# 1 Quantum logic

*The nature of light is a subject of no material importance to the concerns of life or to the practice of the arts, but it is in many other respects extremely interesting.*
Thomas Young

In this chapter, we will continue our discussion of the connections between preparation devices, physical systems and measuring instruments, started in the Introduction. In particular, we will try to understand what is actually measured by the instruments, and how these results depend on the state of the observed system.

Until the end of the 19th century such questions could only raise eyebrows. In classical mechanics and in all pre-quantum physics, it was tacitly assumed that in each state the physical system possesses a set of quantities (position, momentum, mass, etc.). These quantities simply show up in measurements. They can be measured simultaneously, accurately and reproducibly. Yes, of course, every measurement is limited by a certain imprecision, but this is only a technical difficulty that can and should be neglected in a fundamental theory. All this was considered so obvious that it was not even mentioned in textbooks.

However, since the end of the 19th century, these traditional ideas began to disintegrate under the onset of new discoveries such as the radiation spectrum of heated bodies, the discrete spectrum of atoms and the photoelectric effect. Solutions of all these problems have been found within the framework of *quantum mechanics* – a completely new approach to physics that emerged in the first third of the 20th century as a result of joint efforts and passionate debates of such outstanding scientists as Bohr, Born, de Broglie, Dirac, Einstein, Fermi, Fock, Heisenberg, Pauli, Planck, Schrödinger, Wigner and many others. Out of all these studies, a completely unexpected and paradoxical picture of the physical world has emerged that was completely unlike the orderly and transparent classical picture. In spite of their counterintuitive strangeness, predictions of quantum mechanics are extraordinarily accurate: they are checked daily in countless physical and chemical laboratories around the world, and have never been refuted. This makes quantum mechanics the most successful physical theory of all time.

There are dozens of good textbooks explaining the laws of quantum mechanics and how to use them to analyze systems and predict observations in each particular case. We assume that the reader is fairly familiar with these laws. We will be more interested in the deeper meaning and interpretation of the quantum formalism, which still generates bitter controversies. Why does nature behave in a random way? Or is this randomness only apparent, but in fact there is a deeper level of reality, where quantum uncertainty gives way to some new laws? How can different states (alive and dead Schrödinger's cats) exist in a superposition? Is it possible to change the rules of quantum mechanics (for example, by adding some nonlinearity to the Hilbert space) without being in contradiction with experiments? People are increasingly asking such

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questions recently, when the search for the quantum theory of gravitation has intensified, and one popular trend is to look for alternative formulations of quantum mechanics in order to "harmonize" it with the general theory of relativity [**Khrennikov??**].

In this chapter we will present a rather old, but not well-known point of view on the origin of quantum laws. This approach seeks explanation of the quantum behavior in the fundamental logical structure of physics. In particular, this approach asserts that the true logical relationships between results of measurements are different from the classical laws of Aristotle and Boole. The usual classical logic needs to be replaced by the so-called *quantum logic*.

Since ancient Greece, logic has been considered the queen of sciences, perhaps not even a science as such, but something even more fundamental: a metascience, a framework for our perception of the world and for the construction of all other sciences. Therefore, it is difficult to imagine anything more revolutionary and provocative than an encroachment on the laws of logic. Nevertheless, there are enough convincing reasons to make just this step.

> In introductory quantum physics classes (especially in the United States), students are informed *ex cathedra* that the state of a physical system is represented by a complex-valued wave function $\psi$, that observables correspond to self-adjoint operators, that the temporal evolution of the system is governed by a Schrödinger equation and so on. Students are expected to accept all this uncritically, as their professors probably did before them. Any question of why is dismissed with an appeal to authority and an injunction to wait and see how well it all works. Those students whose curiosity precludes blind compliance with the gospel according to Dirac and von Neumann are told that they have no feeling for physics and that they would be better off studying mathematics or philosophy. A happy alternative to teaching by dogma is provided by basic quantum logic, which furnishes a sound and intellectually satisfying background for the introduction of the standard notions of elementary quantum mechanics – D. J. Foulis [**Foulis??**].

The idea that the most fundamental difference between classical and quantum mechanics lies in their different logical structures belongs to Birkhoff and von Neumann. In this chapter, we briefly outline their ideas of *quantum logic* [**Birkhoff??**] as well as later contributions made especially by Mackey [**Mackey??**] and Piron [**Piron2??Piron??**]; see also [**Curcuraci??**].

We will argue that the formalism of quantum mechanics (including the algebras of state vectors and Hermitian operators in the Hilbert space) follows almost inevitably from the simplest properties of measurements and logical relationships between them. These properties and relationships are so simple and fundamental that it seems impossible to modify them, and therefore it would be almost impossible to change quantum laws without violating their internal consistency and agreement with experiment. The practical conclusion is that the unification of quantum mechanics and relativity will not be achieved by changing or modifying quantum laws.[1]

In Section 1.1 we will examine limitations of classical approaches by analyzing the two-holes (two-slits) interference experiment from the points of view of the wave and corpuscular theories of light.

The logical structure of classical physics will be presented in Sections 1.2 and 1.3. In particular, we will discuss the close relationship between the classical Boolean logic and the phase space formalism. In Section 1.4 we will note the remarkable fact that the only difference between classical and quantum logics (and, therefore, between classical and quantum physics in general) lies in two inconspicuous axioms of *distributivity*. This postulate of classical logic should be replaced by the *orthomodular* postulate of quantum logic. In Section 1.5 this will lead us (via Piron's theorem) to the standard formalism of quantum mechanics with its Hilbert spaces, Hermitian operators, wave functions, etc. In Section 1.6 we will add some thoughts to the endless philosophical debate about interpretations of quantum mechanics.

## 1.1 Why do we need quantum mechanics?

The inadequacy of the classical concepts becomes clear if we analyze the dispute between corpuscular and wave theories of light. Let us illustrate the essence of this, without exaggeration, centuries-old debate by the example of a thought experiment with the *camera obscura*.

### 1.1.1 Corpuscular theory of light

You may have seen or heard about a simple optical device called *camera obscura* or pinhole camera. It is easy to make this device yourself. Take a lightproof box, make a small hole in one of its walls and place a photographic plate at the opposite wall, as shown in Figure 1.1. The light entering the inside of the box through the hole will create a clear inverted image of the outside world on the photographic plate.

You can achieve even greater clarity by reducing the size of the hole. But this, of course, will reduce the brightness of the image. This behavior of light has been known for centuries. The first scientific explanation for this and many other properties of light (reflection, refraction, etc.) was suggested by Newton. In a slightly modernized language, his *corpuscular theory* explained the formation of the image as follows:

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**1** In Volume 3 we will explain how one should change the formalism of special relativity to make it compatible with quantum mechanics.

> ***Corpuscular theory:*** Light is a stream of tiny particles (photons) flying along straight classical trajectories (light rays). [For example, the ray that lands at the point $A'$ in Figure 1.1 was emitted from the point $A$ and passed right through the hole.] Each such particle carries a certain amount of energy. When the particle collides with the photographic plate, this energy is released within one grain of the emulsion and creates a single image point. Bright light contains so many photons that their individual spots flood the photographic plate. All these points merge into one continuous image, and the density of the image is proportional to the number of particles hitting the plate during the time of exposure.

Let us continue our experiment with the pinhole camera, making the hole size smaller and smaller. Corpuscular theory asserts that shrinking holes will produce a clearer, but dimmer image. However, the experiment shows something completely different! At some point, as the size of the hole is reduced, the image will begin to blur; and in the limit of a very small hole all the details will disappear, and the picture will turn into one circular diffuse spot, as in Figure 1.2 (a). The shape and size of this blur are no longer dependent on the light source outside the camera. It would seem that the light rays, passing through the small hole, are randomly scattered in all directions. This effect was discovered by Grimaldi in the middle of the 17th century and was subsequently dubbed *diffraction*.

Diffraction does not fit in the corpuscular theory. Why on earth do light corpuscles deviate from straight-line trajectories? Maybe this is due to their interaction with the material of the walls surrounding the hole? However, this explanation should be rejected, if only because the diffraction pattern does not depend on the material – paper or steel – from which the walls of the box are made.

The most striking evidence of the fallacy of the naïve corpuscular theory of light is the *interference* effect, discovered by Young in 1802. To see the interference, we can slightly modify our pinhole camera: instead of one hole, make two holes that are close to each other so that their diffraction blurs on the photographic plate overlap. We already know that if we leave open the left hole and close the right one, then we get a diffuse blur $L$ (the left dashed line in Figure 1.3 (a)). If, on the contrary, we close the left hole and open the right one, we get another diffuse blur $R$. Let us now try to predict what would happen if both holes are opened.

**Figure 1.1:** The image in the pinhole camera is created by rectilinear beams (rays) of light.

**Figure 1.2:** (a) The image in the pinhole camera with a very small hole. (b) Image density along the line *AB*.

**Figure 1.3:** Image density in a two-hole camera. (a) In the naïve corpuscular theory. (b) In reality.

Following the logic of the corpuscular theory, we could conclude that the photons reaching the photographic plate are of two kinds: those that have passed through the left and the right hole, respectively. If the two holes are open simultaneously, then the density of "left" photons should add with the density of "right" photons, and the resulting image $L + R$ must be a superposition of the two images (solid line in Figure 1.3 (a)). Right? No, wrong! This seemingly logical reasoning is at odds with the experiment. The actual image on the photographic plate has an additional structure (brighter and darker areas shown by solid line in Figure 1.3 (b)), called the *interference pattern*. There are regions where the image density is higher than $L + R$ (constructive interference) and regions with the density lower than $L + R$ (destructive interference).

How can corpuscular theory explain this strange interference pattern? For example, we could assume that there is some interaction between light corpuscles, so that the passage of particles through the left and right holes are not independent events, and the law of addition of probabilities is not applicable to them. However, this idea should be rejected, because the interference pattern does not disappear even if we release the photons one by one, so that their interaction is excluded.

For example, in a two-hole interference experiment performed by Taylor in 1909 [**Taylor??**], the light intensity was so low that no more than one photon was present in the camera at any given time. This removed any possibility of interaction between photons and any effect of this interaction on the interference pattern. Does this mean that a single photon can interfere with itself? Maybe the photon somehow splits apart, passes through both holes and then reconnects before colliding with the photographic plate? This explanation also does not stand up to criticism, because one photon can blacken only one grain of the emulsion. Nobody has ever seen a "half-photon".

Perhaps, a particle passing through the right hole somehow knows whether the left hole is open or closed, and adjusts its trajectory, accordingly? This just does not make sense, and we have to admit that our simple corpuscular theory does not have a logical explanation for all these observations.

### 1.1.2 Wave theory of light

The inability to explain such fundamental properties of light as diffraction and interference was a heavy blow to the Newtonian corpuscular theory. These effects, like other light properties known in the pre-quantum era (reflection, refraction, polarization, etc.) were brilliantly explained by the *wave theory* of light developed by Huygens, Young, Fresnel and others. During the 19th century, the wave theory gradually supplanted the Newtonian corpuscles. The idea that light is a wave process received its strongest support from the Maxwell theory, which combined optics with electromagnetic phenomena. This theory explained that light is made of oscillating electric $\boldsymbol{E}(t, \boldsymbol{r})$ and magnetic $\boldsymbol{B}(t, \boldsymbol{r})$ fields – sinusoidal waves propagating with the speed of light $c$. According to Maxwell, the energy of this wave and, accordingly, the intensity of light $I$ is proportional to the square of the amplitude of the field vectors: $I \propto \boldsymbol{E}^2$. Then, from the point of view of the wave theory, the formation of the photographic image can be explained as follows.

> ***Wave theory:*** Light is a continuous oscillating wave or field propagating through space. When a light wave meets with molecules of the photographic emulsion, charged parts of the molecules begin to oscillate under the action of the electric and magnetic vectors in the light field. In those places where the amplitude of the electromagnetic oscillations is maximal, the charges of the molecules are subjected to the strongest force, and the density of the photographic image is the highest.

This model explains both diffraction and interference in a fairly natural way: diffraction simply means that light waves are capable of going around obstacles, just like other types of waves (sea waves, sound waves, etc.) do.[2] To explain the interference at two holes, it is sufficient to note that when two parts of a monochromatic wave pass

**2** The wavelength of the visible light varies between 0.4 micron for violet light and 0.7 microns for red light. So, for large obstacles or holes, the effect of diffraction is very small, and the corpuscular theory of light works quite well.

through different apertures and meet on a photographic plate, their electric (and magnetic) vectors add up. However, the wave intensities are proportional to the squares of the vectors and, therefore, are not additive: $I \propto (\boldsymbol{E}_1 + \boldsymbol{E}_2)^2 = \boldsymbol{E}_1^2 + 2\boldsymbol{E}_1 \cdot \boldsymbol{E}_2 + \boldsymbol{E}_2^2 \neq \boldsymbol{E}_1^2 + \boldsymbol{E}_2^2 \propto I_1 + I_2$. From simple geometric considerations it follows that there are places where these two waves always come with the same phase ($\boldsymbol{E}_1 \uparrow\uparrow \boldsymbol{E}_2$ and $\boldsymbol{E}_1 \cdot \boldsymbol{E}_2 > 0$), which means *constructive interference*, and there are other places where the waves come out of phase ($\boldsymbol{E}_1 \uparrow\downarrow \boldsymbol{E}_2$ and $\boldsymbol{E}_1 \cdot \boldsymbol{E}_2 < 0$), i. e., *destructive interference*.

### 1.1.3 Light of low intensity and other experiments

In the 19th century physics, the particle–wave dispute was resolved in favor of the wave theory of light. However, further experiments showed that the victory was declared prematurely. To understand the problems of the wave theory, let us continue our thought experiment with the interference pattern. This time, we will gradually reduce the intensity of the light source. At first we will not notice anything unusual: quite predictably the image density on the photographic plate will decrease. However, from a certain point we will notice that the image ceases to be uniform and continuous, as before. We will see that it consists of separate dots, as if light were incident on some grains of the emulsion and did not touch others. This observation is difficult to explain from the point of view of the wave theory. How can a continuous wave create this dotted image? But the corpuscular theory copes easily: obviously, these dots are created by separate particles (photons), which bombard the surface of the photographic plate.

In the late 19th and early 20th centuries, other experiments appeared that challenged the wave theory of light. The most famous of them was the photoelectric effect: it was found out that when light falls on a piece of metal, it can knock electrons out of the metal into the vacuum. In itself, this discovery was not surprising. However, it was surprising how the number of knocked-out electrons depended on the frequency of light and its intensity. It was found out that only light with a frequency above a certain threshold $\omega_0$ could knock out electrons from the metal. Light of a lower frequency was unable to do this, even if its intensity was very high. Why was this observation so surprising? From the point of view of the wave theory, it could be assumed that the electrons are emitted from the metal by the forces originated from the electric $\boldsymbol{E}$ and magnetic $\boldsymbol{B}$ fields in the light wave. The higher intensity of light (= the larger magnitudes of the vectors $\boldsymbol{E}$ and $\boldsymbol{B}$) naturally means a higher force acting on the electrons and a greater probability of the electron emission. So why could not intensive low-frequency light cope with this work?

In 1905, Einstein explained the photoelectric effect by returning to the long-forgotten Newtonian corpuscles in the form of light quanta, later called *photons*. Einstein described light as "... consisting of finite number of energy quanta which are localized at points in space, which move without dividing and which can only be produced and absorbed as complete units." [**Arons??**]. According to Einstein, each photon carries the energy $\hbar\omega$, where $\omega$ is the light frequency[3] and $\hbar$ is the *Planck*

**3** $\omega$ is the so-called *cyclic frequency* (measured in radians per second), which is related to the ordinary frequency $\nu$ (measured in cycles per second) by formula $\omega = 2\pi\nu$.

*constant*. Each photon has a chance to encounter only one electron in the metal and transfer its energy to it. Only high-energy photons (i. e., photons present in high-frequency light) can transmit enough energy to the electron to overcome the energy barrier $E_b$ between the volume of the metal and the vacuum. Low-frequency light has low-energy photons $\hbar\omega < E_b \approx \hbar\omega_0$. Hence, regardless of the intensity (= the number of photons) of such light, its photons are simply too weak and unable to kick the electrons strong enough to overcome the barrier.[4]

In the Compton experiments (1923), the interaction of X-ray radiation with free electrons was studied in much detail, and indeed, this interaction was more like a collision of two particles than a shaking of the electron by a periodic electromagnetic wave.

These observations should confirm our conclusion that light is a stream of corpuscles, as Newton said. But how about interference? We have already established that corpuscular theory is unable to give a logical explanation for this effect!

So, the young quantum theory faced the seemingly impossible task of reconciling two classes of experiments with light. Some experiments (diffraction, interference) were easily explained within the framework of the wave theory of light, but did not agree with the corpuscles – photons. Other experiments (photoelectric effect, Compton scattering) contradicted the wave properties and clearly indicated that light consists of particles. To all this confusion, in 1924 de Broglie added the hypothesis that the *particle–wave dualism* is characteristic not only of photons. He argued that all material particles – for example, electrons – have wave properties. This "crazy" idea was soon confirmed by Davisson and Germer, who observed the interference of electron beams in 1927.

Without a doubt, in the first quarter of the 20th century, physics approached the greatest crisis in its history. Heisenberg described this situation as follows.

> I remember discussions with Bohr which went through many hours till very late at night and ended almost in despair; and when at the end of the discussion I went alone for a walk in the neighboring park I repeated to myself again and again the question: Can nature possibly be as absurd as it seemed to us in those atomic experiments? – W. Heisenberg [**Heisenberg??**].

## 1.2 Classical logic

In order to advance in our understanding of the paradoxes mentioned above, we need to go beyond the framework of classical physics. Therefore, to begin with, we are going to outline this framework, i. e., to look at classical mechanics. For simplicity, we consider the classical description of a single particle in a one-dimensional space.

4 In fact, low-frequency light can lead to the electron emission when two low-energy photons collide simultaneously with the same electron. But such events are unlikely and become noticeable only at very high light intensities.

### 1.2.1 Phase space of one classical particle

The *state* $\phi$ of a classical particle is completely and uniquely determined by specifying the particle's position $x$ and momentum $p$.[5] Such states will be called *pure classical*.

Thus, all possible states of a one-particle system are labeled by a pair of numbers $(x, p)$ and can be represented by points on a plane, which will be called the *phase space* of the system and denoted $\mathcal{S}$ (see Figure 1.4). Then particle dynamics is represented by lines (= trajectories) in the phase space $\mathcal{S}$.

**Figure 1.4:** Phase space of a particle in one spatial dimension.

### 1.2.2 Propositions in phase space

In order to make it easier to switch to the quantum description in the future, let us introduce the concept of experimental (or logical) *proposition*, sometimes also called *yes–no question*. The most obvious are propositions about individual observables. For example, the proposition $\mathcal{A}$ = "the particle position is in the interval $a$" is meaningful. In each pure state of the system, this proposition can be either true or false (the question answered "yes" or "no").

Experimentally, such a proposition can be realized using a one-dimensional "Geiger counter", which occupies the region $a$ of space. The counter clicks (= the proposition is true) if the particle passes through the counter's discharge chamber and does not click (= the proposition is false) if the particle is outside the region $a$.

Propositions can be represented in the phase space. For example, the above proposition $\mathcal{A}$ is associated with the strip $A$ in Figure 1.4. The proposition is true if the point representing the state $\phi$ is inside the strip $A$. Otherwise, the proposition is false.

Similarly, propositions about the momentum are represented by strips parallel to the $x$ axis. For example, the strip $B$ in Figure 1.4 corresponds to the logical proposition $\mathcal{B}$ = "the particle momentum belongs to the interval $b$."

5 It might seem more natural to work with the particle's velocity $v$ instead of its momentum. However, we will see later that our choice of primary observables has a special meaning, because $x$ and $p$ are "canonically conjugated" variables.

We will denote by $\mathcal{L}$ the set of all propositions about the physical system.[6] In the rest of this section we will study the structure of this set and establish its connection with the classical *Boolean logic*. The set of all possible states of the system will be denoted $\mathfrak{S}$. In this chapter, our goal is to study the mathematical relationships between the elements $\mathcal{X} \in \mathcal{L}$ and $\phi \in \mathfrak{S}$ in these two sets.

### 1.2.3 Operations with propositions

In classical theory, in addition to the above propositions $\mathcal{A}$ and $\mathcal{B}$ about single observables, we can also associate a proposition with each region of the phase space. For example, in Figure 1.4 we showed a rectangle $C$, which is an intersection of the two strips $C = A \cap B$. Apparently, this rectangle also corresponds to an admissible proposition $\mathcal{C} = \mathcal{A} \wedge \mathcal{B}$ = 'the particle position is in the interval *a and* its momentum is in the interval *b*." In other words, this proposition is obtained by applying the logical operation "AND" to the two elementary propositions $\mathcal{A}$ and $\mathcal{B}$. We denote this logical operation (*meet*) by the symbol $\mathcal{A} \wedge \mathcal{B}$.[7]

The four other logical operations listed in Table 1.1 are also naturally defined in the language of propositions–regions. For example, the rectangle $C = A \cap B$ lies en-

**Table 1.1:** Five operations and two special elements in the theory of subsets of the phase space $\mathcal{S}$, in the classical logic and in the lattice theory.

| **Symbol for subsets in $\mathcal{S}$** | **Name in logic** | **Meaning in classical logic** | **Name in lattice theory** | **Symbol in lattice theory** |
|---|---|---|---|---|
| *Operations with subsets/propositions* | | | | |
| $X \subseteq Y$ | implication | $\mathcal{X}$ IMPLIES $\mathcal{Y}$ | less or equal | $\mathcal{X} \le \mathcal{Y}$ |
| $X \subseteq Y, X \neq Y$ | implication | $\mathcal{X}$ IMPLIES $\mathcal{Y}$ | less | $\mathcal{X} < \mathcal{Y}$ |
| $X \cap Y$ | conjunction | $\mathcal{X}$ AND $\mathcal{Y}$ | meet | $\mathcal{X} \wedge \mathcal{Y}$ |
| $X \cup Y$ | disjunction | $\mathcal{X}$ OR $\mathcal{Y}$ | join | $\mathcal{X} \vee \mathcal{Y}$ |
| $\mathcal{S} \setminus X$ | negation | NOT $\mathcal{X}$ | orthocomplement | $\mathcal{X}^\perp$ |
| *Special subsets/propositions* | | | | |
| $\mathcal{S}$ | tautology | always true | maximal element | $\mathcal{I}$ |
| $\emptyset_{\mathcal{S}}$ | absurdity | always false | minimal element | $\emptyset$ |

6 $\mathcal{L}$ is also called the *propositional system* or *logic*.

7 This symbol differs from the symbol $A \cap B$ for the intersection of two regions in the phase space, thereby emphasizing that we are dealing with the logical operation "AND", which relates specifically to propositions. In classical logic, there is an equivalence between propositions and regions in the phase space $\mathcal{S}$, so having two different notations may seem superfluous. However, in the quantum case, such an equivalence is lost, the idea of the phase space is not applicable and only the logical notation $\mathcal{A} \wedge \mathcal{B}$ makes sense.

tirely inside the strip $A$. From the point of view of logic, we can say that proposition $\mathcal{C}$ "IMPLIES" proposition $\mathcal{A}$. Indeed, in any state where $\mathcal{C}$ is true, the proposition $\mathcal{A}$ is also true. This logical connection will be denoted by $\mathcal{C} \leq \mathcal{A}$.[8]

The proposition $\mathcal{A}$ "OR" $\mathcal{B}$ corresponds to the union ($A \cup B$) of two regions in the phase space. This proposition will be written as $\mathcal{A} \vee \mathcal{B}$. If either $\mathcal{A}$ or $\mathcal{B}$ is true, then the *join* $\mathcal{A} \vee \mathcal{B}$ is definitely true.

The last operation is the complement of a phase-space region $A$.[9] Obviously its logical equivalent is the negation of the proposition $\mathcal{A}$, which we denote by $\mathcal{A}^{\perp}$ (= "NOT"$\mathcal{A}$, *orthocomplement*).

In addition to these four operations, we will need two special propositions, listed in Table 1.1.

*Maximal* proposition (or *tautology*) $\mathcal{I} \in \mathcal{L}$ corresponds to the whole phase space, i. e., the maximal subset of $\mathcal{S}$. This proposition can be expressed in different verbal forms. For example: $\mathcal{I}$ = "particle position is somewhere on the real axis" or $\mathcal{I}$ = "particle momentum is somewhere on the real axis." Both these propositions are always true for any state.[10]

Propositions like "the value of the observable is not on the real axis" or "the value of the observable lies in the empty subset of the real axis" are always false and equal to the single *minimal* (or *absurd*) proposition $\emptyset$ in the set $\mathcal{L}$.

### 1.2.4 Axioms of logic

Five operations and two special propositions, presented above, define a rich mathematical structure. To work with these objects, it is necessary to establish their mutual relations, i. e., laws (or axioms) of logic.

The easiest way to establish these laws is to use the equivalence between logical propositions and subsets of the phase space. This means that the properties of logical operations ("IMPLIES," "AND," "OR," "NOT") coincide with the properties of operations on subsets ("inclusion," "intersection," "union," "complement"). From this analogy, it is not difficult to obtain the laws of classical logic listed in lines 1 through 19 of Table 1.2.[11]

**8** If $\mathcal{C}$ "IMPLIES" $\mathcal{A}$ and definitely $\mathcal{A} \neq \mathcal{C}$, then we will use the symbol $\mathcal{C} < \mathcal{A}$.

**9** That is, the region consisting of phase-space points not belonging to $A$.

**10** Measurements of observables always yield *some* result, because we agreed in the Introduction that an ideal measuring device never misfires.

**11** Actually, the choice of the axioms of logic is rather arbitrary. There are different approaches to the axiomatization of logic, and our approach is not the most economical. We tried to select our axioms so that they had the most transparent meaning.

**Table 1.2:** Basic axioms of classical and quantum logics.

| | Name | Formula |
|---|---|---|
| *Axioms of orthocomplemented lattices* | | |
| 1 | Reflectivity | $\mathcal{X} \le \mathcal{X}$ |
| 2 | Symmetry | $(\mathcal{X} \le \mathcal{Y})$ & $(\mathcal{Y} \le \mathcal{X}) \Rightarrow \mathcal{X} = \mathcal{Y}$ |
| 3 | Transitivity | $(\mathcal{X} \le \mathcal{Y})$ & $(\mathcal{Y} \le \mathcal{Z}) \Rightarrow \mathcal{X} \le \mathcal{Z}$ |
| 4 | Definition of $\mathcal{I}$ | $\mathcal{X} \le \mathcal{I}$ |
| 5 | Definition of $\emptyset$ | $\emptyset \le \mathcal{X}$ |
| 6 | Definition of $\wedge$ | $\mathcal{X} \wedge \mathcal{Y} \le \mathcal{X}$ |
| 7 | Definition of $\wedge$ | $(\mathcal{Z} \le \mathcal{X})$ & $(\mathcal{Z} \le \mathcal{Y}) \Rightarrow \mathcal{Z} \le (\mathcal{X} \wedge \mathcal{Y})$ |
| 8 | Definition of $\vee$ | $\mathcal{X} \le \mathcal{X} \vee \mathcal{Y}$ |
| 9 | Definition of $\vee$ | $(\mathcal{X} \le \mathcal{Z})$ & $(\mathcal{Y} \le \mathcal{Z}) \Rightarrow (\mathcal{X} \vee \mathcal{Y}) \le \mathcal{Z}$ |
| 10 | Commutativity | $\mathcal{X} \vee \mathcal{Y} = \mathcal{Y} \vee \mathcal{X}$ |
| 11 | Commutativity | $\mathcal{X} \wedge \mathcal{Y} = \mathcal{Y} \wedge \mathcal{X}$ |
| 12 | Associativity | $(\mathcal{X} \vee \mathcal{Y}) \vee \mathcal{Z} = \mathcal{X} \vee (\mathcal{Y} \vee \mathcal{Z})$ |
| 13 | Associativity | $(\mathcal{X} \wedge \mathcal{Y}) \wedge \mathcal{Z} = \mathcal{X} \wedge (\mathcal{Y} \wedge \mathcal{Z})$ |
| 14 | Noncontradiction | $\mathcal{X} \wedge \mathcal{X}^\perp = \emptyset$ |
| 15 | Noncontradiction | $\mathcal{X} \vee \mathcal{X}^\perp = \mathcal{I}$ |
| 16 | Double negation | $(\mathcal{X}^\perp)^\perp = \mathcal{X}$ |
| 17 | Contraposition | $\mathcal{X} \le \mathcal{Y} \Rightarrow \mathcal{Y}^\perp \le \mathcal{X}^\perp$ |
| *Additional assertions of classical logic* | | |
| 18 | Distributivity | $\mathcal{X} \vee (\mathcal{Y} \wedge \mathcal{Z}) = (\mathcal{X} \vee \mathcal{Y}) \wedge (\mathcal{X} \vee \mathcal{Z})$ |
| 19 | Distributivity | $\mathcal{X} \wedge (\mathcal{Y} \vee \mathcal{Z}) = (\mathcal{X} \wedge \mathcal{Y}) \vee (\mathcal{X} \wedge \mathcal{Z})$ |
| *Additional postulate of quantum logic* | | |
| 20 | Orthomodularity | $\mathcal{X} \le \mathcal{Y} \Rightarrow \mathcal{X} \leftrightarrow \mathcal{Y}$ |

For example, the transitivity property 3 from Table 1.2 allows us to build syllogisms, such as the one analyzed by Aristotle:

*If all humans are mortal,*
*and all Greeks are humans,*
*then all Greeks are mortal.*

Indeed, we have three propositions: $\mathcal{X}$ = "this is a Greek," $\mathcal{Y}$ = "this is a human being" and $\mathcal{Z}$ = "this is mortal." We know that $\mathcal{X}$ implies $\mathcal{Y}$ (i. e., $\mathcal{X} \le \mathcal{Y}$). We also know that $\mathcal{Y}$ implies $\mathcal{Z}$ ($\mathcal{Y} \le \mathcal{Z}$). Then the transitivity property tells us that $\mathcal{X}$ implies $\mathcal{Z}$ ($\mathcal{X} \le \mathcal{Z}$, i. e., "every Greek is mortal").

Property 14 says that a proposition $\mathcal{X}$ and its negation $\mathcal{X}^\perp$ cannot be true at the same time, i. e., their meet $\mathcal{X} \wedge \mathcal{X}^\perp$ is equal to the absurd proposition $\emptyset$. Property 15 is the famous *tertium non datur* law of logic: either the proposition $\mathcal{X}$ or its negation $\mathcal{X}^\perp$ is true, and *the third is not given*.

A set of objects with operations and special elements from Table 1.1, subject to properties 1–17 from Table 1.2, is referred to as the *orthocomplemented lattice* by mathematicians.

Many useful logical relationships can be derived from the axioms of orthocomplemented lattices. Some of them are formulated in the form of lemmas and theorems in Appendix ??. However, these axioms 1–17 are still not enough to describe the classical logic of propositions unequivocally. Subsets of the phase space and propositions of classical logic are subject to additional *distributive laws* 18 and 19 in Table 1.2.

Like other properties in the upper portion of Table 1.2, the distributive laws are easily derived from our analogy "proposition" ↔ "region of the phase space." Nevertheless, we put these laws in a separate category. As we shall see in Section 1.4, it is these laws that determine the difference between classical and quantum logics. In Table 1.2 we call them "assertions," because we do not consider them to be true in fundamental quantum theory.[12]

### 1.2.5 Phase space from axioms of classical logic

Thus, we have shown that in the phase space of classical mechanics the set of all propositions $\mathcal{L}$ is an orthocomplemented lattice with distributive laws 18 and 19. Such lattices will be called *Boolean algebras* or *classical logics*.[13]

For us, it is very important that one can prove the converse statement, which is the following.

**Theorem 1.1** (representation of classical logic). *For each classical logic $\mathcal{L}$ defined by properties* 1–19 *from Table* 1.2*, there exist a set $\mathcal{S}$*[14] *and an isomorphism $h(\mathcal{X})$ between logical propositions $\mathcal{X} \in \mathcal{L}$ and subsets of $\mathcal{S}$, such that logical operations in $\mathcal{L}$ match with set-theoretical operations in $\mathcal{S}$, as follows:*

$$
\begin{aligned}
\mathcal{X} \le \mathcal{Y} &\Leftrightarrow h(\mathcal{X}) \subseteq h(\mathcal{Y}),\\
h(\mathcal{X} \wedge \mathcal{Y}) &= h(\mathcal{X}) \cap h(\mathcal{Y}),\\
h(\mathcal{X} \vee \mathcal{Y}) &= h(\mathcal{X}) \cup h(\mathcal{Y}),\\
h(\mathcal{X}^{\perp}) &= \mathcal{S} \setminus h(\mathcal{X}),\\
h(\mathcal{I}) &= \mathcal{S},\\
h(\emptyset) &= \emptyset_{\mathcal{S}};
\end{aligned}
$$

*see Table* 1.1.

12 In our book we distinguish *postulates*, *statements* and *assertions*. Postulates form the basis of our theory. In many cases, they undoubtedly follow from experiments, and we do not question their validity. Statements follow logically from the Postulates, and we consider them to be correct. Assertions refer to claims that are made in other theories, but do not have place in our approach (RQD).

13 Strictly speaking, the definition of classical logic involves also a technical condition of the lattice *atomicity*. In our case this means the existence of "minimal nonzero" propositions – *atoms*, which correspond to points in the phase space.

14 In classical mechanics, the set $\mathcal{S}$ is called the *phase space*.

The importance of this theorem lies in the possibility to derive foundations of classical physics (e. g., the structure of the phase space) from axioms of logic. Starting with the Boolean logic, we come to the idea of the phase space, where states are represented by points. From here it is not far to other elements of classical mechanics, such as, for example, the description of dynamics by trajectories.

### 1.2.6 Classical observables

In classical mechanics, an observable (= physical quantity) $F$ is represented by a real function $f : \mathcal{S} \to \mathbb{R}$ on the phase space. To each point (= state) of the phase space the function $f$ associates a single number – the value of the observable in this state. Three examples of such observables/functions are shown in Figure 1.5. They are the position $x$, the momentum $p$ and the energy $H$ of a one-dimensional oscillator (a pendulum) with a quadratic Hamiltonian. The values taken by the corresponding functions $f$ are from the spectra of the observables. In the case of $x$ and $p$, the spectrum is the entire real axis $\mathbb{R} = (-\infty, +\infty)$, and the spectrum of $H$ is the set of nonnegative numbers $[0, +\infty)$.

**Figure 1.5:** Observables in the language of propositions in the phase space: (a) position $x$, (b) momentum $p$, (c) energy of the harmonic oscillator $H(x,p) = p^2/(2m) + \alpha x^2$.

Each such function-observable $f$ defines a set of constant-value lines $x, p, H = \dots, 1, 2, 3, 4, \dots$ in $\mathcal{S}$ (shown in Figure 1.5), which in turn can be interpreted as subsets $\mathcal{S}_f$ or logical propositions in the phase space. The proposition $\mathcal{S}_f \in \mathcal{L}$ is pronounced "the observable $F$ has the value $f$." Thus, each observable can be equivalently described as a map $\mathcal{F}$ from the spectrum of the observable into the set of all propositions $\mathcal{L}$. This map[15] has the following properties:

(1) The function $\mathcal{F}$ associates to each point $f$ of the spectrum of the observable $F$ one and only one logical proposition $\mathcal{S}_f \in \mathcal{L}$.
(2) Propositions corresponding to different points ($f \neq f'$) of the spectrum are disjoint.[16] On the phase plane, such disjoint propositions correspond to nonintersecting regions, otherwise we would have absurd states possessing two different values of the same observable simultaneously.

---

**15** It is also called the *proposition-valued measure*.

**16** Two propositions $\mathcal{X}$ and $\mathcal{Y}$ are called *disjoint*, if $\mathcal{X} \leq \mathcal{Y}^{\perp}$ (or, equivalently, $\mathcal{Y} \leq \mathcal{X}^{\perp}$).

**Figure 1.6:** Observable $H$ as a mapping $\mathcal{F}$ from the spectrum of $H$ into the set of propositions in the phase space $S$. The function $\mathcal{F}$ maps the spectrum interval $a = [2, 4]$ to the subset-proposition $\mathcal{A}$.

(3) The join (union) of the propositions $\mathcal{S}_f$ over all spectrum points is equal to the maximal (trivial) proposition ($\vee_f \mathcal{S}_f = \mathcal{I}$), which is equivalent to the entire phase space. This condition indicates that in each state it is possible to measure some value of the observable. There are no states (= points in the phase space) where the observable is not measurable.

So, with each observable $F$ and with each subset $a$ of the real axis $\mathbb{R}$ we associate an experimental *proposition* $\mathcal{A}$ = "the value of the observable $F$ is inside the subset $a \subseteq \mathbb{R}$." Obviously, $\mathcal{A}$ is equal to the join of elementary propositions $\mathcal{S}_f$ over all points of the spectrum lying inside the interval $a$. The mapping subset→proposition is illustrated in Figure 1.6.

## 1.3 Measurements and probabilities

In the previous section, we developed the classical logic of strictly deterministic states in which the answer to any yes–no question could be either definite "yes" or definite "no." However, such states are rarely found in real experiments. As a rule, measurements are associated with randomness, uncertainties, errors, etc. To describe such unpredictable outcomes we need the concepts of an ensemble and a probability measure.

### 1.3.1 Ensembles and measurements

We will call *experiment* a procedure for preparing an *ensemble*[17] and measuring the same observable in each member of the ensemble.[18]

**17** Ensemble is a set of identical copies of the physical system, made in – as much as possible – the same conditions.

**18** It is important to note that in this book we do not consider repeated measurements performed on the same copy of the physical system. We will assume that after the measurement has been made, the

So, let us prepare many copies of the system, all in one state $\phi$ (= ensemble) and perform measurements of the same proposition $\mathcal{X}$ in all these copies. As we already know, there is no guarantee that the outcomes of these measurements will be the same. Hence, for some members of the ensemble the proposition $\mathcal{X}$ will be found true, and for other members it will be false. Using these data, we can introduce a function $(\phi|\mathcal{X})$, which we call *probability measure* and which associates to each state $\phi$ and each proposition $\mathcal{X}$ the probability that $\mathcal{X}$ is true in the state (ensemble) $\phi$. The value of this function (a real number in the interval between 0 and 1) is obtained as a result of the following steps:

(i) prepare an instance of the system in the state $\phi$;
(ii) make a measurement and determine whether the proposition $\mathcal{X}$ is true or false;
(iii) repeat steps (i) and (ii) $N$ times and calculate the probability by the formula

$$(\phi|\mathcal{X}) = \lim_{N\to\infty} \frac{M}{N},$$

where $M$ is the number of times the proposition $\mathcal{X}$ was found true.

In order to obtain the most complete description of the physical system, it is necessary to perform such experiments with all possible propositions $\mathcal{X} \in \mathcal{L}$ for all possible ensembles (= states) $\phi \in \mathfrak{S}$.

### 1.3.2 States as probability measures

If we are not too lazy to complete all such measurements, we will notice that the probability measure $(\phi|\mathcal{X})$ has the following properties:

– The probability corresponding to the maximal (trivial) proposition is 1 in all states, so

$$(\phi|\mathcal{I}) = 1. \tag{1.1}$$

– The probability corresponding to the minimal (absurd) proposition is 0 in all states, so

$$(\phi|\emptyset) = 0. \tag{1.2}$$

– The probability corresponding to the join of disjoint propositions is the sum of individual probabilities, so

$$(\phi|\mathcal{X} \vee \mathcal{Y}) = (\phi|\mathcal{X}) + (\phi|\mathcal{Y}), \quad \text{if } \mathcal{X} \le \mathcal{Y}^{\perp}. \tag{1.3}$$

---

used copy of the system is discarded. A fresh copy is required for each new measurement. In particular, this means that we are not interested in the state of the system after the measurement. The description of successive measurements in one instance of a physical system is an interesting task, but it is beyond the scope of our book.

The first two statements follow directly from definitions of special logic elements $\mathcal{I}$ and $\emptyset$. The third statement is known as the *third Kolmogorov probability axiom*: the probability of observing either one of the two (or several) mutually exclusive events is equal to the sum of event probabilities.

### 1.3.3 Probability distributions and statistical mechanics

In classical physics, the description of random events is handled by *statistical mechanics*. In this discipline, states that have an element of randomness are called *mixed classical* states. Mathematically, they are represented by *probability distributions*, which are functions $\rho(x,p)$ on the phase space that

(1) are nonnegative: $\rho(x,p) \geq 0$;
(2) normalized (their integral over the entire phase space is equal to 1),

$$\int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dp\rho(x,p) = 1;$$

(3) express the probability of the answer *yes* to the question $\mathcal{X}$ by the formula

$$(\phi|\mathcal{X}) = \int_X dxdp\rho(x,p), \tag{1.4}$$

where $X$ is the region of the phase space corresponding to the question (= proposition) $\mathcal{X}$.

By combining the notion of probability distribution with the laws of logic 1–19 from Table 1.2, we can arrive at the classical theory of probability. But we will not dwell on it here. In the next two sections, we will be more interested in quantum logic and quantum probability theory. Here, we will finish our discussion of classical probabilities with a few remarks about determinism.

The randomness present in mixed classical states is usually associated with our inability to provide identical preparation conditions for all members in the ensemble. For example, when we throw a die, it falls in an accidental, unpredictable manner. However, we believe that this unpredictability is simply due to our inability to strictly control the movement of our hand. Thus, classical randomness and probabilities are technical in nature rather than fundamental.

Therefore, classical physics is based on one tacitly assumed axiom, which we formulate here as an assertion.

**Assertion 1.2** (full determinism). *It is possible to prepare such ensembles (states) of the physical system where measurements of all observables produce the same result every time. In other words, we assume the existence of pure classical states representable by points in the phase space.*

Pure states are also representable in the language of probability distributions. They correspond to delta-like functions $\rho(x,p) = \delta(x-x_0)\delta(p-p_0)$ on the phase space. Then formula (1.4) confirms that in such states all experimental results are deterministic. A proposition $\mathcal{X}$ is either true ($(\phi|\mathcal{X}) = 1$), if the point $(x_0, p_0)$ of the phase space belongs to the subset $X$, or false ($(\phi|\mathcal{X}) = 0$) otherwise, without any intermediate possibility.

## 1.4 Logic of quantum mechanics

To clarify the basic ideas of quantum mechanics, let us return to the experiment with photons passing through one hole (see Subection 1.1.3). We found out that in the low intensity regime, when the photons are emitted one by one, the image on the screen (or photographic plate) consists of individual dots, which are randomly distributed sites of particle (photon) hits. This means that results of measuring the photon position are not reproducible, even if the state preparation conditions are controlled in the most careful way!

From this we conclude that the behavior of photons involves some *random* element. This is the most fundamental statement of quantum mechanics.[19]

**Statement 1.3** (fundamental randomness). *Measurements in the microworld have an element of randomness. This randomness is fundamental and cannot be explained or reduced* (*as we did in the classical case*) *to some inaccuracies in the preparation of initial states or experimental errors.*

By adopting this Statement, we conclude that classical Assertion 1.2 (complete determinism) is incorrect. In the pinhole camera setup, it is impossible to prepare such an ensemble of photons, in which all of them hit the same point on the screen. What is the reason for this scatter? Honestly, no one knows. This is one of the greatest mysteries of nature. Quantum theory does not even attempt to explain the physical causes of such a random behavior of microsystems. This theory takes randomness as a given and simply tries to find its mathematical description. To proceed, we have to move beyond the simple declaration of randomness and introduce more precise statements and definitions.

### 1.4.1 Partial determinism of quantum mechanics

We begin our construction of the formalism from the following postulate.

19 In Subection 1.5.2, we will see that this statement is, in fact, a consequence of the even more fundamental Postulate 1.6.

**Postulate 1.4** (connection between states and propositions). For each yes–no question there is an ensemble in which the answer "yes" is found with certainty (the probability = 1).

Indeed, it makes no sense to talk about an experimental proposition, if there is not a single ensemble in which this proposition can be unambiguously measured.

In Subsection 1.2.6, we identified observables with mappings from $\mathbb{R}$ into the set of yes–no questions. Thus, each value $f$ of the observable $F$ maps to a proposition $S_f$. According to Postulate 1.4, we can always prepare an ensemble in which this proposition is 100 % true.

**Statement 1.5** (partial determinism). *For each observable $F$ and each value $f$ from its spectrum, an ensemble can be prepared in which measurements of this observable are reproducible, i. e., repeatedly yield the same value $f$.*

Postulate 1.4 and Statement 1.5 are weakened versions of the classical Assertion 1.2. Instead of requiring the reproducibility of measurements for all observables and propositions at once, we limit this property to single observables.[20] Hence the quantum postulate is a softer requirement, and quantum mechanics is a more general theory than classical mechanics. Moreover, we expect the quantum theory to include classical mechanics as a special case.

So, in quantum mechanics we do not question the existence of propositions about one observable. This means that propositions represented by the strips $A$ and $B$ in Figure 1.4 continue to have well-defined meanings.[21] However, the quantum and classical approaches diverge when it comes to propositions involving more than one observable. For example, in quantum mechanics we cannot guarantee that the proposition corresponding to the rectangle $C = A \cap B$ in Figure 1.4[22] exists and can be realized in the form of an instrumental setup.

Heisenberg was the first to question the simultaneous measurability of certain pairs of observables. He gave the following heuristic arguments. Imagine that we want to accurately measure both the position and the momentum of an electron. For this, we have to look through the microscope. To see the electron, we have to illuminate it. For a more accurate determination of the position we should use light with a short wavelength. However, photons of this light have high energy (momentum). Colliding

20 Notice also that Statement 1.5 does not forbid the existence of certain groups of (*compatible*) observables, whose measurements can be reproducible within the same ensemble. For example, in Chapter ??, we will see that three components $(p_x, p_y, p_z)$ of the particle momentum are compatible observables. The same is true for three components $(r_x, r_y, r_z)$ of the particle position. However, the pairs $(p_x, x)$, $(p_y, y)$ and $(p_z, z)$ are incompatible.

21 That is, there are ensemble states in which these propositions are true.

22 This is a proposition about simultaneous measurement of both the position and the momentum of one particle.

with the electron under study, such photons will inevitably give it a kick, making it impossible to accurately determine the electron's velocity or momentum.

So, we suspect that for sufficiently narrow strips $A$ and $B$ in Figure 1.4 their intersection $C = A \cap B$ may be just "too small" to correspond to any real experimental proposition. Thus, there are no experimental propositions about points in the phase space. In other words, in nature there is no device that could realize the proposition "the particle's position is $x_0$ and the particle's momentum is $p_0$." This also means that the true lattice of propositions cannot coincide with the Boolean lattice of subsets in the phase space. What can we do? What lattice should we take to build the logic of questions in quantum physics?

Our plan for constructing quantum theory is as follows:

- First, we will establish the logic of propositions in our theory. We will call it *quantum logic*. As we saw above, we expect it to differ from the classical Boolean logic (= orthocomplemented distributive lattice).
- Next, we will formulate the representation theorem of Piron, which asserts that propositions of quantum logic can be represented by subspaces in a Hilbert space.
- From this result it will be easy to derive all basic properties of the quantum formalism: the superposition principle, the probability interpretation of wave functions, observables as Hermitian operators, etc.

### 1.4.2 Axioms of quantum logic from probability measures

Note that in our derivation of the axioms of classical logic we used the equivalence between logical propositions and subsets of the phase space. In the quantum case, this equivalence does not work, and we have to look for other ways. To implement the first point of our plan, we will show that many axioms from Table 1.2 can be derived even without reference to the phase space.[23] For these derivations we will need only the simplest properties of probability measures $(\phi|\mathcal{X})$.

Suppose that we have prepared two state ensembles $\phi$ and $\psi$ of our physical system and measured values of the probability measures $(\phi|\mathcal{X})$ and $(\psi|\mathcal{X})$ by going over all possible experimental propositions $\mathcal{X}$. If, as a result of this gigantic work, we find that $(\phi|\mathcal{X}) = (\psi|\mathcal{X})$ for all $\mathcal{X}$, then the states $\phi$ and $\psi$ will be regarded as equal ($\phi = \psi$). Indeed, there is no physical difference between these two states, where measurements give the same results (= probabilities).

For similar reasons, we will say that two propositions $\mathcal{X}$ and $\mathcal{Y}$ are equal ($\mathcal{X} = \mathcal{Y}$) if for all states $\phi$

$$(\phi|\mathcal{X}) = (\phi|\mathcal{Y}). \tag{1.5}$$

23 These axioms will be transferred without changes from the classical logic to the quantum one.

It then follows that the probability measure $(\phi|\mathcal{X})$, considered as a function on the set of all states $\mathfrak{S}$, is a unique representative of the proposition $\mathcal{X}$.[24] Hence, we can study properties of propositions by analyzing properties of probability measures $(\phi|\mathcal{X})$. For this, there is no need to deal with regions of the phase space, which is exactly what we want.

For example, we will say that $\mathcal{X} \leq \mathcal{Y}$ if measurements for all states $\phi \in \mathfrak{S}$ show that $(\phi|\mathcal{X}) \leq (\phi|\mathcal{Y})$. The relation $\mathcal{X} \leq \mathcal{Y}$ defines the *partial ordering* on the propositional system $\mathcal{L}$.

After the partial ordering $\leq$ is established on the entire set $\mathcal{L}$, it is not difficult to define the *meet* operation $\mathcal{X} \wedge \mathcal{Y}$ for all pairs $\mathcal{X}, \mathcal{Y}$. For example, if $\mathcal{X}$ and $\mathcal{Y}$ are given, we should be able to find a set of all propositions $\mathcal{Z}'$ that are “less than or equal to” both $\mathcal{X}$ and $\mathcal{Y}$,[25] $\mathcal{Z}' \leq \mathcal{X}$ and $\mathcal{Z}' \leq \mathcal{Y}$. It is reasonable to assume that there is a single maximal proposition $\mathcal{Z}$ in this set. We shall call it the *meet* of $\mathcal{X}$ and $\mathcal{Y}$: $\mathcal{Z} = \mathcal{X} \wedge \mathcal{Y}$.

The *join* $\mathcal{X} \vee \mathcal{Y}$ for all pairs $\mathcal{X}, \mathcal{Y}$ is defined in a similar way: it is the unique smallest proposition that is greater than or equal to both $\mathcal{X}$ and $\mathcal{Y}$. These definitions are formalized as properties 6–9 in Table 1.2. Properties 10–13 follow naturally from these definitions.

Further, suppose that for some pair of propositions $\mathcal{X}$, $\mathcal{Y}$ we notice that for all states $(\phi|\mathcal{X}) = 1 - (\phi|\mathcal{Y})$. Then we will say that the two propositions are orthocomplemented: $\mathcal{X} = \mathcal{Y}^{\perp}$ or, equivalently, $\mathcal{Y} = \mathcal{X}^{\perp}$.

Reasoning in this way, it is not difficult to derive (see Appendix ??) all axioms 1–17 of classical logic from Table 1.2, except for the axioms of distributivity 18 and 19. The latter two axioms cannot be justified using our approach with probability measures $(\phi|\mathcal{X})$. For this reason, we regard distributive laws as less valid and call them simply assertions.

So, in quantum mechanics, we are not allowed to use the distributive laws of logic. However, in order to obtain a nontrivial theory, it is necessary to find some kind of substitute for these two laws. On the one hand, this new postulate must be sufficiently specific, so that it can be used to develop a nontrivial logical and physical theory. On the other hand, it must be general enough and include distributive laws as a special case, because we want to have the classical theory as a limiting case of the quantum one.

So, how should we formulate this new quantum axiom?

### 1.4.3 Compatibility of propositions

To answer this question, let us turn to the important concept of compatibility. We will say that two propositions $\mathcal{X}$ and $\mathcal{Y}$ are *compatible* (denoted $\mathcal{X} \leftrightarrow \mathcal{Y}$) if

$$\mathcal{X} = (\mathcal{X} \wedge \mathcal{Y}) \vee (\mathcal{X} \wedge \mathcal{Y}^{\perp}) \tag{1.6}$$

24 That is, different propositions define different functions $(\phi|\mathcal{X})$ on the set of states $\mathfrak{S}$.

25 At least one such proposition $\emptyset$ always exists.

and

$$\mathcal{Y} = (\mathcal{X} \wedge \mathcal{Y}) \vee (\mathcal{X}^{\perp} \wedge \mathcal{Y}). \tag{1.7}$$

In Subection 1.5.3 we will see that two experimental propositions can be measured simultaneously if and only if they are compatible. Therefore, we should not be surprised by Theorem **??**, which states that the compatibility (= simultaneous measurability) of all propositions is a characteristic property of classical Boolean lattices.

### 1.4.4 Logic of quantum mechanics

From the Heisenberg microscope example it should be clear that, unlike in the classical case, in quantum mechanics not all propositions are measurable simultaneously (= compatible). Then, as the basic statement of quantum logic, we postulate that two propositions are definitely compatible if one follows from the other, and we leave it to mathematics to decide on the compatibility of other pairs.[26]

**Postulate 1.6** (orthomodularity). Propositions about physical systems obey the *orthomodular law*: If $\mathcal{B}$ follows from $\mathcal{A}$, then $\mathcal{A}$ and $\mathcal{B}$ are compatible, i. e.,

$$\mathcal{A} \leq \mathcal{B} \quad \Rightarrow \quad \mathcal{A} \leftrightarrow \mathcal{B}. \tag{1.8}$$

Orthocomplemented lattices[27] with the additional orthomodular Postulate 1.6 (property 20 in Table 1.2) are called *orthomodular lattices*. With the addition of technical conditions of atomicity and irreducibility, these lattices become the so-called *quantum logics*. The relationships between different types of lattices and logics are shown in Figure 1.7.

**Figure 1.7:** Relationships between different types of lattices and logics.

**26** The author does not know any deeper justification for this postulate. The strongest argument is that this postulate really works, i. e., it leads to the well-known mathematical structure of quantum mechanics, which has been tested extensively in experiments.

**27** That is, described by properties 1–17 in Table 1.2.

### 1.4.5 Quantum logic and Hilbert space

We have seen that in classical mechanics we do not have to use the exotic lattice theory. Instead, we can apply Theorem 1.1 and go over to the physically transparent language of the phase space. Is there a similar equivalence theorem in the quantum case? The answer is "yes." It is not difficult to notice close analogies between the quantum system of propositions described above and the algebra of projections on closed subspaces in a complex Hilbert space $\mathscr{H}$ (see Appendices **??** and **??**). In particular, if operations between projections (or subspaces) in the Hilbert space are translated into the language of lattice operations in accordance with Table 1.3,[28] then all axioms of quantum logic are easily verified.

**Table 1.3:** Translations of symbols between equivalent languages: (i) subspaces in the Hilbert space $\mathscr{H}$, (ii) projections on these subspaces and (iii) propositions in quantum logic $\mathcal{L}$.

| Subspaces in $\mathscr{H}$ | Projections in $\mathscr{H}$ | Propositions in $\mathcal{L}$ |
|---|---|---|
| $\mathscr{X} \subseteq \mathscr{Y}$ | $P_{\mathscr{X}} P_{\mathscr{Y}} = P_{\mathscr{Y}} P_{\mathscr{X}} = P_{\mathscr{X}}$ | $\mathcal{X} \leq \mathcal{Y}$ |
| $\mathscr{X} \cap \mathscr{Y}$ | $P_{\mathscr{X} \cap \mathscr{Y}}$ | $\mathcal{X} \wedge \mathcal{Y}$ |
| $\mathscr{X} \uplus \mathscr{Y}$ | $P_{\mathscr{X} \uplus \mathscr{Y}}$ | $\mathcal{X} \vee \mathcal{Y}$ |
| $\mathscr{X}'$ | $1 - P_{\mathscr{X}}$ | $\mathcal{X}^{\perp}$ |
| $\mathscr{X}$ and $\mathscr{Y}$ compatible | $[P_{\mathscr{X}}, P_{\mathscr{Y}}] = 0$ | $\mathcal{X} \leftrightarrow \mathcal{Y}$ |
| $\mathscr{X} \perp \mathscr{Y}$ | $P_{\mathscr{X}} P_{\mathscr{Y}} = P_{\mathscr{Y}} P_{\mathscr{X}} = 0$ | $\mathcal{X} \leq \mathcal{Y}^{\perp}$ |
| $\emptyset$ | 0 | $\emptyset$ |
| $\mathscr{H}$ | 1 | $\mathcal{I}$ |
| ray x | $\lvert x\rangle\langle x\rvert$ (1D projection) | $x$ is an atom |

For example, the violation of distributive laws follows from the fact that in $\mathscr{H}$ there are pairs of incompatible subspaces (see Appendix **??**). The validity of the orthomodular law in $\mathscr{H}$ is proved in Theorem **??**.

### 1.4.6 Piron's theorem

Thus, we have established that the set of subspaces in a complex Hilbert space $\mathscr{H}$ is indeed a representative of some quantum logic. Can we claim also the opposite, i. e., that for each quantum logic one can construct a representation by subspaces in some Hilbert space? The (positive) answer to this question is given by the famous *Piron*

28 We have denoted by $\mathscr{X} \uplus \mathscr{Y}$ the *linear span* of two subspaces $\mathscr{X}$ and $\mathscr{Y}$ (see Appendix **??**). Here $\mathscr{X} \cap \mathscr{Y}$ denotes the intersection of these subspaces and $\mathscr{X}'$ is the orthogonal complement of $\mathscr{X}$. In the Hilbert space *atoms* are one-dimensional subspaces. They are also referred to as *rays*.

*theorem* [**Piron2??Piron??**], which forms the basis of the mathematical formalism of quantum mechanics and is a quantum analog of the classical representation Theorem 1.1.

**Theorem 1.7** (Piron's theorem). *Each quantum logic $\mathcal{L}$ is isomorphic to the lattice of closed subspaces in a Hilbert space $\mathscr{H}$. The correspondences proposition $\leftrightarrow$ subspace are defined by the rules shown in Table* 1.3.

The proof of this theorem is beyond the scope of our book.[29]

### 1.4.7 Should we abandon classical logic?

So, we came to the paradoxical conclusion that classical logic and classical probability theory are not suitable for describing quantum microscopic systems. How could this be? After all, classical logic is the foundation of all mathematics and indeed of the whole scientific method! All proofs of mathematical theorems use the laws of Boolean logic, including the distributive laws that were discarded by us.[30] Even theorems of quantum mechanics are being proved in the framework of classical logic. Are we not entering into a contradiction when we claim that the true logic of experimental statements is not classical, but quantum [**Putnam??**]?

In everyday life, as in ordinary mathematics, we have the right to use inaccurate classical logic, because we usually deal with fixed objects that are not subject to quantum fluctuations. Theorems of Euclidean geometry speak of well-defined circles and triangles, not of statistical ensembles of figures with randomly distributed parameters. Therefore, in the proofs of such theorems, it is perfectly acceptable to use the laws of classical logic. However, when we go to the microworld, where results of measurements are subject to randomness and observables may be incompatible with each other, then we have to admit that classical distributive laws are no longer valid and that quantum logic should take over.

**29** Piron's theorem does not specify the nature of scalars in the Hilbert space. It leaves the possibility of choosing any *division ring with involutive antiautomorphism* as the collection of scalars in $\mathscr{H}$. We can substantially reduce this unwanted freedom if we recall the important role played by real numbers in physics (for example, the values of observables always lie in $\mathbb{R}$). Therefore, it makes sense to consider only those rings that include $\mathbb{R}$ as a subring. In 1877 Frobenius proved that there are only three such rings. These are *real numbers* $\mathbb{R}$, *complex numbers* $\mathbb{C}$ and *quaternions* $\mathbb{H}$. Although there is fairly extensive literature on the real and, especially, quaternionic quantum mechanics [**Stueckelberg??quaternionic??Moretti??Moretti2??**], the significance of these exotic theories for physics remains unclear. Therefore, in our book we will adhere to the standard quantum mechanics in complex Hilbert spaces.

**30** Note, however, attempts [**Dejonghe??**] to develop the so-called *quantum mathematics*, which is based on the laws of quantum logic.

The theory of orthomodular lattices is well known to mathematicians. In principle, we could make all constructions and calculations in quantum theory, based on this formalism. Such an approach would have certain advantages, since all its components would have a clear physical meaning: the elements $\mathcal{X}$ of the lattice are experimental propositions implementable in the laboratory and the probabilities $(\phi|\mathcal{X})$ can be measured directly in experiments. However, such a theory would encounter insurmountable difficulties, mainly because lattices are rather exotic mathematical objects; we lack experience and intuition to work with them. In addition, this approach would require us to abandon the familiar distributive laws of logic and thus would greatly complicate our reasoning.

Historically, the development of quantum theory took another route. Thanks to Piron's theorem,[31] the physically transparent but mathematically cumbersome lattices could be replaced by physically obscure but mathematically convenient Hilbert spaces, wave functions and Hermitian operators. In the next section we will briefly summarize this traditional formalism.

## 1.5 Physics in Hilbert space

In the previous section, we established a one-to-one correspondence between experimental propositions and subspaces of the Hilbert space. In this section, we will use this fact to construct the mathematical formalism of quantum mechanics. In particular, we will see that, in accordance with textbooks, observables are expressed by Hermitian operators in $\mathscr{H}$, and pure quantum states are unit length vectors in the same space.

### 1.5.1 Quantum observables

In Subsection 1.2.6, we saw that, in the language of logic, an observable $F$ is a mapping $\mathcal{F}$ associating a proposition in $\mathcal{L}$ (= a subspace in $\mathscr{H}$) with each point of the spectrum of $F$. The points $f$ in the spectrum of the observable $F$ are called *eigenvalues* of this observable. The subspace $\mathscr{F}_f \subseteq \mathscr{H}$ corresponding to the eigenvalue $f$ is called the *eigensubspace*, and the projection $P_f$ on this subspace is called the *spectral projection*. Each vector in the eigensubspace $\mathscr{F}_f$ will be called an *eigenvector* of the observable.

Let us consider two distinct eigenvalues $f$ and $g$ of one observable $F$. According to the definition from Subsection 1.2.6, the corresponding propositions $\mathcal{F}_f$ and $\mathcal{F}_g$ are disjoint, and their eigensubspaces are orthogonal. The linear span of the subspaces

31 Of course, the fathers of quantum mechanics did not know about this theorem, which was formulated only in the 1960s.

$\mathscr{F}_f$, where $f$ runs through the entire spectrum of the observable $F$, coincides with the entire Hilbert space $\mathscr{H}$. Consequently, spectral projections $P_f$ of any observable form a *resolution of the identity* (see Appendix ??). Thus, according to formula (??), we can associate an Hermitian operator

$$F = \sum_f f P_f \tag{1.9}$$

with each observable $F$. In the following, we often use the terms “observable” and “Hermitian operator” as synonyms.

### 1.5.2 States

As we know from Subsection 1.3.2, each state $\phi$ defines a probability measure $(\phi|\mathcal{X})$ on propositions in $\mathcal{L}$. In accordance with the quantum isomorphism “proposition” ↔ “subspace,” the state $\phi$ also defines a *probability measure* $(\phi|\mathscr{X})$ on subspaces $\mathscr{X}$ in the Hilbert space $\mathscr{H}$. This probability measure is a function that maps subspaces into the interval $[0,1] \subseteq \mathbb{R}$ and has the following properties:

– The probability corresponding to the entire space $\mathscr{H}$ is equal to 1 in all states,

$$(\phi|\mathscr{H}) = 1. \tag{1.10}$$

– The probability corresponding to the zero (empty) subspace is 0 in all states,

$$(\phi|\emptyset) = 0. \tag{1.11}$$

– The probability corresponding to the direct sum of orthogonal subspaces is the sum of probabilities for each subspace,[32]

$$(\phi|\mathscr{X} \oplus \mathscr{Y}) = (\phi|\mathscr{X}) + (\phi|\mathscr{Y}), \quad \text{if } \mathscr{X} \perp \mathscr{Y}. \tag{1.12}$$

The following important theorem [**Gleason??**] provides a classification of all such probability measures (= all possible states of a quantum system).

**Theorem 1.8** (Gleason's theorem). *If $(\phi|\mathscr{X})$ is a probability measure on subspaces in the Hilbert space $\mathscr{H}$ with properties (1.10)–(1.12), then there is a nonnegative[33] Hermitian operator $\hat{\rho}$ in $\mathscr{H}$ such that*

$$\mathrm{Tr}(\hat{\rho}) = 1 \tag{1.13}$$

**32** This is equivalent to the third Kolmogorov probability axiom (1.3). By the symbol ⊕ (direct sum) we denote the linear span (⊎) of two subspaces in the case where these subspaces are orthogonal.

**33** A Hermitian operator is called *nonnegative* if all its eigenvalues are greater than or equal to zero.

*and for any subspace $\mathscr{X}$ and its projection $P_{\mathscr{X}}$, the value of the probability measure is*[34]

$$(\phi|\mathscr{X}) = \mathrm{Tr}(P_{\mathscr{X}}\hat{\rho}). \tag{1.14}$$

The operator $\hat{\rho}$ is usually called the *density operator* or *density matrix*.

Proving Gleason's theorem is not easy, and we refer the curious reader to the original papers [**Gleason??Richman_Bridges??**]. Here we will only touch upon the physical interpretation of this result. First, in accordance with the spectral Theorem **??**, one can find an orthonormal basis $|e_i\rangle$ in $\mathscr{H}$ where the density operator $\hat{\rho}$ reduces to the diagonal form

$$\hat{\rho} = \sum_i \rho_i |e_i\rangle\langle e_i|, \tag{1.15}$$

where the eigenvalues $\rho_i$ have the properties

$$0 \le \rho_i \le 1, \tag{1.16}$$

$$\sum_i \rho_i = 1. \tag{1.17}$$

Among all states with properties (1.16)–(1.17), one can select those in which only one coefficient $\rho_i$ is nonzero and $\rho_j = 0$ for all other indices $j \neq i$. In this case the density operator reduces to the projection onto a one-dimensional subspace

$$\hat{\rho} = |e_i\rangle\langle e_i|. \tag{1.18}$$

Such states will be referred to as *pure quantum* states. For pure states, the formula (1.14) for calculating probabilities is simplified. Formally, using Lemma **??** and Theorem **??**, we find that the probability for the proposition $\mathcal{X}$ to be true in the state (1.18) is equal to the square of the modulus of the projection of $|e_i\rangle$ onto the subspace $\mathscr{X}$, i. e.,

$$(\phi|\mathcal{X}) = \mathrm{Tr}(P_{\mathscr{X}}|e_i\rangle\langle e_i|) = \mathrm{Tr}(\langle e_i|P_{\mathscr{X}}|e_i\rangle) = \langle e_i|P_{\mathscr{X}}P_{\mathscr{X}}|e_i\rangle = \|P_{\mathscr{X}}|e_i\rangle\|^2. \tag{1.19}$$

Therefore, it is customary to describe a pure state by a vector $|e_i\rangle$ of unit length chosen arbitrarily from the corresponding one-dimensional subspace (= ray).[35]

**34** Tr means the *trace* of the matrix of the operator $\hat{\rho}$, i. e., the sum of its diagonal elements; see Appendix **??**.

**35** Obviously, the vector $|e_i\rangle$ is defined only up to a *phase factor* $e^{i\alpha}$, which has a unit modulus ($|e^{i\alpha}| = 1$, where $\alpha \in \mathbb{R}$). Indeed, being substituted in (1.19), the vector $e^{i\alpha}|e_i\rangle$ leads to the same probability value,

$$\left\|P_{\mathscr{X}}\left(e^{i\alpha}|e_i\rangle\right)\right\|^2 = |e^{i\alpha}|^2\|P_{\mathscr{X}}|e_i\rangle\|^2 = \|P_{\mathscr{X}}|e_i\rangle\|^2,$$

so that both vectors $|e_i\rangle$ and $e^{i\alpha}|e_i\rangle$ are legitimate representatives of the state $\phi$.

In Subsection 1.5.1 we introduced the notion of an eigenvector of the observable $F$. Pure states corresponding to such eigenvectors will be called *eigenstates* of the observable $F$. Obviously, observables have definite values (= eigenvalues) in their eigenstates. This means that the eigenstates are precisely those states (= ensembles) whose existence was guaranteed by Statement 1.5.

Importantly, there are no quantum probability measures (= states) that give definite answers to all experimental questions. Thus, by assuming the orthomodularity of the propositional lattice (Postulate 1.6), we automatically explained the probabilistic nature of quantum states (Statement 1.3).

*Mixed* quantum states are expressed as mixtures (1.15) of pure states. The coefficients $\rho_i$ in this formula reflect the probabilities of the pure states in the mixture. Thus, in quantum mechanics there are two types of uncertainties. The first type is present in mixed states. This is the same uncertainty familiar to us from classical (statistical) physics. It appears in situations where the experimenter does not have complete control over the preparation of the system, for example, when he throws a die. The second type of uncertainty is present even in pure quantum states (1.18). It has no analog in classical physics, it cannot be gotten rid of by improved control of the initial conditions. This uncertainty reflects the unavoidable presence of chance in microscopic phenomena.

We will not discuss mixed quantum states in this book. Therefore, we will only deal with uncertainties of the second fundamental type. Hence, speaking of a quantum state $\phi$, we always have in mind a certain state vector $|\phi\rangle$, determined up to a phase factor $e^{i\alpha}$. In the following, we will use the terms "quantum state" and "state vector" as synonyms.

### 1.5.3 Complete sets of commuting observables

In Subsection 1.4.3 we defined the idea of compatible propositions, and in Lemma ?? we showed that the compatibility of propositions is equivalent to the commutativity of the corresponding projections. For physics, these properties are important because for a pair of compatible propositions (= projections, = subspaces), there are states in which both these propositions have certain values, i. e., they are simultaneously measurable without any statistical randomness. Similar claims can be made about two compatible (= commuting) Hermitian operators (= observables). In accordance with Theorem ??, a pair of such operators has a common basis of eigenvectors (= eigenstates). In these eigenstates both observables have definite (eigen)values.

We assume that for every physical system one can always find at least one minimal and complete set of mutually commuting observables $K, L, M, \ldots$.[36] Then we can

36 A set $K, L, M, \ldots$ is called *minimal* if not one observable from this set can be expressed as a function of other observables in the set. The set is *complete* if no new observable can be added to it without

construct an orthonormal basis $|e_i\rangle$ of common eigenvectors of these operators so that each such eigenvector is uniquely marked by eigenvalues $k_i, l_i, m_i, \dots$ of the operators $K, L, M, \dots$. That is, if $|e_i\rangle$ and $|e_j\rangle$ are two different basis vectors, then their sets of eigenvalues $\{k_i, l_i, m_i, \dots\}$ and $\{k_j, l_j, m_j, \dots\}$ are not the same.

### 1.5.4 Wave functions

Each state vector $|\phi\rangle$ can be represented as a linear combination of the basis vectors constructed in the previous subsection,

$$|\phi\rangle = \sum_i \phi_i |e_i\rangle, \tag{1.20}$$

where in the bra-ket notation (see Appendix ??) the coefficients are expressed as

$$\phi_i = \langle e_i|\phi\rangle. \tag{1.21}$$

The set of complex numbers $\phi_i$ can be considered as a function $\phi(k, l, m, \dots)$ on the common spectrum of the observables $K, L, M, \dots$. This is called the *wave function* of the state $|\phi\rangle$ in the *representation* defined by the observables $K, L, M, \dots$. We will discuss examples of one-particle wave functions in Sections ??–??.

### 1.5.5 Expectation values

Formula (1.9) defines a spectral resolution of the observable $F$, where index $f$ runs through all eigenvalues of $F$. The spectral projections $P_f$ can be expanded through basis eigenvectors, so we have

$$P_f \equiv \sum_{i=1}^{m} |e_i^f\rangle\langle e_i^f|. \tag{1.22}$$

Here $|e_i^f\rangle$ are orthogonal eigenvectors of the operator $F$ that are inside the eigensubspace $\mathscr{F}_f$, and $m$ is the dimension of this subspace.[37] Then from (1.19) one can find the probability for measuring $f$ in each pure state $\phi$,

$$(\phi|P_f) = \left\| \sum_{i=1}^{m} |e_i^f\rangle\langle e_i^f|\phi\rangle \right\|^2 = \sum_{i=1}^{m} |\langle e_i^f|\phi\rangle|^2. \tag{1.23}$$

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destroying the minimality property. An example of a complete set of mutually commuting observables for one massive particle is $\{M, P_x, P_y, P_z, S_z\}$, where $M$, $\boldsymbol{P}$ and $\boldsymbol{S}$ are the operators of mass, momentum and spin, respectively (see Section ??).

37 If $m > 1$, then the eigenvalue $f$ is called *degenerate*.

Sometimes we need to know the weighted average, or the *expectation value*, $\langle F \rangle$ of the observable $F$ in the state $|\phi\rangle$,

$$\langle F \rangle \equiv \sum_f (\phi|P_f) f.$$

Substituting here equation (1.23), we obtain

$$\langle F \rangle = \sum_{j=1}^{n} |\langle e_j|\phi\rangle|^2 f_j \equiv \sum_{j=1}^{n} |\phi_j|^2 f_j,$$

where the summation is carried out over the entire basis of eigenvectors $|e_j\rangle$. From expansions (1.20), (1.9) and (1.22) it follows that the combination $\langle\phi|F|\phi\rangle$ is a more compact notation for the expectation value $\langle F \rangle$. Indeed

$$\begin{aligned}
\langle\phi|F|\phi\rangle &= \Big(\sum_i \phi_i^* \langle e_i|\Big)\Big(\sum_j |e_j\rangle f_j \langle e_j|\Big)\Big(\sum_k \phi_k |e_k\rangle\Big) \\
&= \sum_{ijk} \phi_i^* f_j \phi_k \langle e_i|e_j\rangle\langle e_j|e_k\rangle = \sum_{ijk} \phi_i^* f_j \phi_k \delta_{ij}\delta_{jk} \\
&= \sum_j |\phi_j|^2 f_j = \langle F \rangle. \qquad (1.24)
\end{aligned}$$

### 1.5.6 Basic rules of classical and quantum mechanics

The results obtained in this chapter can be summarized as follows. If the physical system is prepared in a pure state $\phi$ and we want to calculate the probability $\omega$ to measure the observable $F$ within the interval $E \subseteq \mathbb{R}$, then we need to perform the following steps.

In classical mechanics:
(1) Determine the phase space $\mathcal{S}$ of the physical system.
(2) Find the real function $f : \mathcal{S} \to \mathbb{R}$ corresponding to our observable $F$.
(3) Find the subset $U \subseteq \mathcal{S}$ corresponding to the spectral interval $E$, where $U$ is the collection of all points $s \in \mathcal{S}$ such that $f(s) \in E$ (see Figure 1.6).
(4) Find the point $s_\phi \in \mathcal{S}$ representing the pure classical state $\phi$.
(5) The probability $\omega$ is 1 if $s_\phi \in U$ and $\omega = 0$ otherwise.

In quantum mechanics:
(1) Determine the Hilbert space $\mathscr{H}$ of the physical system.
(2) Find the Hermitian operator $F$ corresponding to our observable in $\mathscr{H}$.
(3) Find the eigenvalues and eigenvectors of $F$.
(4) Find the spectral projection $P_E$ corresponding to the spectral interval $E$.

(5) Find the unit vector $|\phi\rangle$ (defined up to a phase factor) representing the state $\phi$ in the Hilbert space $\mathscr{H}$.
(6) Substitute all these ingredients in the probaility formula $\omega = \langle\phi|P_E|\phi\rangle$.

At the moment, the classical and quantum recipes seem completely unrelated to each other. Nevertheless, we are sure that such a connection must exist, because we know that both these theories are variants of the probability formalism on orthomodular lattices. In Section ??, we will see that in the macroscopic world with massive objects and poor resolution of measuring devices, the classical recipe appears as a reasonable approximation to the quantum one.

## 1.6 Interpretations of quantum mechanics

So far in this chapter, we were occupied with the mathematical formalism of quantum mechanics. Many details of this formalism (wave functions, superpositions of states, Hermitian operators, nonstandard logic, etc.) seem very abstract and detached from reality. This situation has generated a lot of debates about the physical meaning and interpretation of quantum laws. In this section, we will suggest our point of view on these controversies.

### 1.6.1 Quantum nonpredictability

Experiments with quantum microsystems revealed one simple but nonetheless mysterious fact: if we prepare $N$ absolutely identical physical systems under the same conditions and measure the value of the same physical quantity, we can obtain $N$ different results.

Let us illustrate this statement with two examples. From experience we know that each photon passing through the aperture of the pinhole camera will hit some point on the photographic plate. However, the next photon, most likely, will hit another point. And, in general, the locations of hits are randomly distributed over the surface. Quantum mechanics does not even try to predict the fate of each individual photon. It only knows how to calculate the probability density for the points of impact, but the behavior of each individual photon remains completely random and unpredictable.

Another example of this – obviously random – behavior is the decay of radioactive nuclei. The $^{232}$Th nucleus has a half-life of 14 billion years. This means that in any sample containing thorium, approximately half of all $^{232}$Th nuclei will decay during this period. In principle, quantum physicists can calculate the decay probability of a nucleus by solving the corresponding Schrödinger equation.[38] However, they cannot even approximately guess when the given nucleus decays. It can happen today or in 100 billion years.

It would be wrong to think that the probabilistic nature of microscopic systems has little effect on our macroscopic world. Very often the effects of random quantum processes can be amplified and lead to macroscopic phenomena, which are equally random. One well-known example of such amplification is the thought experiment with “Schrödinger’s cat” [**cat??**].

**38** Although our current knowledge of the nature of nuclear forces is completely inadequate to perform this kind of calculation for thorium.

### 1.6.2 Collapse of wave function

In the *orthodox interpretation* of quantum mechanics, the behavior described above is called the "collapse of the quantum probability distribution" and is often surrounded with a certain aura of mystery.[39] In this interpretation, the most controversial point of quantum mechanics is its different attitude to the physical system and the measuring device. The system is regarded as a quantum object that can exist in strange superpositions,[40] while the measuring device is a classical object whose state (readout) is fully unambiguous. It is believed that at the time of measurement, an uncontrolled interaction between the system and the measuring device occurs, so that the superposition collapses into one of its components, which is recorded by the instrument. Inside the theory, this difference of attitudes is expressed in the fact that the system is described by a wave function, but the measuring device is described by an Hermitian operator. This leads to a number of unpleasant questions.

Indeed, the measuring device consists of the same atoms as the physical system and the rest of the universe. Therefore, it is rather strange when such devices are put into a separate category of objects. But if we decided to combine the device and the system into one wave function, when would it collapse? Maybe this collapse would require the participation of a conscious observer? Does this mean that by making observations, we control the course of physical processes?

Sometimes a mystery is seen in the fact that the quantum-mechanical probability distribution (= wave function) has two mutually exclusive laws of evolution. While we are not looking at the system, this distribution develops smoothly and predictably (in accordance with the Schrödinger equation), and at the time of measurement it experiences a random unpredictable collapse.

### 1.6.3 Collapse of classical probability distribution

By itself, the collapse of the probability distribution is not something strange. A similar collapse occurs in the classical world as well. For example, when shooting from a rifle at a target, it is almost impossible to predict the hit location of each specific bullet. Therefore, the state of the bullet before it hits the target (= before the measurement) is conveniently described by the probability distribution. At the moment of the

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**39** To emphasize the analogy with the classical case, here we specifically talk about the collapse of the "probability distribution," and not about the collapse of the "wave function," as in other works. It is precisely the probability distribution that is subject to experimental observation, and the wave function is a purely theoretical concept.

**40** The electron in the previous example is allegedly in a superposition of states smeared over the surface of the photographic plate, and the thorium nucleus is in a superposition of the decayed and undecayed states.

hit, the bullet punches the target at a specific place, and the probability is immediately replaced by certainty. The measurement leads to the "collapse of the probability distribution," exactly as in the formalism of quantum mechanics.

The probability density for the bullet changes smoothly (spreads out) from the moment of the shot and up to the time of impact. The unpredictable collapse of this probability distribution occurs instantaneously in the entire space. These behaviors are completely analogous to the two (allegedly contradictory) variants of quantum evolution, but the classical collapse does not raise any controversy among theorists and philosophers.

We rightly believe that the collapse of classical probability is the natural behavior of any probability distribution. Then, why does the collapse of quantum probability still trouble theoreticians?

The fact is that in the case of the bullet and the target, we are sure that the bullet was *somewhere* at each time instant, even when we did not see it. In all these moments the bullet had a definite position, momentum, rotation speed about its axis (spin) and other properties. Our description of the bullet had some element of randomness only because of our laziness, unwillingness or inability to completely control the act of shooting. By describing the state of the bullet by a probability distribution, we simply admitted the level of our ignorance. When we looked at the pierced target and thus "collapsed" the probability distribution, we had absolutely no influence on the state of the bullet, but simply improved (updated) our knowledge about it. The probability distribution and its collapse are things that occur exclusively in our heads and do not have actual physical existence.

### 1.6.4 Hidden variables

Einstein believed that the same logic should be applied to measurements in the microworld. He wrote:

> I think that a particle must have a separate reality independent of the measurements. That is an electron has spin, location and so forth even when it is not being measured. I like to think that the moon is there even if I am not looking at it.

If we follow this logic blindly, we must admit that even at the microscopic level, nature must be regular and deterministic. Then the observed randomness of quantum processes should be explained by some yet unknown "hidden" variables that cannot be observed and controlled *yet*. If we exaggerate somewhat, the theory of hidden variables reduces to the idea that each electron has a navigation system that directs it to the designated point on the photographic plate. Each nucleus has an alarm clock inside it, and the nucleus decays at the call of this alarm clock. The behavior of quantum systems only seems random to us, since we have not yet penetrated the designs of these “navigation systems” and “alarm clocks”.

According to the theory of “hidden variables,” the randomness in the microworld has no special quantum-mechanical nature. This is the same classic pseudo-randomness that we see when shooting at a target or throwing dice. Then we have to admit that modern quantum mechanics is not the last word. Future theory will teach us how to fully describe the properties of individual systems and to predict events without reference to the quantum chance. Of course, such faith cannot be refuted, but so far no one has succeeded in constructing a convincing theory of hidden variables predicting (at least approximately, but beyond the limits of quantum probabilities) the results of microscopic measurements.

### 1.6.5 Quantum-logical interpretation

The most famous thought experiment in quantum mechanics is the two-hole interference, which demonstrates the limits of classical probability theory. Recall that in this experiment (see Section 1.1 and Subsection **??**) we did not have the right to add the probabilities for passing through alternative holes. Instead, quantum mechanics recommended adding the so-called probability amplitudes and then squaring the resulting sum [**Feynman-lecturesIII??**].

This observation leads to the suspicion that the usual postulates of probability (and logic) do not operate in microsystems. Thus, we naturally approach the idea of quantum logic as the basis of quantum mechanics. It turns out that both fundamental features of quantum measurements – the randomness of outcomes and the addition of probability amplitudes for alternative events – find a simple and concise explanation in quantum logic (see Section 1.4). Both these laws of quantum mechanics follow directly from the orthomodular logic of experimental propositions. As we know from Piron’s theorem, such logic is realized by a system of projections in the Hilbert space, and by Gleason’s theorem any state (= probability measure) on such a system must be stochastic, random.

As we saw in Section 1.4 (see Figure 1.7), the Boolean deterministic logic of classical mechanics is only a particular case of the orthomodular quantum logic with its probabilities. Thus, even in formal reasoning, it is the *particular* classical theory that needs a special explanation and interpretation, and not the *general* quantum mechanics.

> . . . classical mechanics is loaded with metaphysical hypotheses which clearly exceed our everyday experience. Since quantum mechanics is based on strongly relaxed hypotheses of this kind, classical mechanics is less intuitive and less plausible than quantum mechanics. Hence classical mechanics, its language and its logic cannot be the basis of an adequate interpretation of quantum mechanics – P. Mittelstaedt [**Mittelstaedt??**].

### 1.6.6 Quantum randomness and limits of knowledge

So, we came to the conclusion that quantum probability, its collapse and the existence of superpositions of states are inevitable consequences of the special orthomodular nature of the logic of experimental propositions. The laws of probability, built on this logic, differ from the classical laws of probability that are familiar to us. In particular, any state (= a probability measure on logical propositions) must be stochastic, i. e., it is impossible to get rid of the element of chance in measurements. This also means that there is no mystery in the collapse of the wave function, and there is no need to introduce an artificial boundary between the physical system and the measuring apparatus.

The imaginary paradox of the quantum formalism is connected, on the one hand, with the weirdness of quantum logic, and on the other hand with unrealistic expectations about the power of science. Theoretical physicists experience an internal protest when faced with real physically measurable effects,[41] which they are powerless to control and/or predict. These are facts without explanations, effects without causes. It seems that microparticles are subject to some annoying mysterious random force. But in our view, instead of grieving, physicists should have celebrated their success.

To us, the idea of the fundamental, irreducible and fundamentally inexplicable nature of quantum probabilities seems very attractive, because it may signal the fulfillment of the centuries-old dream of scientists searching for deep laws of nature. Perhaps, in such an elegant way, nature has evaded the need to answer our endless questions "why?" Indeed, if at the fundamental level nature were deterministic, then we would face a terrifying prospect of unraveling the endless sequences of cause–effect relationships. Each phenomenon would have its own cause, which, in turn, would have a deeper reason, and so on, *ad infinitum*. Quantum mechanics breaks this chain and at some point gives us the full right to answer: "I don't know. It's just an accident." And if some phenomenon is truly random, then there is no need to seek an explanation for it. The chain of questions "why?" breaks. The quest for understanding ends in a logical, natural and satisfying way.

So, perhaps, the apparent "incompleteness" of quantum theory is not a problem to be solved, but an accurate reflection of the fundamental essence of nature, in particular, its inherent unpredictability? In this connection, the following quote from Einstein seems suitable:

> I now imagine a quantum theoretician who may even admit that the quantum-theoretical description refers to ensembles of systems and not to individual systems, but who, nevertheless, clings to the idea that the type of description of the statistical quantum theory will, in its essential features, be retained in the future. He may argue as follows: True, I admit that the quantum-theoretical description is an incomplete description of the individual system. I even admit that

**41** Such as random hits of electrons on the screen or decay of nuclei.

> a complete theoretical description is, in principle, thinkable. But I consider it proven that the search for such a complete description would be aimless. For the lawfulness of nature is thus constructed that the laws can be completely and suitably formulated within the framework of our incomplete description. To this I can only reply as follows: Your point of view – taken as theoretical possibility – is incontestable – A. Einstein [**incontestable??**].

The most important philosophical lesson of quantum mechanics is the call to abandon speculations about unobservable things and their use in the foundations of the theory. Quantum mechanics does not know whether the moon is there or not. Quantum mechanics says that the moon will be there when we look.