 &:= T_{ij}\, T_{ij} &&= \operatorname{tr}(\underline{\underline{T}} \cdot \underline{\underline{T}}^{\mathrm{T}}), \\
I_4 &:= T_{ij}\, T_{jk}\, T_{ki} &&= \operatorname{tr} \underline{\underline{T}}^3, \\
I_5 &:= T_{ij}\, T_{jk}\, T_{ik} &&= \operatorname{tr}(\underline{\underline{T}}^2 \cdot \underline{\underline{T}}^{\mathrm{T}}), \\
I_6 &:= T_{ij}\, T_{jk}\, T_{lk}\, T_{il} &&= \operatorname{tr}(\underline{\underline{T}}^2 \cdot (\underline{\underline{T}}^{\mathrm{T}})^2), \\
I_7 &:= T_{ij}\, T_{jk}\, T_{lk}\, T_{lm}\, T_{nm}\, T_{in} &&= \operatorname{tr}(\underline{\underline{T}}^2 \cdot \underline{\underline{T}}^{\mathrm{T}} \cdot \underline{\underline{T}} \cdot (\underline{\underline{T}}^{\mathrm{T}})^2).
\end{aligned}
\tag{5.16}
$$

These invariants in fact form an integrity basis. Both $\underline{\underline{T}}$ and $\underline{\underline{T}}^{\mathrm{T}}$ are needed and it is not possible to write all invariants in terms of traces using only $\underline{\underline{T}}$. We can see this, for example, if we compare I_2 and I_3. In index notation we have $I_2 = T_{ij}\, T_{ji}$ and $I_3 = T_{ij}\, T_{ij}$; both are scalars obtained by contracting tensor coordinates, but without the transposed tensor, we cannot write I_3 as a trace.[2]

2 This explains why (5.15) is not a complete set of irreducible invariants of the tensors $\underline{\underline{a}}$ and $\underline{\underline{b}}$; to obtain an integrity basis, we have to include also the transposed tensors $\underline{\underline{a}}^{\mathrm{T}}$ and $\underline{\underline{b}}^{\mathrm{T}}$, i. e. we have to investigate four different tensors.

If we compare our results with Section 5.3.2, we see that, in general, a second-order tensor has six independent invariants, but seven irreducible invariants. We can clearly write all irreducible invariants in terms of the independent invariants from Section 5.3.2, No. 1, if we write the traces in (5.16) with respect to the principal axis system of the symmetric part of $\underline{\underline{T}}$.

Symmetric tensors obey $\underline{\underline{T}}^{\mathrm{T}} = \underline{\underline{T}}$. Then $I_2 = I_3$ and $I_4 = I_5$, and we can reduce $I_6 = \operatorname{tr} \underline{\underline{T}}^4$ and $I_7 = \operatorname{tr} \underline{\underline{T}}^6$, using the Cayley–Hamilton theorem (3.95), to the remaining invariants. So, from the irreducible invariants in (5.16), only the three basic invariants $I_1 = \operatorname{tr} \underline{\underline{T}}$, $I_2 = \operatorname{tr} \underline{\underline{T}}^2$, $I_4 = \operatorname{tr} \underline{\underline{T}}^3$ remain.

Antimetric tensors satisfy $\underline{\underline{T}}^{\mathrm{T}} = -\underline{\underline{T}}$; hence $I_1 = 0$. We know $\underline{\underline{T}}^2$ is symmetric because $T_{ij}\, T_{jk} = (-T_{ji})\,(-T_{kj}) = T_{kj}\, T_{ji}$, so I_2 is nonzero, but we have $I_3 = -I_2$. Here $\underline{\underline{T}}^3$ is again antimetric because $T_{ij}\, T_{jk}\, T_{kl} = (-T_{ji})\,(-T_{kj})\,(-T_{lk}) = -T_{lk}\, T_{kj}\, T_{ji}$, so we have $I_4 = 0$ and thus also $I_5 = -I_4 = 0$. It further follows that $I_6 = \operatorname{tr} \underline{\underline{T}}^4$ and $I_7 = -\operatorname{tr} \underline{\underline{T}}^6$, so the Cayley–Hamilton theorem (3.95) helps us reduce I_6 and I_7 to the only remaining invariant $I_2 = \operatorname{tr} \underline{\underline{T}}^2$.

For orthogonal tensors we have $\underline{\underline{T}}^{\mathrm{T}} = \underline{\underline{T}}^{-1}$. Then $I_3 = I_6 = I_7 = 3$. In other words, I_3, I_6, and I_7 do not contain any information about a particular orthogonal tensor and thus do not count as invariants. It further follows that $I_5 = I_1$, so we are left with the three basic invariants $I_1 = \operatorname{tr} \underline{\underline{T}}$, $I_2 = \operatorname{tr} \underline{\underline{T}}^2$, $I_4 = \operatorname{tr} \underline{\underline{T}}^3$. The determinant of an orthogonal tensor is, according to Section 3.13.2, equal to ± 1, and the cotensor is, according to (3.14), equal to the tensor itself, up to the sign. Thus, we have for the main invariants, according to (3.47)–(3.49), $A'' = \operatorname{tr} \underline{\underline{T}}$, $A' = \pm \operatorname{tr} \underline{\underline{T}}$, $A = \pm 1$; then it follows from (3.100) that $\operatorname{tr} \underline{\underline{T}}^2 = \operatorname{tr}^2 \underline{\underline{T}} \mp 2 \operatorname{tr} \underline{\underline{T}}$, $\operatorname{tr} \underline{\underline{T}}^3 = \operatorname{tr}^3 \underline{\underline{T}} \mp 3 \operatorname{tr}^2 \underline{\underline{T}} \pm 3$, so an orthogonal tensor has only one irreducible invariant $I_1 = \operatorname{tr}(\underline{\underline{T}})$.

5. As our next example we consider the case of two symmetric tensors $\underline{\underline{U}}$ and $\underline{\underline{V}}$, which is also important for physical applications. Now we do not need to distinguish between the tensors and their transposed tensors, so we certainly can obtain an integrity basis from (5.15). We set $\underline{\underline{a}} = \underline{\underline{U}}$, $\underline{\underline{b}} = \underline{\underline{V}}$ in (5.15) and keep the first 10 invariants. Only the invariant of degree six does not belong to the integrity basis because it turns out to be reducible. On the one hand, from the symmetry of the tensors and by taking cyclic permutations and transposition (5.5)–(5.7) into account, it follows that

$$\operatorname{tr}(\underline{\underline{U}}^2 \cdot \underline{\underline{V}} \cdot \underline{\underline{U}} \cdot \underline{\underline{V}}^2) = \operatorname{tr}((\underline{\underline{V}}^2)^{\mathrm{T}} \cdot \underline{\underline{U}}^{\mathrm{T}} \cdot \underline{\underline{V}}^{\mathrm{T}} \cdot (\underline{\underline{U}}^2)^{\mathrm{T}}) = \operatorname{tr}(\underline{\underline{U}}^2 \cdot \underline{\underline{V}}^2 \cdot \underline{\underline{U}} \cdot \underline{\underline{V}}),$$

and on the other hand, from (5.12), we get

$$\operatorname{tr}(\underline{\underline{U}}^2 \cdot \underline{\underline{V}} \cdot \underline{\underline{U}} \cdot \underline{\underline{V}}^2) = \operatorname{tr}(\underline{\underline{U}} \cdot (\underline{\underline{U}} \cdot \underline{\underline{V}}) \cdot \underline{\underline{U}} \cdot \underline{\underline{V}}^2) \equiv -\underbrace{\operatorname{tr}(\underline{\underline{U}}^3 \cdot \underline{\underline{V}}^3)}_{\equiv 0} - \operatorname{tr}(\underline{\underline{U}}^2 \cdot \underline{\underline{V}}^2 \cdot \underline{\underline{U}} \cdot \underline{\underline{V}}).$$

The comparison of these two relations yields

$$\operatorname{tr}(\underline{\underline{U}}^2 \cdot \underline{\underline{V}} \cdot \underline{\underline{U}} \cdot \underline{\underline{V}}^2) \equiv -\operatorname{tr}(\underline{\underline{U}}^2 \cdot \underline{\underline{V}} \cdot \underline{\underline{U}} \cdot \underline{\underline{V}}^2),$$

which is only possible if $\operatorname{tr}(\underline{\underline{U}}^2 \cdot \underline{\underline{V}} \cdot \underline{\underline{U}} \cdot \underline{\underline{V}}^2) \equiv 0$, and thus it is reducible.

So we find that the integrity basis of two symmetric tensors $\underline{\underline{U}}$ and $\underline{\underline{V}}$ consists of ten irreducible invariants:

– the basic invariants
  $\operatorname{tr}\underline{\underline{U}}$, $\operatorname{tr}\underline{\underline{U}}^2$, $\operatorname{tr}\underline{\underline{U}}^3$ and $\operatorname{tr}\underline{\underline{V}}$, $\operatorname{tr}\underline{\underline{V}}^2$, $\operatorname{tr}\underline{\underline{V}}^3$ of each of the tensors and
– the simultaneous invariants
  $\operatorname{tr}(\underline{\underline{U}} \cdot \underline{\underline{V}})$, $\operatorname{tr}(\underline{\underline{U}}^2 \cdot \underline{\underline{V}})$, $\operatorname{tr}(\underline{\underline{U}} \cdot \underline{\underline{V}}^2)$, $\operatorname{tr}(\underline{\underline{U}}^2 \cdot \underline{\underline{V}}^2)$.

Problem 5.2.
Find all irreducible invariants which can be computed from two antimetric tensors $\underline{\underline{A}}$ and $\underline{\underline{B}}$, and compare the result with the invariants of two vectors from Section 5.3.1.

Hint: To show that $\operatorname{tr}(\underline{\underline{A}}^2 \cdot \underline{\underline{B}}^2)$ is reducible, represent the two antimetric tensors $\underline{\underline{A}}$ and $\underline{\underline{B}}$ temporarily by their corresponding vectors.

6. As a last example, we investigate the invariants of a symmetric tensor $\underline{\underline{S}}$ and an antimetric tensor $\underline{\underline{A}}$. Also for this case we can obtain an integrity basis from (5.15), since the tensors and their transposed tensors differ at most by a sign. If we set $\underline{\underline{a}} = \underline{\underline{S}}$, $\underline{\underline{b}} = \underline{\underline{A}}$ in (5.15), some invariants are zero, because not only the trace of an antimetric tensor and its (also antimetric) third power is zero but also, according to (5.8), the trace of the scalar product of a symmetric tensor and an antimetric tensor is zero. Thus, we have $I_{12} = I_{23} = I_{32} = I_{33} = 0$. Contrary to the case of two symmetric tensors, here the invariant of degree six is not reducible. Using cyclic permutations and transposition (5.5)–(5.7), we initially have

$$\operatorname{tr}(\underline{\underline{S}}^2 \cdot \underline{\underline{A}} \cdot \underline{\underline{S}} \cdot \underline{\underline{A}}^2) = \operatorname{tr}((\underline{\underline{A}}^2)^{\mathrm{T}} \cdot \underline{\underline{S}}^{\mathrm{T}} \cdot \underline{\underline{A}}^{\mathrm{T}} \cdot (\underline{\underline{S}}^2)^{\mathrm{T}}) = -\operatorname{tr}(\underline{\underline{A}}^2 \cdot \underline{\underline{S}} \cdot \underline{\underline{A}} \cdot \underline{\underline{S}}^2),$$

and from (5.12) we further obtain that

$$\operatorname{tr}(\underline{\underline{S}}^2 \cdot \underline{\underline{A}} \cdot \underline{\underline{S}} \cdot \underline{\underline{A}}^2) = \operatorname{tr}(\underline{\underline{S}} \cdot (\underline{\underline{S}} \cdot \underline{\underline{A}}) \cdot \underline{\underline{S}} \cdot \underline{\underline{A}}^2) \equiv -\underbrace{\operatorname{tr}(\underline{\underline{S}}^3 \cdot \underline{\underline{A}}^3)}_{=0} - \operatorname{tr}(\underline{\underline{S}}^2 \cdot \underline{\underline{A}}^2 \cdot \underline{\underline{S}} \cdot \underline{\underline{A}}).$$

Comparing these two equations leads to the trivial statement $\operatorname{tr}(\underline{\underline{S}}^2 \cdot \underline{\underline{A}} \cdot \underline{\underline{S}} \cdot \underline{\underline{A}}^2) \equiv \operatorname{tr}(\underline{\underline{S}}^2 \cdot \underline{\underline{A}} \cdot \underline{\underline{S}} \cdot \underline{\underline{A}}^2)$, i. e. we obtain no additional information about this invariant.

We thus obtain for the integrity basis of a symmetric tensor $\underline{\underline{S}}$ and an antimetric tensor $\underline{\underline{A}}$ a total of seven irreducible invariants:

– the basic invariants
  $\operatorname{tr}\underline{\underline{S}}$, $\operatorname{tr}\underline{\underline{S}}^2$, $\operatorname{tr}\underline{\underline{S}}^3$ and $\operatorname{tr}\underline{\underline{A}}^2$ of each of the tensors,
– and also the simultaneous invariants
  $\operatorname{tr}(\underline{\underline{S}} \cdot \underline{\underline{A}}^2)$, $\operatorname{tr}(\underline{\underline{S}}^2 \cdot \underline{\underline{A}}^2)$, $\operatorname{tr}(\underline{\underline{S}}^2 \cdot \underline{\underline{A}} \cdot \underline{\underline{S}} \cdot \underline{\underline{A}}^2)$.

Problem 5.3.
Find all irreducible invariants which can be formed from a polar symmetric tensor $\underline{\underline{S}}$ and a vector $\underline{u}$, and compare the result with the invariants of a symmetric tensor and an antimetric tensor. Also, distinguish the cases where $\underline{u}$ is polar or axial.

5.3.4 Summary

We finally summarize the results of Section 5.3 in a table. Here $\underline{\underline{T}}$ denotes an arbitrary tensor, $\underline{\underline{R}}$ is an orthogonal tensor, $\underline{\underline{S}}$, $\underline{\underline{U}}$, $\underline{\underline{V}}$ are symmetric tensors, $\underline{\underline{A}}$, $\underline{\underline{B}}$ are antimetric tensors, and $\underline{u}$, $\underline{v}$, $\underline{w}$ are vectors. All vectors and tensors are polar.

Arguments	Integrity basis
$\underline{u}$	$\underline{u} \cdot \underline{u}$
$\underline{u}, \underline{v}$	$\underline{u} \cdot \underline{u},\ \underline{v} \cdot \underline{v},\ \underline{u} \cdot \underline{v}$
$\underline{u}, \underline{v}, \underline{w}$	$\underline{u} \cdot \underline{u},\ \underline{v} \cdot \underline{v},\ \underline{w} \cdot \underline{w},\ \underline{u} \cdot \underline{v},\ \underline{u} \cdot \underline{w},\ \underline{v} \cdot \underline{w}$
$\underline{\underline{T}}$	$\mathrm{tr}\,\underline{\underline{T}},\ \mathrm{tr}\,\underline{\underline{T}}^2,\ \mathrm{tr}\,\underline{\underline{T}}^3,\ \mathrm{tr}(\underline{\underline{T}} \cdot \underline{\underline{T}}^{\mathrm{T}}),\ \mathrm{tr}(\underline{\underline{T}}^2 \cdot \underline{\underline{T}}^{\mathrm{T}}),$ $\mathrm{tr}(\underline{\underline{T}}^2 \cdot (\underline{\underline{T}}^{\mathrm{T}})^2),\ \mathrm{tr}(\underline{\underline{T}}^2 \cdot \underline{\underline{T}}^{\mathrm{T}} \cdot \underline{\underline{T}} \cdot (\underline{\underline{T}}^{\mathrm{T}})^2)$
$\underline{\underline{S}}$	$\mathrm{tr}\,\underline{\underline{S}},\ \mathrm{tr}\,\underline{\underline{S}}^2,\ \mathrm{tr}\,\underline{\underline{S}}^3$
$\underline{\underline{A}}$	$\mathrm{tr}\,\underline{\underline{A}}^2$
$\underline{\underline{R}}$	$\mathrm{tr}\,\underline{\underline{R}}$
$\underline{\underline{U}}, \underline{\underline{V}}$	$\mathrm{tr}\,\underline{\underline{U}},\ \mathrm{tr}\,\underline{\underline{U}}^2,\ \mathrm{tr}\,\underline{\underline{U}}^3,\ \mathrm{tr}\,\underline{\underline{V}},\ \mathrm{tr}\,\underline{\underline{V}}^2,\ \mathrm{tr}\,\underline{\underline{V}}^3,$ $\mathrm{tr}(\underline{\underline{U}} \cdot \underline{\underline{V}}),\ \mathrm{tr}(\underline{\underline{U}}^2 \cdot \underline{\underline{V}}),\ \mathrm{tr}(\underline{\underline{U}} \cdot \underline{\underline{V}}^2),\ \mathrm{tr}(\underline{\underline{U}}^2 \cdot \underline{\underline{V}}^2)$
$\underline{\underline{A}}, \underline{\underline{B}}$	$\mathrm{tr}\,\underline{\underline{A}}^2,\ \mathrm{tr}\,\underline{\underline{B}}^2,\ \mathrm{tr}(\underline{\underline{A}} \cdot \underline{\underline{B}})$
$\underline{\underline{S}}, \underline{\underline{A}}$	$\mathrm{tr}\,\underline{\underline{S}},\ \mathrm{tr}\,\underline{\underline{S}}^2,\ \mathrm{tr}\,\underline{\underline{S}}^3,\ \mathrm{tr}\,\underline{\underline{A}}^2,\ \mathrm{tr}(\underline{\underline{S}} \cdot \underline{\underline{A}}^2),$ $\mathrm{tr}(\underline{\underline{S}}^2 \cdot \underline{\underline{A}}^2),\ \mathrm{tr}(\underline{\underline{S}}^2 \cdot \underline{\underline{A}} \cdot \underline{\underline{S}} \cdot \underline{\underline{A}}^2)$
$\underline{\underline{S}}, \underline{u}$	$\mathrm{tr}\,\underline{\underline{S}},\ \mathrm{tr}\,\underline{\underline{S}}^2,\ \mathrm{tr}\,\underline{\underline{S}}^3,\ \underline{u} \cdot \underline{u},\ \underline{u} \cdot \underline{\underline{S}} \cdot \underline{u},\ \underline{u} \cdot \underline{\underline{S}}^2 \cdot \underline{u}$

5.4 Isotropic Tensor Functions

5.4.1 Invariance Conditions

Tensors cannot be related by arbitrary functions, because the function value must satisfy the corresponding transformation law for tensor coordinates under a change to another Cartesian coordinate system. Let polar vectors $\underline{v}$, …, $\underline{w}$ and polar second-order tensors $\underline{\underline{M}}$, …, $\underline{\underline{N}}$ be arguments of tensor functions $\mathscr{F}$ of various orders. Under a change of the Cartesian coordinate system we have for the coordinates of the arguments, according to (2.17), the transformation equations

$$\tilde{v}_i = \alpha_{pi}\, v_p,\ \ldots,\ \tilde{w}_j = \alpha_{qj}\, w_q,$$
$$\widetilde{M}_{kl} = \alpha_{pk}\, \alpha_{ql}\, M_{pq},\ \ldots,\ \widetilde{N}_{mn} = \alpha_{pm}\, \alpha_{qn}\, N_{pq}.$$

Depending on the order of the tensors, we obtain different conditions for the function $\mathscr{F}$:

– A polar scalar $s = f(\underline{v},\ \ldots,\ \underline{w}, \underline{\underline{M}},\ \ldots,\ \underline{\underline{N}})$ must remain unchanged under a transformation of the coordinate system,

$$f(v_i,\ \ldots,\ w_j, M_{kl},\ \ldots,\ N_{mn}) = f(\tilde{v}_i,\ \ldots,\ \tilde{w}_j, \widetilde{M}_{kl},\ \ldots,\ \widetilde{N}_{mn}),$$

and then it follows for the function f that

$$\begin{aligned} &f(v_i,\ \ldots,\ w_j, M_{kl},\ \ldots,\ N_{mn}) \\ &\quad = f(\alpha_{pi}\, v_p,\ \ldots,\ \alpha_{qj}\, w_q, \alpha_{pk}\, \alpha_{ql}\, M_{pq},\ \ldots,\ \alpha_{pm}\, \alpha_{qn}\, N_{pq}). \end{aligned} \tag{5.17}$$

– For a polar vector $\underline{u} = \underline{f}(\underline{v},\ \ldots,\ \underline{w}, \underline{\underline{M}},\ \ldots,\ \underline{\underline{N}})$, substituting the transformed coordinates $\tilde{v}_i,\ \ldots,\ \tilde{w}_j,\ \widetilde{M}_{kl},\ \ldots,\ \widetilde{N}_{mn}$ must give the transformed vector coordinates

$$\tilde{u}_r = \alpha_{ur}\, u_u = f_r(\tilde{v}_i,\ \ldots,\ \tilde{w}_j, \widetilde{M}_{kl},\ \ldots,\ \widetilde{N}_{mn}).$$

In the original coordinates, we have

$$u_u = f_u(v_i,\ \ldots,\ w_j, M_{kl},\ \ldots,\ N_{mn}),$$

so it follows for the function $\underline{f}$ that

$$\begin{aligned} &\alpha_{ur}\, f_u(v_i,\ \ldots,\ w_j, M_{kl},\ \ldots,\ N_{mn}) \\ &\quad = f_r(\alpha_{pi}\, v_p,\ \ldots,\ \alpha_{qj}\, w_q, \alpha_{pk}\, \alpha_{ql}\, M_{pq},\ \ldots,\ \alpha_{pm}\, \alpha_{qn}\, N_{pq}). \end{aligned} \tag{5.18}$$

– For a polar second-order tensor $\underline{\underline{T}} = \underline{\underline{f}}(\underline{v},\ \ldots,\ \underline{w}, \underline{\underline{M}},\ \ldots,\ \underline{\underline{N}})$, substituting the transformed coordinates $\tilde{v}_i,\ \ldots,\ \tilde{w}_j,\ \widetilde{M}_{kl},\ \ldots,\ \widetilde{N}_{mn}$ must give the transformed tensor coordinates

$$\widetilde{T}_{rs} = \alpha_{ur}\, \alpha_{vs}\, T_{uv} = f_{rs}(\tilde{v}_i,\ \ldots,\ \tilde{w}_j, \widetilde{M}_{kl},\ \ldots,\ \widetilde{N}_{mn}).$$

With

$$T_{uv} = f_{uv}(v_i,\ \ldots,\ w_j, M_{kl},\ \ldots,\ N_{mn}),$$

we then obtain for the function $\underline{\underline{f}}$

$$\begin{aligned} &\alpha_{ur}\, \alpha_{vs}\, f_{uv}(v_i,\ \ldots,\ w_j, M_{kl},\ \ldots,\ N_{mn}) \\ &\quad = f_{rs}\,(\alpha_{pi}\, v_p,\ \ldots,\ \alpha_{qj}\, w_q, \alpha_{pk}\, \alpha_{ql}\, M_{pq},\ \ldots,\ \alpha_{pm}\, \alpha_{qn}\, N_{pq}). \end{aligned} \tag{5.19}$$

If the transformation coefficients α_{ij} form an arbitrary orthogonal matrix (we also say that they encompass all orthogonal transformations), we call (5.17), (5.18) and (5.19) isotropic tensor functions.

The functions (5.17), (5.18), and (5.19) can in general further depend on polar scalars; since this does not impose any restrictions on the functions, we did not include scalars in the list of arguments.

We derived the invariance conditions here only for polar tensors, but they can be easily transferred to axial tensors by means of the transformation equations (2.18).

5.4.2 Scalar-Valued Functions

In order to satisfy the invariance condition (5.17), a scalar cannot depend on individual tensor coordinates, because, in general, coordinates change under a transformation of the coordinate system, so a scalar can only depend on combinations of coordinates which are invariants and thus also scalars. If it is possible to form an integrity basis with P irreducible invariants $I_1, \ldots, I_P$ from the coordinates $\underline{v}, \ldots, \underline{w}, \underline{\underline{M}}, \ldots, \underline{\underline{N}}$, then the scalar-valued function f satisfies

$$s = f(\underline{v}, \ldots, \underline{w}, \underline{\underline{M}}, \ldots, \underline{\underline{N}}) = f(I_1, I_2, \ldots, I_P). \tag{5.20}$$

The theory of the representation of tensor functions does not provide any more information about the function f, so for a particular physical process one has to rely on experiments. The type and the number P of the invariants $I_1, \ldots, I_P$ depend on the particular case and must be determined according to the rules from Section 5.3.

5.4.3 Vector-Valued Functions

1. We can obtain a scalar from a vector by means of the scalar product of the vector and another vector. This suggests that, with the help of the scalar product of the vector-valued function $\underline{f}$ and an arbitrary auxiliary vector $\underline{h}$ and by adding the auxiliary vector to the argument list, we can reduce the problem of finding a representation of a vector-valued function to the problem of finding a representation of a scalar-valued function:

$$\underline{u} \cdot \underline{h} = \underline{f}(\underline{v}, \ldots, \underline{w}, \underline{\underline{M}}, \ldots, \underline{\underline{N}}) \cdot \underline{h} = f(\underline{v}, \ldots, \underline{w}, \underline{\underline{M}}, \ldots, \underline{\underline{N}}, \underline{h}).$$

Then f is, according to Section 5.4.2, a function of the P irreducible invariants $I_1, \ldots, I_P$, which can be formed from the $\underline{v}, \ldots, \underline{w}, \underline{\underline{M}}, \ldots, \underline{\underline{N}}$ and the simultaneous invariants with a vector $\underline{h}$. However, since $\underline{h}$ is only an auxiliary vector, which must cancel out from the vector-valued function $\underline{f}$ at the end, we need only those simultaneous invariants which are linear in $\underline{h}$. These simultaneous invariants have the form $\underline{J}_i \cdot \underline{h}$, where $\underline{J}_i$ is a set of Q vectors, which are computed from the $\underline{v}, \ldots, \underline{w}, \underline{\underline{M}}, \ldots, \underline{\underline{N}}$ and which have to be determined for each case individually. The vectors $\underline{J}_i$ are also called the generators of the representation and a complete set of generators is called a function basis. We start by writing $\underline{u} \cdot \underline{h}$ as a linear combination of the $\underline{J}_i \cdot \underline{h}$, i. e.

$$\underline{u} \cdot \underline{h} = k_1 \underline{J}_1 \cdot \underline{h} + \cdots + k_Q \underline{J}_Q \cdot \underline{h}, \tag{a}$$

where the coefficients $k_1, \ldots, k_Q$ are scalars which can still depend on the invariants $I_1, \ldots, I_P$ of the integrity basis for $\underline{v}, \ldots, \underline{w}, \underline{\underline{M}}, \ldots, \underline{\underline{N}}$, i. e. we have

$$k_i = k_i(I_1, \ldots, I_P).$$

Since each term in (a) is a scalar product with the auxiliary vector $\underline{h}$, we can cancel $\underline{h}$ out, and thus we obtain a representation of the original vector-valued function $\underline{f}$, which inherently satisfies the invariance condition (5.18) due to its construction. We have

$$\underline{u} = k_1 \underline{J}_1 + \cdots + k_Q \underline{J}_Q. \tag{5.21}$$

We show by means of two examples how to determine the generators $\underline{J}_i$ of this representation for a particular case.

2. In the first example, we seek a representation of a vector $\underline{u}$ which depends only on another vector $\underline{v}$,

$$\underline{u} = \underline{f}(\underline{v}).$$

Scalar multiplication with an auxiliary vector $\underline{h}$ gives

$$\underline{u} \cdot \underline{h} = f(\underline{v}, \underline{h}).$$

According to Section 5.3.1, the vector $\underline{v}$ has only one irreducible invariant, i. e. its square $\underline{v} \cdot \underline{v}$, and from the vectors $\underline{v}$ and $\underline{h}$ we can form also only one irreducible simultaneous invariant, and this is linear in $\underline{h}$, namely the scalar product $\underline{v} \cdot \underline{h}$. So the representation of the vector $\underline{u}$ has only one generator, namely the vector $\underline{v}$ itself, and we have

$$\underline{u} = k(\underline{v} \cdot \underline{v})\, \underline{v}.$$

The function $k(\underline{v} \cdot \underline{v})$ cannot be determined further from the theory of the representation of tensor functions. For example, for a physical process, the missing information must be obtained from experiments.

3. In a second example, we extend the function $\underline{f}$ from No. 2 and include a symmetric tensor $\underline{\underline{S}}$, so we seek a representation for

$$\underline{u} = \underline{f}(\underline{v}, \underline{\underline{S}})$$

or, after scalar multiplication by an auxiliary vector $\underline{h}$,

$$\underline{u} \cdot \underline{h} = f(\underline{v}, \underline{\underline{S}}, \underline{h}).$$

As in No. 2, the scalar product $\underline{v} \cdot \underline{h}$ is a simultaneous invariant which is linear in $\underline{h}$. In addition, other linear simultaneous invariants can be formed with the help of the scalar product of $\underline{\underline{S}}$ and $\underline{v}$ on the left and $\underline{h}$ on the right; since $\underline{\underline{S}}$ is symmetric, the order does not matter. Doing the same with the integer powers of $\underline{\underline{S}}$ yields initially the invariants $\underline{v} \cdot \underline{\underline{S}} \cdot \underline{h}$, $\underline{v} \cdot \underline{\underline{S}}^2 \cdot \underline{h}$, $\underline{v} \cdot \underline{\underline{S}}^3 \cdot \underline{h}$, etc. Since $\underline{v} \cdot \underline{h}$ can also be written as $\underline{v} \cdot \underline{\underline{\delta}} \cdot \underline{h}$, we see that $\underline{v} \cdot \underline{\underline{S}}^3 \cdot \underline{h}$ (and correspondingly any expression with higher powers of $\underline{\underline{S}}$) is reducible, because we can write $\underline{\underline{S}}^3$, using the Cayley–Hamilton theorem (3.94), in terms of $\underline{\underline{S}}^2$, $\underline{\underline{S}}$, and $\underline{\underline{\delta}}$. Hence there are three generators for the representation of the vector $\underline{u}$, i. e.

$$\underline{u} = k_1\, \underline{v} + k_2\, \underline{v} \cdot \underline{\underline{S}} + k_3\, \underline{v} \cdot \underline{\underline{S}}^2.$$

The coefficients k_1, k_2, k_3 are scalar-valued functions of the invariants of $\underline{v}$ and $\underline{\underline{S}}$. Since here we consider only polar vectors and tensors, it follows from the result to Problem 5.3 that

$$k_i = f(\operatorname{tr}\underline{\underline{S}}, \operatorname{tr}\underline{\underline{S}}^2, \operatorname{tr}\underline{\underline{S}}^3, \underline{v} \cdot \underline{v}, \underline{v} \cdot \underline{\underline{S}} \cdot \underline{v}, \underline{v} \cdot \underline{\underline{S}}^2 \cdot \underline{v}).$$

Problem 5.4.
Find the representation of a vector $\underline{u}$ which depends on a polar tensor $\underline{\underline{T}}$ for the following cases:

- $\underline{\underline{T}}$ is symmetric or antimetric;
- $\underline{u}$ is polar or axial.

5.4.4 Tensor-Valued Functions

1. It is easy to generalize our work from Section 5.4.3 for vector-valued functions to tensor-valued functions; we simply introduce an auxiliary tensor $\underline{\underline{H}}$ and use the double scalar product; this again ensures that the result inherently satisfies the invariance condition (5.19) for tensor-valued functions. Then, from

$$\underline{\underline{T}} = \underline{\underline{f}}(\underline{v}, \ldots, \underline{w}, \underline{\underline{M}}, \ldots, \underline{\underline{N}})$$

we first get

$$\underline{\underline{T}} \cdot\cdot\, \underline{\underline{H}} = f(\underline{v}, \ldots, \underline{w}, \underline{\underline{M}}, \ldots, \underline{\underline{N}}, \underline{\underline{H}}).$$

The generators $\underline{\underline{J}}_i$ are now second-order tensors, because in order for $\underline{\underline{H}}$ to cancel out at the end, the simultaneous invariants with the auxiliary tensor must have the linear form $\underline{\underline{J}}_i \cdot\cdot \underline{\underline{H}}$. Then we can represent $\underline{\underline{T}} \cdot\cdot \underline{\underline{H}}$ as the linear combination of all Q simultaneous invariants, i. e. we have

$$\underline{\underline{T}} \cdot\cdot \underline{\underline{H}} = k_1 \underline{\underline{J}}_1 \cdot\cdot \underline{\underline{H}} + \cdots + k_Q \underline{\underline{J}}_Q \cdot\cdot \underline{\underline{H}},$$

and, after canceling $\underline{\underline{H}}$ out, we obtain for the original tensor-valued function $\underline{\underline{f}}$

$$\underline{\underline{T}} = k_1 \underline{\underline{J}}_1 + \cdots + k_Q \underline{\underline{J}}_Q. \tag{5.22}$$

As in Section 5.4.3, the coefficients $k_1, \ldots, k_Q$ are scalars which can depend on the invariants $I_1, \ldots, I_P$ of an integrity basis for $\underline{v}, \ldots, \underline{w}, \underline{\underline{M}}, \ldots, \underline{\underline{N}}$, i. e.

$$k_i = k_i(I_1, \ldots, I_P).$$

The type and number Q of the generators $\underline{\underline{J}}_1, \ldots, \underline{\underline{J}}_Q$ can only be determined for each particular case; we will again explain this with two examples.

2. We first consider a second-order tensor $\underline{\underline{T}}$ which depends only on a symmetric second-order tensor $\underline{\underline{S}}$, so we seek a representation for

$$\underline{\underline{T}} = \underline{\underline{f}}(\underline{\underline{S}}).$$

The double scalar product with an auxiliary tensor $\underline{\underline{H}}$ gives

$$\underline{\underline{T}} \cdot\cdot \underline{\underline{H}} = f(\underline{\underline{S}}, \underline{\underline{H}}).$$

Since $\underline{\underline{S}}$ is symmetric and because we only need the invariants which are linear in $\underline{\underline{H}}$, we can start from (5.15) and set there $\underline{\underline{a}} = \underline{\underline{S}}$ and $\underline{\underline{b}} = \underline{\underline{H}}$. From this we obtain the three basic invariants $\operatorname{tr} \underline{\underline{S}}, \operatorname{tr} \underline{\underline{S}}^2, \operatorname{tr} \underline{\underline{S}}^3$ of the symmetric tensor $\underline{\underline{S}}$ and, using the symmetry of $\underline{\underline{S}}$, the invariants which are linear in $\underline{\underline{H}}$,

$$\operatorname{tr} \underline{\underline{H}} = H_{ii} = \delta_{ij} H_{ij} = \underline{\underline{\delta}} \cdot\cdot \underline{\underline{H}},$$

$$\operatorname{tr}(\underline{\underline{S}} \cdot \underline{\underline{H}}) = S_{ij} H_{ji} = S_{ij} H_{ij} = \underline{\underline{S}} \cdot\cdot \underline{\underline{H}},$$

$$\operatorname{tr}(\underline{\underline{S}}^2 \cdot \underline{\underline{H}}) = S_{ij} S_{jk} H_{ki} = S_{kj} S_{ji} H_{ki} = \underline{\underline{S}}^2 \cdot\cdot \underline{\underline{H}}.$$

Hence we have three generators $\underline{\underline{\delta}}, \underline{\underline{S}}, \underline{\underline{S}}^2$, and the representation of the tensor $\underline{\underline{T}}$ is

$$\underline{\underline{T}} = k_1 \underline{\underline{\delta}} + k_2 \underline{\underline{S}} + k_3 \underline{\underline{S}}^2,$$

where the coefficients k_1, k_2, k_3 are scalar-valued functions of the three basic invariants of $\underline{\underline{S}}$, i. e.

$$k_i = f(\operatorname{tr} \underline{\underline{S}}, \operatorname{tr} \underline{\underline{S}}^2, \operatorname{tr} \underline{\underline{S}}^3).$$

Since we assumed that the tensor $\underline{\underline{S}}$ is symmetric, we immediately obtain from the theory of the representation of tensor functions that the tensor $\underline{\underline{T}}$ must also be symmetric.

3. As a second example, we consider a tensor $\underline{\underline{T}}$ which depends on a symmetric tensor $\underline{\underline{S}}$ and also on a vector $\underline{v}$, i. e. we are looking for a representation for

$$\underline{\underline{T}} = \underline{\underline{f}}(\underline{\underline{S}}, \underline{v}).$$

The double scalar product with the auxiliary tensor $\underline{\underline{H}}$ gives

$$\underline{\underline{T}} \cdot\cdot\, \underline{\underline{H}} = f(\underline{\underline{S}}, \underline{v}, \underline{\underline{H}}).$$

We can keep the invariants of $\underline{\underline{S}}$ and $\underline{v}$ from Section 5.4.3, No. 3, so we only need to compute the simultaneous invariants which are linear in $\underline{\underline{H}}$. As in No. 2 we initially find

$$\underline{\underline{\delta}} \cdot\cdot\, \underline{\underline{H}},\ \underline{\underline{S}} \cdot\cdot\, \underline{\underline{H}},\ \underline{\underline{S}}^2 \cdot\cdot\, \underline{\underline{H}}.$$

Another simultaneous invariant can be formed with the vector $\underline{v}$:

$$\underline{v}\,\underline{v} \cdot\cdot\, \underline{\underline{H}}.$$

In order to form the simultaneous invariants with both $\underline{\underline{S}}$ and $\underline{v}$, we start from $\underline{v}\,\underline{v} \cdot\cdot\, \underline{\underline{H}}$ and form scalar products with $\underline{v}\,\underline{v}$ and $\underline{\underline{S}}$ or $\underline{\underline{S}}^2$ on the right or on the left; we can neglect higher powers of $\underline{\underline{S}}$, because of the Cayley–Hamilton theorem (3.95). This gives

$$(\underline{\underline{S}} \cdot \underline{v}\,\underline{v}) \cdot\cdot\, \underline{\underline{H}},\ (\underline{v}\,\underline{v} \cdot \underline{\underline{S}}) \cdot\cdot\, \underline{\underline{H}},\ (\underline{\underline{S}}^2 \cdot \underline{v}\,\underline{v}) \cdot\cdot\, \underline{\underline{H}},\ (\underline{v}\,\underline{v} \cdot \underline{\underline{S}}^2) \cdot\cdot\, \underline{\underline{H}},$$
$$(\underline{\underline{S}} \cdot \underline{v}\,\underline{v} \cdot \underline{\underline{S}}) \cdot\cdot\, \underline{\underline{H}},\ (\underline{\underline{S}}^2 \cdot \underline{v}\,\underline{v} \cdot \underline{\underline{S}}) \cdot\cdot\, \underline{\underline{H}},\ (\underline{\underline{S}} \cdot \underline{v}\,\underline{v} \cdot \underline{\underline{S}}^2) \cdot\cdot\, \underline{\underline{H}},\ (\underline{\underline{S}}^2 \cdot \underline{v}\,\underline{v} \cdot \underline{\underline{S}}^2) \cdot\cdot\, \underline{\underline{H}}.$$

It remains to check if these simultaneous invariants are irreducible. Clearly, $(\underline{\underline{S}} \cdot \underline{v}\,\underline{v}) \cdot\cdot\, \underline{\underline{H}}$ and $(\underline{v}\,\underline{v} \cdot \underline{\underline{S}}) \cdot\cdot\, \underline{\underline{H}}$ are irreducible. However, the scalar products $\underline{\underline{S}} \cdot \underline{v}\,\underline{v} \cdot \underline{\underline{S}}$, $\underline{\underline{S}}^2 \cdot \underline{v}\,\underline{v}$, and $\underline{v}\,\underline{v} \cdot \underline{\underline{S}}^2$ are linked via the generalized Cayley–Hamilton theorem (5.1), for example, if we set $\underline{\underline{a}} = \underline{\underline{c}} = \underline{\underline{S}}$ and $\underline{\underline{b}} = \underline{v}\,\underline{v}$. Thus, from the three invariants $(\underline{\underline{S}}^2 \cdot \underline{v}\,\underline{v}) \cdot\cdot\, \underline{\underline{H}}$, $(\underline{v}\,\underline{v} \cdot \underline{\underline{S}}^2) \cdot\cdot\, \underline{\underline{H}}$, and $(\underline{\underline{S}} \cdot \underline{v}\,\underline{v} \cdot \underline{\underline{S}}) \cdot\cdot\, \underline{\underline{H}}$ only two are irreducible, and we choose $\underline{\underline{S}}^2 \cdot \underline{v}\,\underline{v}$ and $\underline{v}\,\underline{v} \cdot \underline{\underline{S}}^2$ for the function basis. The remaining invariants $(\underline{\underline{S}}^2 \cdot \underline{v}\,\underline{v} \cdot \underline{\underline{S}}) \cdot\cdot\, \underline{\underline{H}}$, $(\underline{\underline{S}} \cdot \underline{v}\,\underline{v} \cdot \underline{\underline{S}}^2) \cdot\cdot\, \underline{\underline{H}}$, and $(\underline{\underline{S}}^2 \cdot \underline{v}\,\underline{v} \cdot \underline{\underline{S}}^2) \cdot\cdot\, \underline{\underline{H}}$ are reducible, since, according to the generalized Cayley–Hamilton theorem (5.1), the scalar products $\underline{\underline{S}}^2 \cdot \underline{v}\,\underline{v} \cdot \underline{\underline{S}}$, $\underline{\underline{S}} \cdot \underline{v}\,\underline{v} \cdot \underline{\underline{S}}^2$, and $\underline{\underline{S}}^2 \cdot \underline{v}\,\underline{v} \cdot \underline{\underline{S}}^2$ can be written in terms of $\underline{\underline{\delta}}$, $\underline{\underline{S}}$, $\underline{\underline{S}}^2$, $\underline{v}\,\underline{v}$, $\underline{\underline{S}} \cdot \underline{v}\,\underline{v}$, $\underline{v}\,\underline{v} \cdot \underline{\underline{S}}$, $\underline{\underline{S}}^2 \cdot \underline{v}\,\underline{v}$, $\underline{v}\,\underline{v} \cdot \underline{\underline{S}}^2$.

Hence the function $\underline{\underline{f}}$ has eight generators, and we can represent the tensor $\underline{\underline{T}}$ as

$$\underline{\underline{T}} = k_1\, \underline{\underline{\delta}} + k_2\, \underline{\underline{S}} + k_3\, \underline{\underline{S}}^2 + k_4\, \underline{v}\,\underline{v} + k_5\, \underline{\underline{S}} \cdot \underline{v}\,\underline{v} + k_6\, \underline{v}\,\underline{v} \cdot \underline{\underline{S}} + k_7\, \underline{\underline{S}}^2 \cdot \underline{v}\,\underline{v} + k_8\, \underline{v}\,\underline{v} \cdot \underline{\underline{S}}^2.$$

The coefficients $k_1, \ldots, k_8$ are, as in Section 5.4.3, No. 3, scalar-valued functions of the invariants of $\underline{\underline{S}}$ and $\underline{v}$, i. e.

$$k_i = f(\mathrm{tr}\,\underline{\underline{S}}, \mathrm{tr}\,\underline{\underline{S}}^2, \mathrm{tr}\,\underline{\underline{S}}^3, \underline{v}\cdot\underline{v}, \underline{v}\cdot\underline{\underline{S}}\cdot\underline{v}, \underline{v}\cdot\underline{\underline{S}}^2\cdot\underline{v}).$$

In contrast to No. 2, here we cannot conclude from the symmetry of $\underline{\underline{S}}$ that $\underline{\underline{T}}$ is also symmetric. This is only true if $k_5 = k_6$ and $k_7 = k_8$. But if $\underline{\underline{T}}$ is symmetric, then we can represent $\underline{\underline{T}}$ also in the form

$$\underline{\underline{T}} = k_1^*\,\underline{\underline{\delta}} + k_2^*\,\underline{\underline{S}} + k_3^*\,\underline{\underline{S}}^2 + k_4^*\,\underline{v}\,\underline{v} + k_5^*\,\left(\underline{\underline{S}}\cdot\underline{v}\,\underline{v} + \underline{v}\,\underline{v}\cdot\underline{\underline{S}}\right) + k_6^*\,\underline{\underline{S}}\cdot\underline{v}\,\underline{v}\cdot\underline{\underline{S}}.$$

If $\underline{\underline{T}}$ is symmetric, then $\underline{\underline{S}}^2\cdot\underline{v}\,\underline{v}$ and $\underline{v}\,\underline{v}\cdot\underline{\underline{S}}^2$ can only appear as a sum, and according to the generalized Cayley–Hamilton theorem (5.1), we can replace this sum by the generator $\underline{\underline{S}}\cdot\underline{v}\,\underline{v}\cdot\underline{\underline{S}}$, which is symmetric from the outset. The coefficients $k_1^*, \ldots, k_6^*$ are again scalar-valued functions of the invariants of $\underline{\underline{S}}$ and $\underline{v}$.

Problem 5.5.
Find the representation of a polar second-order tensor $\underline{\underline{T}}$ which depends on a polar vector or on an axial vector $\underline{v}$.

5.4.5 Summary

We summarize the results of Section 5.4 in a table. Here $\underline{\underline{S}}$ denotes a polar symmetric tensor, $\underline{\underline{A}}$ a polar antimetric tensor, $\underline{v}$ a polar vector, and $\underline{u}$ an axial vector.

Argument	Function basis	
	vector-valued functions	
	polar	axial
$\underline{v}$	$\underline{v}$	
$\underline{u}$		$\underline{u}$
$\underline{\underline{S}}$	–	
$\underline{\underline{A}}$		$\underline{\underline{\epsilon}}\cdot\cdot\underline{\underline{A}}$
$\underline{\underline{S}}, \underline{v}$	$\underline{v}, \underline{\underline{S}}\cdot\underline{v}, \underline{\underline{S}}^2\cdot\underline{v}$	
	tensor-valued functions	
	polar	axial
$\underline{v}$	$\underline{\underline{\delta}}, \underline{v}\,\underline{v}$	$\underline{\underline{\epsilon}}\cdot\underline{v}$
$\underline{u}$	$\underline{\underline{\delta}}, \underline{u}\,\underline{u}, \underline{\underline{\epsilon}}\cdot\underline{u}$	
$\underline{\underline{S}}$	$\underline{\underline{\delta}}, \underline{\underline{S}}, \underline{\underline{S}}^2$	
$\underline{\underline{S}}, \underline{v}$	$\underline{\underline{\delta}}, \underline{\underline{S}}, \underline{\underline{S}}^2, \underline{v}\,\underline{v}, \underline{\underline{S}}\cdot\underline{v}\,\underline{v}, \underline{v}\,\underline{v}\cdot\underline{\underline{S}}, \underline{\underline{S}}^2\cdot\underline{v}\,\underline{v}, \underline{v}\,\underline{v}\cdot\underline{\underline{S}}^2$	

5.5 Considering Anisotropy

1. In the previous section, we called a tensor function isotropic if the transformation coefficients in the invariance conditions (5.17), (5.18) and (5.19) include all orthogonal transformations. We could also consider only a subset of certain permissible orthogonal transformations, e. g. only rotations about a fixed axis. Such considerations are important in physics and material science, if the problem is to describe the properties of a material in more detail; many materials, such as crystals and composite materials, are characterized by a directional dependency, i. e. they behave differently under rotations, depending on the direction of the axis of rotation. In such cases, often certain rotations about distinguished axes and angles of rotation exist, where the behavior of these materials remains the same. These rotations can be used to classify a material; we also say that the material possesses a certain symmetry or belongs to a certain symmetry group. Mathematically, such a symmetry group is defined as the set of orthogonal transformations under which the behavior of the material remains unchanged.

2. The invariance conditions from Section 5.4.1 can also be used if only a subset of all orthogonal transformations is allowed. Without further elaboration, we only mention here that, in general, the number of the invariants increases if the permissible transformations are restricted. An example is the restriction to only proper orthogonal transformations. Then we also speak of hemitropic invariants and hemitropic tensor functions. They are formed similarly as in Sections 5.3 and 5.4, but now we also have to take the simultaneous invariants with the ε-tensor into account, because for proper orthogonal transformations, we do not need to distinguish between polar and axial tensors. Another example are rotations only about a certain axis. If we choose this axis as the z-axis of a Cartesian coordinate system, then the matrix of the transformation coefficients has, according to (3.75) and with Section 3.13.4, the form

$$\alpha_{ij} = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then it follows from the transformation equations (2.17) that the coordinate u_3 of a vector and the coordinate T_{33} of a second-order tensor remain unchanged under the transformation and thus they have to be counted as invariants.

3. All tensors which are invariant under the transformations of a certain symmetry group form an anisotropy class. We explain this by means of three examples with second-order tensors.

The anisotropy class of general anisotropy includes all tensors; its symmetry group consists of the identical transformation $\alpha_{ij} = \delta_{ij}$ and the inversion $\alpha_{ij} = -\delta_{ij}$, because only under these transformations the coordinates do not change.

We have

$$\widetilde{T}_{ij} = \alpha_{mi}\,\alpha_{nj}\,T_{mn} = \delta_{mi}\,\delta_{nj}\,T_{mn} = T_{ij}.$$

On the other hand, the anisotropy class of isotropy includes all tensors whose coordinates are invariant under arbitrary orthogonal transformations. We already introduced such tensors as isotropic tensors; they have the form

$$\underline{\underline{T}} = k\,\underline{\underline{\delta}},$$

because for the transformed coordinates, we have, according to the orthogonality relation (2.6),

$$\widetilde{T}_{ij} = \alpha_{mi}\,\alpha_{nj}\,(k\,\delta_{mn}) = k\,\alpha_{mi}\,\alpha_{mj} = k\,\delta_{ij}.$$

As a third example, we consider the anisotropy class of transverse isotropy. Its symmetry group includes all rotations and rotary reflections with a fixed axis. The general form of a transversely isotropic tensor of second order follows from Problem 5.5. A transversely isotropic tensor can be seen as a tensor which depends only on the direction $\underline{n}$ of the axis of rotation or rotary reflection: $\underline{\underline{T}} = \underline{\underline{f}}(\underline{n})$. Thus, it remains to ensure in the result to Problem 5.5 that $\underline{n}$ is an (axial) unit vector, and we have

$$\underline{\underline{T}} = \alpha\,\underline{\underline{\delta}} + \beta\,\underline{n}\,\underline{n} + \gamma\,\underline{\underline{\underline{\varepsilon}}}\cdot\underline{n}. \tag{5.23}$$

Since $\underline{n}\cdot\underline{n} = 1$, here the α, β, γ are, unlike in Problem 5.5, not functions, but arbitrary constants.

If we choose the z-axis of the Cartesian coordinate system as the axis of rotation or rotary reflection, i. e. if $n_1 = n_2 = 0$, $n_3 = 1$, then $\underline{\underline{T}}$ has the coordinate matrix

$$\begin{aligned} T_{ij} &= \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \beta \end{pmatrix} + \begin{pmatrix} 0 & \gamma & 0 \\ -\gamma & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} \alpha & \gamma & 0 \\ -\gamma & \alpha & 0 \\ 0 & 0 & \alpha+\beta \end{pmatrix}. \end{aligned}$$

It is easy to see that, if we evaluate the transformation equations, the coordinates of $\underline{\underline{T}}$ remain unchanged under a rotation of the coordinate system about the z-axis, i. e.

we have

$$
\begin{aligned}
\widetilde{T}_{ij} &= \alpha_{mi}\,\alpha_{nj}\,T_{mn} = \alpha^{\mathrm{T}}_{im}\,T_{mn}\,\alpha_{nj} \\
&= \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}
\begin{pmatrix} \alpha & \gamma & 0 \\ -\gamma & \alpha & 0 \\ 0 & 0 & \alpha+\beta \end{pmatrix}
\begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} \begin{array}{l}\alpha\cos\varphi \\ \quad -\gamma\sin\varphi\end{array} & \begin{array}{l}\alpha\sin\varphi \\ \quad +\gamma\cos\varphi\end{array} & 0 \\ \begin{array}{l}-\alpha\sin\varphi \\ \quad -\gamma\cos\varphi\end{array} & \begin{array}{l}\alpha\cos\varphi \\ \quad -\gamma\sin\varphi\end{array} & 0 \\ 0 & 0 & \alpha+\beta \end{pmatrix}
\begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} \begin{array}{l}\alpha\cos^2\varphi - \gamma\sin\varphi\,\cos\varphi \\ \quad +\alpha\sin^2\varphi + \gamma\sin\varphi\,\cos\varphi\end{array} & \begin{array}{l}-\alpha\cos\varphi\,\sin\varphi + \gamma\sin^2\varphi \\ \quad +\alpha\cos\varphi\sin\varphi + \gamma\cos^2\varphi\end{array} & 0 \\ \begin{array}{l}-\alpha\cos\varphi\,\sin\varphi - \gamma\cos^2\varphi \\ \quad +\alpha\cos\varphi\,\sin\varphi - \gamma\sin^2\varphi\end{array} & \begin{array}{l}\alpha\sin^2\varphi + \gamma\sin\varphi\,\cos\varphi \\ \quad +\alpha\cos^2\varphi - \gamma\sin\varphi\,\cos\varphi\end{array} & 0 \\ 0 & 0 & \alpha+\beta \end{pmatrix} \\
&= \begin{pmatrix} \alpha & \gamma & 0 \\ -\gamma & \alpha & 0 \\ 0 & 0 & \alpha+\beta \end{pmatrix}.
\end{aligned}
$$

The class of transversely isotropic tensors has some subclasses of tensors which we have already seen earlier. If we set $\alpha = \cos\vartheta$, $\beta = \pm 1 - \cos\vartheta$, $\gamma = -\sin\vartheta$, then we obtain for $\underline{\underline{T}}$, according to (3.77) and (3.81), the general form of an orthogonal tensor, with the angle of rotation ϑ and the axis of rotation or rotary reflection $\underline{n}$. If we set $\alpha = \beta = 0$, then $\underline{\underline{T}}$ is antimetric and $\gamma\,\underline{n}$ is the corresponding vector of $\underline{\underline{T}}$. If we choose $\gamma = 0$, then the tensor is symmetric with an eigenvalue α of multiplicity two and another eigenvalue $\alpha+\beta$ of multiplicity one; $\underline{n}$ is the eigendirection corresponding to the eigenvalue $\alpha+\beta$ of multiplicity one.

We summarize the results of our discussion on anisotropy classes of second-order tensors in a table. For the anisotropy classes, the number of tensors $\underline{\underline{T}}$, which are contained in the class, increases from the first class to the last; the tensors of a particular anisotropy class are always included in the next anisotropy class. For the symmetry groups, the number of orthogonal matrices α_{ij} which are included in the class increases from the last class to the first; the matrices of a symmetry group are always contained in the previous symmetry group.

4. Anisotropies can also be considered by isotropic tensor functions. To explain this we consider the stress–strain relation for an elastic body. When we introduced the function $\underline{\underline{T}} = \underline{\underline{f}}(\underline{\underline{D}})$ between the symmetric stress tensor $\underline{\underline{T}}$ and the symmetric strain

Anisotropy class	**General tensor $\underline{\underline{T}}$**	**Symmetry group α_{ij}**
Isotropy	$k\,\underline{\underline{\delta}}$	arbitrary
Transverse isotropy	$\alpha\,\underline{\underline{\delta}} + \beta\,\underline{n}\,\underline{n} + \gamma\,\underline{\underline{\underline{\varepsilon}}}\cdot\underline{n}$	$\begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $\underline{n} = \underline{e}_z$
General anisotropy	arbitrary	$\pm\delta_{ij}$

tensor $\underline{\underline{D}}$ in Section 5.1, we silently assumed an isotropic solid, and, according to Section 5.4.4, No. 2, the function $\underline{\underline{f}}$ has the following representation:

$$\underline{\underline{T}} = k_1\,\underline{\underline{\delta}} + k_2\,\underline{\underline{D}} + k_3\,\underline{\underline{D}}^2, \qquad k_i = f(\operatorname{tr}\underline{\underline{D}}, \operatorname{tr}\underline{\underline{D}}^2, \operatorname{tr}\underline{\underline{D}}^3).$$

If, on the other hand, the solid is transversely isotropic, then it has a distinguished direction $\underline{n}$, which we have to include in the argument list of the function $\underline{\underline{f}}$, i. e. we are looking for a representation for $\underline{\underline{T}} = \underline{\underline{f}}(\underline{\underline{D}}, \underline{n})$. We can use the result from Section 5.4.4, No. 3, if in addition we take into account that $\underline{\underline{T}}$ is symmetric and that $\underline{n}$ is a unit vector, i. e. that $\underline{n}\cdot\underline{n}$ is no invariant, i. e.

$$\begin{aligned}\underline{\underline{T}} &= k_1^*\,\underline{\underline{\delta}} + k_2^*\,\underline{\underline{D}} + k_3^*\,\underline{\underline{D}}^2 + k_4^*\,\underline{n}\,\underline{n} + k_5^*(\underline{\underline{D}}\cdot\underline{n}\,\underline{n} + \underline{n}\,\underline{n}\cdot\underline{\underline{D}}) + k_6^*\,\underline{\underline{D}}\cdot\underline{n}\,\underline{n}\cdot\underline{\underline{D}},\\ k_i^* &= f(\operatorname{tr}\underline{\underline{D}}, \operatorname{tr}\underline{\underline{D}}^2, \operatorname{tr}\underline{\underline{D}}^3, \underline{n}\cdot\underline{\underline{D}}\cdot\underline{n}, \underline{n}\cdot\underline{\underline{D}}^2\cdot\underline{n}).\end{aligned} \tag{a}$$

This expression includes the stress–strain relation for an isotropic solid as a special case, if we set $k_4^* = k_5^* = k_6^* = 0$ and if k_1^*, k_2^*, and k_3^* do not depend on $\underline{n}$.

For small strains we can approximate (a) by a linear relation between the coordinates of $\underline{\underline{T}}$ and $\underline{\underline{D}}$. Then $k_3^* = k_6^* = 0$, $k_2^* = \alpha$, and $k_5^* = \beta$ are constant, and k_1^* and k_4^* are linear functions of the invariants $\operatorname{tr}\underline{\underline{D}}$ and $\underline{n}\cdot\underline{\underline{D}}\cdot\underline{n}$ which are linear in $\underline{\underline{D}}$, so we have

$$k_1^* = \kappa\operatorname{tr}\underline{\underline{D}} + \lambda\,\underline{n}\cdot\underline{\underline{D}}\cdot\underline{n}, \quad k_4^* = \mu\operatorname{tr}\underline{\underline{D}} + \nu\,\underline{n}\cdot\underline{\underline{D}}\cdot\underline{n}.$$

Then (a) simplifies in index notation to

$$\begin{aligned}T_{ij} = {}&(\kappa D_{pp} + \lambda n_p D_{pq} n_q)\,\delta_{ij} + (\mu D_{pp} + \nu n_p D_{pq} n_q)\,n_i n_j\\ &+ \alpha D_{ij} + \beta\,(D_{ip} n_p n_j + n_i n_p D_{pj}).\end{aligned}$$

If we introduce appropriate Kronecker symbols, we can factor out D_{kl} as follows:

$$\begin{aligned}T_{ij} = {}&(\kappa\,\delta_{pk}\,\delta_{pl}\,D_{kl} + \lambda\,n_p\,\delta_{pk}\,D_{kl}\,\delta_{lq}\,n_q)\,\delta_{ij}\\ &+ (\mu\,\delta_{pk}\,\delta_{pl}\,D_{kl} + \nu\,n_p\,\delta_{pk}\,D_{kl}\,\delta_{lq}\,n_q)\,n_i\,n_j + \alpha\,\delta_{ik}\,\delta_{jl}\,D_{kl}\\ &+ \beta\,(\delta_{ik}\,D_{kl}\,\delta_{lp}\,n_p\,n_j + n_i\,n_p\,\delta_{pk}\,D_{kl}\,\delta_{lj})\\ = {}&\{\kappa\,\delta_{ij}\,\delta_{kl} + \lambda\,\delta_{ij}\,n_k\,n_l + \mu\,n_i\,n_j\,\delta_{kl} + \nu\,n_i\,n_j\,n_k\,n_l + \alpha\,\delta_{ik}\,\delta_{jl}\\ &+ \beta\,(\delta_{ik}\,n_j\,n_l + n_i\,n_k\,\delta_{jl})\}D_{kl}.\end{aligned}$$

Since both T_{ij} and D_{kl} are symmetric, we can write

$$T_{ij} = \{\kappa\, \delta_{ij}\, \delta_{kl} + \lambda\, \delta_{ij}\, n_k\, n_l + \mu\, n_i\, n_j\, \delta_{kl} + \nu\, n_i\, n_j\, n_k\, n_l + \tfrac{1}{2}\,\alpha\,(\delta_{ik}\,\delta_{jl} + \delta_{il}\,\delta_{jk})$$
$$+ \tfrac{1}{2}\,\beta\,(\delta_{ik}\, n_j\, n_l + n_i\, n_k\, \delta_{jl} + \delta_{il}\, n_j\, n_k + n_i\, n_l\, \delta_{jk})\}D_{kl}.$$

The expression in the curly brackets can be interpreted as the coordinates of a fourth-order elasticity tensor $\underline{\underline{\underline{\underline{E}}}}$ with the symmetry properties

$$E_{ijkl} = E_{jikl} = E_{ijlk},$$

so we can shorten this to

$$T_{ij} = E_{ijkl}\, D_{kl}.$$

The constants α, β, κ, λ, μ, and ν must be determined from experiments (possibly by making further physical assumptions).