1 Abstract

We compare the asymptotic behavior of various exponential and factorial/gamma functions. In particular, for parameter $a \geq 1$, we study how $\Gamma(ax+1)$ grows as $x \to \infty$. As part of this study, we study the function $f(x) = \frac{x^x}{\Gamma(ax+1)}$ for various values of a, proving that $\lim_{x\to\infty} f(x) = 0$ in two different ways. We also examine the behavior of $f(x)$ near its critical point as $a \to 1$, to include applying asymptotics to identify the location of that critical point.

2 Summary

We explore the growth rate of $\Gamma(ax+1)$, where a is a parameter greater than 1, compared to other functions a^x, $(ax)^x$, x^{ax}, x^x, $\Gamma(x)$, and a^{ax}. When $a = 1$, the gamma function (and the factorial function) follows a recursive formula which can be exploited in proofs or calculations. ($\Gamma(x+1) = x \cdot \Gamma(x)$). Without knowing what a is, we lose this recursive relation, and so it becomes challenging to compare the modified gamma function with other functions or for substitutions. This paper demonstrates an alternative route in showing how the growth rate of $\Gamma(ax+1)$ exceeds that of x^x, and further examines the relationships between a^x, $(ax)^x$, x^{ax}, $\Gamma(x)$, and a^{ax}.

3 Introduction

3.1 Background

$\Gamma(n+1)$ grows faster than a^n as $n \to \infty$ and slower than n^n for any parameter a. Formally, this can be described using Big O notation: If $\frac{a^n}{n^n} = 0$ then $a^n = O(n^n)$.

These can be proven using the ratio test and treating the functions as sequences. Because the gamma function is equal to the factorial function for all integers, we can substitute the factorial function when using the ratio test. Proving that the sequence of a function converges via the ratio test implies that the function approaches 0 as $n \to \infty$. In the case that our function can be expressed as a fraction, this also means that the denominator grows faster than the numerator. Therefore to demonstrate the growth rates, we use the ratio test on a function where the growth rate of the numerator is believed to be smaller than that of the denominator.

3.2 $(an)^n$ vs $\Gamma(n+1)$

Depending on what a is, either function can have the highest growth rate. We use the ratio test on $\frac{(an)^n}{n!}$.

$$\lim_{x\to\infty} \frac{(an+a)^{n+1}}{(n+1)\cdot n!} \cdot \frac{n!}{(an)^n} = \frac{(an+a)^n \cdot a(n+1)}{(an)^n \cdot (n+1)} = \frac{(an+a)^n \cdot a}{(an)^n} =$$

$$\frac{a^n \cdot (n+1)^n \cdot a}{(a^n) \cdot n^n} = \frac{(n+1)^n \cdot a}{n^n} = a \cdot e$$

If a is less than $\frac{1}{e}$, then the series converges and $n!$ exceeds $\frac{1}{e}$. When a is larger than $\frac{1}{e}$, the series diverges and we cannot determine whether the sequence diverges. We use the ratio test on its reciprocal:

$$\lim_{x\to\infty} \frac{(n+1)n!}{(an+a)^{n+1}} \cdot \frac{(an)^n}{n!} = \frac{(an)^n \cdot (n+1)}{(an+a)^n \cdot a(n+1)} = \frac{(an)^n}{(an+a)^n \cdot a}$$

$$= \frac{(a^n) \cdot n^n}{a^n \cdot (n+1)^n \cdot a} = \frac{n^n}{(n+1)^n \cdot a} = a \cdot \frac{1}{e}$$

.

$(an)^n$ grows faster. When $a = \frac{1}{e}$, the ratio test is inconclusive, so we must try something else.

If we were to use Stirling's approximation, we can show that the limit of the function $\frac{\frac{n}{e}^n}{n!} = \frac{n^n}{e^n * n!} = 0$. Observe $\frac{n^n}{e^n * \sqrt{2\pi n}\frac{n}{e}^n} = \frac{n^n}{\sqrt{2\pi n} n^n} = \frac{1}{\sqrt{2\pi n}}$ The limit of this as n approaches infinity is 0. Therefore, when $a = \frac{1}{e}$, the function $n!$ grows faster than $(an)^n$. We assume that the behavior of the limit using Stirling's approximation matches that of the gamma function without going into the proofs.

3.3 n^{an} vs $(an)^n$

Observe that $\frac{n^{an}}{(an)^n} = (\frac{n^a}{(an)})^n = (\frac{n^{a-1}}{a})^n$. The limit of this as n approaches infinity is infinity. n^{an} grows faster than $(an)^n$. The intuition here is that the exponent is weighted more than the base.

3.4 $\Gamma(n+1)$ vs a^n

To prove that $\Gamma(n+1)$ exceeds a^n, we use the ratio test on $\frac{a^n}{n!}$. We have

$$\lim_{n\to\infty} \frac{a \cdot a^n}{(an+1) \cdot n!} \cdot \frac{n!}{a^n} = \frac{a}{n+1}$$

. The $\lim_{n\to\infty} \frac{a}{n+1} = 0$ shows the convergence of this series and implies the denominator grows faster than the numerator.

3.5 n^n vs $\Gamma(n+1)$

To prove that n^n exceeds $\Gamma(n+1)$, we use the ratio test on

$$\frac{n!}{n^n}$$

$$\lim_{n\to\infty} \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!} = \lim_{n\to\infty} \frac{(n+1) \cdot n!}{(n+1)^n \cdot (n+1)} \cdot \frac{n^n}{n!} = \lim_{n\to\infty} \frac{n^n}{(n+1)^n}$$

Since the limit is less than 1, the function converges, and the denominator must exceed the numerator without bound.

3.6 Alternate definition of the Euler-Mascheroni constant

Let us study the $\lim_{n\to 0}\Gamma(n+1)^{\frac{1}{n}}$. Keep in mind this will be the same limit as $n!^{\frac{1}{n}}$

$$\Gamma(n+1)^{\frac{1}{n}} = e^{\ln(\Gamma(n+1)^{\frac{1}{n}})} = e^{\frac{\ln(\Gamma(n+1))}{n}}$$

.

Using L' Hospital's on the exponent gives us

$$\frac{\ln(\Gamma'(n+1))}{1}$$

. This is the digamma function, the logarithmic derivative of the gamma function, which when evaluated at 1 (that is, when $n = 0$), gives the negative of the euler-mascheroni constant (an already established definition): $\frac{\ln(\Gamma'(0+1))}{1} = \frac{\ln(\Gamma'(1))}{1}$. Therefore, the original function can be expressed as $e^{-\gamma}$.Taking the natural log of $e^{-\gamma}$ gives us $\ln(e^{-\gamma}) = -\gamma$. Finally, multiplying it by -1 gives us the euler-mascheroni constant.

Why is definition aesthetic you may ask. Imagine the function $(n^n)^{1/n}$, which can be simplified to having the same growth rate as n. In this function, the growth rate of the inner function can be seen as negated by the growth rate of the outer function. But if we replace the exponential function x^x with the gamma function, the function still approaches infinity, but it must be at a slower rate, since $n!$ does not grow as fast as n^n. I believe that the euler-mascheroni constant somehow characterizes this difference in growth rates. Evaluating the limit at 0 instead of infinity maybe gives us the minimum difference in growth rates? I am not sure, this part is me trying to come up with a intuitive grasp without refining the logic first.

We can go further and observe the limit with a modifier on the gamma function. We can apply the same method to $\lim_{n\to 0}\Gamma(an)^{\frac{1}{n}}$. The steps are exactly the same, until we take the derivative of $\ln\Gamma(an)$, where we use the chain rule to get

$$\frac{a * \Gamma'(an+1)}{1}$$

. Regardless of what a is,

$$\ln\Gamma'(a*0+1) = \ln(\Gamma'(1))$$

This limit is $e^{-a\gamma}$. Taking the natural log of $e^{-a\gamma}$ gives us

$$\ln(e^{-a\gamma}) = -a\gamma$$

Notice how the a modifier can increase the growth rate beyond x^x if a is large enough. I suspect the product $a * /gamma$ must exceed 1 if the growth rate of $gamma(ax+1)^{1/n}$ is to exceed that of $(x^x)^{1/n}$. Not sure. Hoping for other people's insight.

4 $\Gamma(an+1)$ vs n^n: Conjecture

We show that

$$\lim_{x\to\infty} \frac{x^x}{\Gamma(ax+1)} = 0$$

We can construct an inductive proof in the field of integers to show that if a is 2, there exists an x where $\gamma(ax+1) > x^x$, and consequently show how if a is any integer larger than or equal to 2, the new factorial function exceeds x^x. Now suppose a was any real number larger than 1. Even without understanding the gamma function, which extends the factorial function to all real numbers, we can illustrate the pattern of the factorial function depending on what a is. If a is 1.5, we let x equal a multiple of 2 so that we stay in the realm of integers. If $a = 1.25$, then x must be a multiple of 4. Observe that $(ax)!$ is equal to $(x+c)!$, where c is equal to $x \cdot (a-1)$ with $2 > a >= 1$, and we can increase x in intervals of $\frac{1}{a-1}$ to have an idea of how the factorial function increases based on its coefficient.

4.1 example with $a = 1.5$

x	c	(x+c)!	(ax)!
0	0	0!	0!
2	1	(2+1)!	3!
4	2	(4+2)!	6!

5 Representing the problem

Let $\Gamma(ax+1)$ equal a number x^y where y is any real number, and a function of x. Then, if $y > x$,

$$\lim_{x\to\infty} \frac{x^x}{\Gamma(ax+1)} = 0$$

holds, because

$$\lim_{x\to\infty} \frac{x^x}{x^y} = \lim_{x\to\infty} (x)^{(x-y)} = 0$$

If we can prove that at some x, y exceeds x, then we prove the limit. Our next step is to represent y in terms of x:

$$\frac{\ln(\Gamma(ax+1)}{\ln(x)} = y$$

. We define an equality to compare the limit of x and the limit of y as x approaches infinity, and we divide by x, to get what we want to prove, which is that there exists an x where $\frac{y}{x}$ exceeds 1:

$$\frac{\ln((ax)!)}{\ln(x)} > x = \frac{\ln(\Gamma(ax+1))}{\ln(x)\cdot x} > 1$$

6 Proving the limit using the squeeze theorem

From 3, we understand that we can prove the theorem by showing that if a is greater than 1, then the limit as x approaches infinity of $\frac{\ln(\Gamma(ax+1))}{\ln(x)\cdot x}$ is also larger than 1, and we hypothesize that this limit and a are equal. It is important to keep in mind that, from what we currently know, it is possible that a does not equal the limit, and for the theorem to be true - for instance, if the limit was $a+.01$ or $a-\frac{a-1}{10}$. It is sufficient that $a>1$ while $\lim_{x\to\infty}\frac{\ln(\gamma(ax+1))}{\ln(x)*x}>1$. We choose to show the limit to be equal to a for simplicity.

The two functions which we want to find for the squeeze theorem should approach a. Let $f(x)$ equal $\frac{\ln(\Gamma(ax+1))}{\ln(x)\cdot x}$ and let $d(x)$ equal $\frac{\ln(((ax)^{ax})}{ln(x)\cdot x}$. We need to find a lower bound function $g(x)$ that approaches a. It is found that this lower bound is $\frac{\ln((bx)^{ax})}{ln(x)\cdot x}$, where b is any coefficient smaller than the inverse of $\frac{1}{e}$. $d(x)$ approaches a, because $\frac{\ln(((ax)^{ax})}{\ln(x)\cdot x}=\frac{axln(ax)}{ln(x)\cdot x}=\frac{a\cdot\ln(ax)}{\ln(x)}$, and since $\frac{\ln(ax)}{ln(x)}=\frac{\ln(a)+\ln(x)}{\ln(x)}$ approaches 1, the function approaches a. The limit of $g(x)$ is calculated the same way. To use the squeeze theorem to prove the limit, we are not restricted in our domain of x, since x approaches infinity. In other words, we are only required to show the existence of an x where, beyond that number, $d(x)$ will always be larger than $f(x)$ and $g(x)$ will be smaller than $f(x)$. For the squeeze theorem, when we compare the functions $d(x)$, $f(x)$, and $g(x)$, we will only compare the the sub-functions in the numerator, which are respectively, $(bx)^{ax}, (ax)!$ and $(ax)^{ax}$, since the denominators are the same when x is in the field of positive numbers and when $a>=1$, which are constraints we already have.

7 The upper bound

For the rest of this proof, a is any real number greater than or equal to 1. This restriction on a will not interfere with our theorem and exempts us from unneeded comparisons, since we know when $a<=1$, the limit stated in the theorem does not approach 0. $g(x)$ can be shown larger than or equal to $f(x)$ for all values of x by showing that $(ax)^{(ax)}$ exceeds $(ax)!$ at some x. If we just consider the gamma function in the realm of integers, this is easy to show through inspection, since $(ax)^{ax}$ will have at least the same number of factors + 1 than $\gamma(ax)$, but each of those factors will exceed or equal each factor of $\gamma(ax)$, similar to how $x!$ is always less than x^x. Given an a, we can find an x that would allow us to find an integer representation of $(ax)!$ and $(ax)^{ax}$, and proving that the sequence of $(ax)^{ax}$ exceeds $(ax)!$ also proves the same relationship of the functions' limits, if the functions are continuous. We can also prove it more straightforwardly by using the ratio test on $\frac{\Gamma(ax+1)}{(ax)^{ax}}$. We let $t=a*x$, and the result is $\frac{t^t}{(t+1)^t}$. The limit of this approaches $\frac{1}{e}$, and the convergence of the series shows that the sequence approaches 0, which shows that $(ax)^{ax}$ exceeds $\Gamma(ax+1)$. The sequence approaches a as x approaches infinity-the value of the limit does not change if we change the factorial function with the gamma function.

8 The lower bound

Again, we will compare $d(x)$ and $f(x)$ by showing that $\Gamma(ax)$ is greater than or equal to $(bx)^{ax}$ if b is less than the inverse of e ($\approx .37$), at some number x and beyond. First, we will prove this if a is equal to 1, and then we will show how increasing a further decreases the ratio of $\frac{(bx)^{ax}}{\gamma(ax)}$ To show that

$$\gamma(x)$$

exceeds $(bx)^{ax}$ at some x, we show that

$$\sum_{n=1}^{\infty} \frac{(bn)^{an}}{(an)!}$$

converges. If we can prove the series converges, then we prove the sequence of the function converges (not only the sequence of sums but the sequence itself), and proving the sequence of the function converges shows that the limit of that function approaches 0. If the limit of the ratio approaches 0, it follows that the function in the denominator exceeds the function in the numerator at a given point, satisfying the squeeze theorem.

We start by showing that the series $\sum_{n=1}^{\infty} \frac{(bn))^{an}}{\Gamma(an)}$ converges with a equal to 1 and b equal to some constant, using the ratio test:

$$\lim_{x\to\infty} \frac{(bx+b)^{x+1} * x!}{\Gamma(x+1) * (bx)^x} = \lim_{x\to\infty} \frac{(bx+b)^x * (bx+b) * \Gamma(x)}{(x+1) * \Gamma(x) * (bx)^x} = \lim_{x\to\infty} \frac{(bx+b)^x * b}{(bx)^x}$$

$$= \lim_{x\to\infty} \frac{(x+1)^x * b^x * b}{x^x * b^x} = \lim_{x\to\infty} \frac{(x+1)^x * b}{x^x} = b * e$$

Looking at this closely, we see that the limit calculated using the ratio test is a constant times e. Therefore, for the series to converge, b must be a number less than $\frac{1}{e}$. We are successful in showing that $(bx)^{ax}$ is less than $\Gamma(ax)$ for some number x and $b < \frac{1}{e}$, but only if $a = 1$. We suspect that increasing a should make it converge even faster. $\Gamma(x) > (bx)^x$ implies $\ln(\Gamma(x)) > x * \ln(bx)$ which implies $a * \ln(\Gamma(x)) > ax * \ln(bx)$ or else the negation of our theorem would be true (if a is less than 1, then the limit equals 0 should not be a true statement), and we verify this by letting $x = \frac{u}{a}$ and making the limit a function of u. When a is equal to 1, the limit is identical to the original:

$$\frac{(bu)^u}{u!}$$

In this case, a is a hidden variable - this function has the same exact properties as the original. If we were to increase a, notice that only one of the u's changesthe one being multiplied by b. It decreases, as it is being divided by a number larger than 1. The other 2 remain as u since we have $\frac{a \cdot u}{a}$. We have established that $\frac{(bu)^u}{u!}$ when $a = 1$, grows faster than $\frac{(bu)^u}{u!}$ when $a > 1$. By comparison, we have established $(bx)^{ax}$ with a greater than or equal to 1 and b less than $\frac{1}{e}$, as the lower bound, completing our use of the squeeze theorem.

9 comparisons of x^{ax} and $(ax)^x$

To determine which function grows faster, observe the function

$$\frac{x^{ax}}{(ax)^x} = (\frac{x^a}{ax})^x$$

$$(\frac{x^{(a-1)}}{a})^x$$

.

The $\lim_{x\to\infty} (\frac{x^{(a-1)}}{a})^x = \infty$, so x^{ax} exceeds $(ax)^x$.

10 Proof by Stirling's approximation

We earlier determined that solving the limit $\frac{\ln(\gamma(ax))}{x * \ln(x)}$, is sufficient for proving the theorem.

Stirling's approximation would have $\Gamma(ax) = \sqrt{2\pi ax}\frac{ax}{e}^{ax}$

We use l' Hospitals rule. Take the derivative of the numerator and denominator twice.

$\ln((ax)!) = \ln\sqrt{2\pi ax}\frac{ax}{e}^{ax} = 0.5\ln(2\pi a) + 0.5\ln x + ax\ln a + ax\ln x - ax\ln e$

Then $\frac{\ln((ax)!)}{x * \ln x} = \frac{0.5\ln(2\pi a)+0.5\ln x + ax\ln a + ax\ln x - ax}{x * \ln x}$

Calculate the 1st derivative of the numerator: $\frac{0.5}{x} + a\ln a + a\ln x + a - a = a\ln a + a\ln x$

Calculate 2nd: $\frac{a}{x}$

Calculate the 1st derivative of the denominator: $1 + \ln x$.

Calculate 2nd: $\frac{1}{x}$

The ratio is a.

11 Fixed point iteration to determine max value

We have demonstrated that $\lim_{x\to\infty}\frac{x^x}{\Gamma(ax+1)} = 0$, but interestingly, if $a - 1$ is a very small number, the function reaches extremely high values before it converges to 0. We are interested in the peak of this function based on the parameter a. Let us substitute Stirling's approximation for the gamma function and take its derivative to have an estimation on where the peak occurs.

Using Stirling's approximation and taking the first derivative we have

$$\frac{d}{dx}\frac{\sqrt{2\pi ax}\frac{(ax)}{e}^{ax}}{x^x}$$

=

$$\frac{d}{dx}\sqrt{2\pi a}\frac{(ax)}{e}^{ax}x^{0.5-x} =$$

$$\sqrt{2\pi a}\frac{(ax)}{e}^{ax} \cdot x^{0.5-x} \cdot (-a + \frac{1}{2x} + -1 + -\ln x + a\ln(a) + a\ln(x) + a) = 0$$

$$\sqrt{2\pi a}\frac{(ax)}{e}^{ax} \cdot x^{0.5-x} \cdot (\frac{1}{2x} + -1 + -\ln x + a\ln(a) + a\ln(x)) = 0$$

Divide by coefficients:

$$\frac{1}{2x} + -1 + -\ln x + a\ln(a) + a\ln(x) = 0$$

$$2(a-1)x\ln(x) = -1 + 2x - 2ax\ln(a)$$

$$\ln(x) = \frac{-1 + 2x - 2ax\ln(a)}{2x(a-1)}$$

$$x = e^{\frac{-1+2x-2ax\ln(a)}{2x(a-1)}}$$

$$x = e^{\frac{-1}{2x(a-1)}} * e^{\frac{2x}{2x(a-1)}} * e^{\frac{-2ax}{2x(a-1)}}$$

$$x = e^{\frac{-1}{2x(a-1)}} * e^{\frac{1}{a-1}} * e^{\frac{-a}{a-1}}$$

The fastest growing term as x approaches infinity is

$$e^{\frac{1}{a-1}}$$

Conclusion TBC