Facets of the Union-Closed Polytope

Dan Gallagher

July 28, 2023

Outline

The Frankl Union-Closed Sets Conjecture

$\mathcal{P}([n])$:=power set of $[n] := \{1, \ldots, n\}$

The Frankl Union-Closed Sets Conjecture

$\mathcal{P}([n])$:=power set of $[n] := \{1, \ldots, n\}$

$\mathcal{F} \subseteq \mathcal{P}([n])$ is a union-closed family if for all $S, T \in \mathcal{F}$, we also have $S \cup T \in \mathcal{F}$.

The Frankl Union-Closed Sets Conjecture

$\mathcal{P}([n])$:=power set of $[n] := \{1, \ldots, n\}$

$\mathcal{F} \subseteq \mathcal{P}([n])$ is a union-closed family if for all $S, T \in \mathcal{F}$, we also have $S \cup T \in \mathcal{F}$.

Conjecture (Frankl, 1979)

Let $\mathcal{F} \subseteq \mathcal{P}([n])$ be a non-empty union-closed family. Then there exists an element in $[n]$ present in at least half of the sets in $\mathcal{F}$.

The Frankl Union-Closed Sets Conjecture

$\mathcal{P}([n])$:=power set of $[n] := \{1, \ldots, n\}$

$\mathcal{F} \subseteq \mathcal{P}([n])$ is a union-closed family if for all $S, T \in \mathcal{F}$, we also have $S \cup T \in \mathcal{F}$.

Conjecture (Frankl, 1979)

Let $\mathcal{F} \subseteq \mathcal{P}([n])$ be a non-empty union-closed family. Then there exists an element in $[n]$ present in at least half of the sets in $\mathcal{F}$.

Example

$\mathcal{F} = \{\{1\}, \{2\}, \{1, 2\}, \{2, 3\}, \{3, 4\}, \{1, 2, 3\}, \{1, 3, 4\}, \{2, 3, 4\}, \{1, 2, 3, 4\}\}$

- $\mathcal{F}$ is union-closed
- $1, 2, 3$ are present in at least half of the sets in $\mathcal{F}$

Previous results

Theorem (Vučković-Živković, 2017)

The Frankl conjecture holds for any union-closed family $\mathcal{F} \subseteq \mathcal{P}([n])$ *with* $n \leq 12$.

Previous results

Theorem (Vučković-Živković, 2017)

The Frankl conjecture holds for any union-closed family $\mathcal{F} \subseteq \mathcal{P}([n])$ *with* $n \leq 12$.

Theorem (Knill, 1994)

In any union-closed family $\mathcal{F}$, *there exists an element in at least* $\frac{|\mathcal{F}|-1}{\log_2 |\mathcal{F}|}$ *sets.*

Previous results

Theorem (Vučković-Živković, 2017)

The Frankl conjecture holds for any union-closed family $\mathcal{F} \subseteq \mathcal{P}([n])$ with $n \leq 12$.

Theorem (Knill, 1994)

In any union-closed family $\mathcal{F}$, there exists an element in at least $\frac{|\mathcal{F}|-1}{\log_2 |\mathcal{F}|}$ sets.

Theorem (Gilmer, 2022)

Let $\mathcal{F} \subseteq \mathcal{P}([n])$ be a union-closed family with $\mathcal{F} \neq \emptyset$, then there exists an element that appears in at least $0.01|\mathcal{F}|$ sets in $\mathcal{F}$.

Previous results

Theorem (Vučković-Živković, 2017)

The Frankl conjecture holds for any union-closed family $\mathcal{F} \subseteq \mathcal{P}([n])$ with $n \leq 12$.

Theorem (Knill, 1994)

In any union-closed family $\mathcal{F}$, there exists an element in at least $\frac{|\mathcal{F}|-1}{\log_2 |\mathcal{F}|}$ sets.

Theorem (Gilmer, 2022)

Let $\mathcal{F} \subseteq \mathcal{P}([n])$ be a union-closed family with $\mathcal{F} \neq \emptyset$, then there exists an element that appears in at least $0.01|\mathcal{F}|$ sets in $\mathcal{F}$.

Theorem (Sawin, 2022)

Let $\mathcal{F} \subseteq \mathcal{P}([n])$ be a union-closed family with $\mathcal{F} \neq \emptyset$. Then there exists an element contained in at least $\frac{3-\sqrt{5}}{2}|\mathcal{F}| \approx 0.382|\mathcal{F}|$ sets.

An equivalent conjecture

For a union-closed family $\mathcal{F} \subseteq \mathcal{P}([n])$ and $i \in [n]$,
let $\mathcal{F}_i := \{S \in \mathcal{F} | i \in S\}$.

Conjecture (Maximization Conjecture)

For any positive integers a and n, let

$$\mathfrak{F}(a, n) = \{\mathcal{F} \subseteq \mathcal{P}([n]) \,|\mathcal{F} \textit{ is a non-empty union-closed family} \\ \textit{and } \max_{i\in[n]} |\mathcal{F}_i| \leq a\}.$$

Then $\max_{\mathcal{F}\in\mathfrak{F}(a,n)} |\mathcal{F}| \leq 2a$ *for all* $a, n \in \mathbb{Z}_{>0}$.

The Maximization Conjecture as an integer program

$$
\begin{aligned}
f(n,a) :=& \max \sum_{S \in \mathcal{P}([n])} x_S & \\
\text{such that } & x_S + x_T \leq 1 + x_{S \cup T} & \text{for all } S, T \in \mathcal{P}([n]) \\
& \sum_{\substack{S \in \mathcal{P}([n]):\\ i \in S}} x_S \leq a & \text{for all } i \in [n] \\
& x_S \in \{0,1\} & \text{for all } S \in \mathcal{P}([n])
\end{aligned}
$$

The Maximization Conjecture as an integer program

$$
\begin{aligned}
f(n,a) := \max \sum_{S \in \mathcal{P}([n])} x_S & \\
\text{such that } x_S + x_T \le 1 + x_{S\cup T} & \qquad \text{for all } S, T \in \mathcal{P}([n]) \\
\sum_{\substack{S \in \mathcal{P}([n]):\\ i \in S}} x_S \le a & \qquad \text{for all } i \in [n] \\
x_S \in \{0,1\} & \qquad \text{for all } S \in \mathcal{P}([n])
\end{aligned}
$$

Conjecture (Maximization Conjecture)

For any positive integers a and n, $f(n,a) \le 2a$.

The Maximization Conjecture as an integer program

$$
\begin{aligned}
f(n,a) := \max & \sum_{S \in \mathcal{P}([n])} x_S & \\
\text{such that } & x_S + x_T \leq 1 + x_{S \cup T} & \text{for all } S, T \in \mathcal{P}([n]) \\
& \sum_{\substack{S \in \mathcal{P}([n]): \\ i \in S}} x_S \leq a & \text{for all } i \in [n] \\
& x_S \in \{0, 1\} & \text{for all } S \in \mathcal{P}([n])
\end{aligned}
$$

Conjecture (Maximization Conjecture)

For any positive integers a and n, $f(n,a) \leq 2a$.

Let $P(n,a)$ denote the polytope defined by the convex hull of the feasible region of the integer program. We call $P(n,a)$ the *union-closed polytope*.

f-conjecture

Conjecture (f-*conjecture*, Pulaj-Raymond-Theis, 2016)

Fix $a \in \mathbb{N}$. *Then* $f(n, a) = f(n + 1, a)$ *for every* $n \in \mathbb{N}$ *such that* $n \geq \lceil \log_2 a \rceil + 1$.

f-conjecture

Conjecture (f*-conjecture*, Pulaj-Raymond-Theis, 2016)

Fix $a \in \mathbb{N}$*. Then* $f(n, a) = f(n+1, a)$ *for every* $n \in \mathbb{N}$ *such that* $n \geq \lceil \log_2 a \rceil + 1$*.*

Theorem (Pulaj-Raymond-Theis, 2016)

Fix $a \in \mathbb{N}$*. Then* $f(n, a) = f(n+1, a)$ *for every* $n \in \mathbb{N}$ *such that* $n \geq a$*.*

f-conjecture

Conjecture (f*-conjecture*, Pulaj-Raymond-Theis, 2016)

Fix $a \in \mathbb{N}$. *Then* $f(n, a) = f(n+1, a)$ *for every* $n \in \mathbb{N}$ *such that* $n \geq \lceil \log_2 a \rceil + 1$.

Theorem (Pulaj-Raymond-Theis, 2016)

Fix $a \in \mathbb{N}$. *Then* $f(n, a) = f(n+1, a)$ *for every* $n \in \mathbb{N}$ *such that* $n \geq a$.

We will focus on understanding $P(n, n)$.

$P(3,3)$

It was already known that $P(3,3)$ had the following facets (up to symmetry):

1. Union-closed inequalities
2. Frequency inequalities
3. $x_{\{1\}} + x_{\{2\}} + x_{\{3\}} - x_{\{1,2\}} - x_{\{1,3\}} - x_{\{2,3\}} \leq 1$
4. $x_{\{1\}} + x_{\{2\}} + x_{\{1,3\}} \leq 2$
5. $x_{\{1\}} + x_{\{1,2\}} + x_{\{1,3\}} \leq 2$
6. $x_{\{1\}} + x_{\{1,2\}} + x_{\{1,3\}} + 2x_{\{2,3\}} - 2x_{\{1,2,3\}} \leq 2$
7. $x_{\{1,2\}} + x_{\{1,3\}} + 2x_{\{2,3\}} - 2x_{\{1,2,3\}} \leq 1$
8. $2x_{\{1\}} + x_{\{2\}} + x_{\{3\}} + x_{\{1,3\}} \leq 3$
9. $x_{\{1\}} + x_{\{2\}} + x_{\{1,2\}} + x_{\{1,3\}} + 2x_{\{2,3\}} - x_{\{1,2,3\}} \leq 3$
10. $x_{\{1\}} + x_{\{2\}} + x_{\{1,2\}} + 2x_{\{1,3\}} + 2x_{\{2,3\}} - x_{\{1,2,3\}} \leq 3$
11. $x_{\{1\}} + x_{\{2\}} + x_{\{3\}} + x_{\{1,2\}} + x_{\{1,3\}} + x_{\{2,3\}} \leq 3$

$P(3,3)$

It was already known that $P(3,3)$ had the following facets (up to symmetry):

1. Union-closed inequalities
2. Frequency inequalities
3. $x_{\{1\}} + x_{\{2\}} + x_{\{3\}} - x_{\{1,2\}} - x_{\{1,3\}} - x_{\{2,3\}} \leq 1$
4. $x_{\{1\}} + x_{\{2\}} + x_{\{1,3\}} \leq 2$
5. $x_{\{1\}} + x_{\{1,2\}} + x_{\{1,3\}} \leq 2$
6. $x_{\{1\}} + x_{\{1,2\}} + x_{\{1,3\}} + 2x_{\{2,3\}} - 2x_{\{1,2,3\}} \leq 2$
7. $x_{\{1,2\}} + x_{\{1,3\}} + 2x_{\{2,3\}} - 2x_{\{1,2,3\}} \leq 1$
8. $2x_{\{1\}} + x_{\{2\}} + x_{\{3\}} + x_{\{1,3\}} \leq 3$
9. $x_{\{1\}} + x_{\{2\}} + x_{\{1,2\}} + x_{\{1,3\}} + 2x_{\{2,3\}} - x_{\{1,2,3\}} \leq 3$
10. $x_{\{1\}} + x_{\{2\}} + x_{\{1,2\}} + 2x_{\{1,3\}} + 2x_{\{2,3\}} - x_{\{1,2,3\}} \leq 3$
11. $x_{\{1\}} + x_{\{2\}} + x_{\{3\}} + x_{\{1,2\}} + x_{\{1,3\}} + x_{\{2,3\}} \leq 3$

$P(n,n)$ for $n \geq 4$ was unknown.

Ideas to compute $P(n, n)$

A polytope P can be described by either its facets or its vertices.

Ideas to compute $P(n, n)$

A polytope P can be described by either its facets or its vertices.

Definition

The polar of a polytope $P \subset \mathbb{R}^n$ is given by
$P^\Delta = \{\mathbf{c} \in (\mathbb{R}^n)^* : \mathbf{cx} \leq 1 \text{ for all } \mathbf{x} \in P\}$.

Facets of P correspond to vertices of P^Δ and vice-versa if the origin is in the interior of P. In our case, we need to first translate $P(n, n)$ to achieve this.

Computational results [G., 2023]

Up to symmetry, we found

- 2001 classes of facet-defining inequalities for $P(4,4)$ (corresponding to 38011 facets)
- 26862 classes of facet-defining inequalities for $P(5,5)$ (corresponding to 2742316 facets)
- 2566 classes of facet-defining inequalities for $P(6,6)$ (corresponding to 1492991 facets)

Computational results [G., 2023]

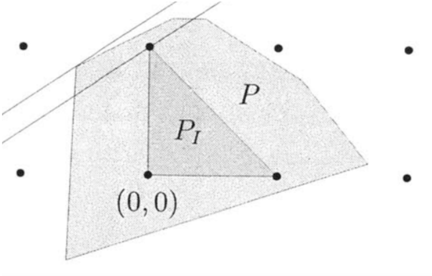
Up to symmetry, we found

- 2001 classes of facet-defining inequalities for $P(4,4)$ (corresponding to 38011 facets)
- 26862 classes of facet-defining inequalities for $P(5,5)$ (corresponding to 2742316 facets)
- 2566 classes of facet-defining inequalities for $P(6,6)$ (corresponding to 1492991 facets)

Facets get wild!
$14x_{\{1\}} + 20x_{\{2\}} + x_{\{3\}} + 14x_{\{4\}} - 14x_{\{1,2\}} + 9x_{\{1,3\}} + 4x_{\{1,4\}} + 15x_{\{2,3\}} - 14x_{\{2,4\}} + 14x_{\{3,4\}} + 20x_{\{1,2,3\}} + 33x_{\{1,2,4\}} + 9x_{\{1,3,4\}} + 15x_{\{2,3,4\}} - 17x_{\{1,2,3,4\}} \leq 65$ is a facet for $P(4,4)$.

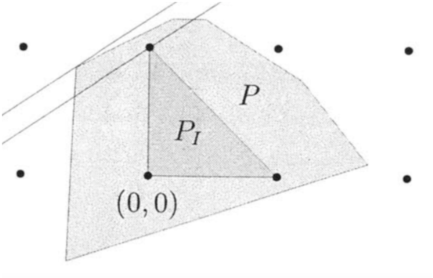
Standard methods and Chvátal rank of $P(n, n)$

Let P^I be the convex hull of the feasible region of an integer program, and let $P := \{\mathbf{x} \in \mathbb{R}^n | A\mathbf{x} \leq \mathbf{b}\}$ be the feasible region of its linear relaxation.



Standard methods and Chvátal rank of $P(n, n)$

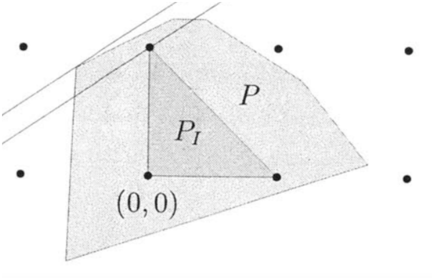
Let P^I be the convex hull of the feasible region of an integer program, and let $P := \{\mathbf{x} \in \mathbb{R}^n | A\mathbf{x} \leq \mathbf{b}\}$ be the feasible region of its linear relaxation.



Many cut procedures exist; we looked at the Chvátal-Gomory procedure to better understand the complexity of $P(n, n)$.

Standard methods and Chvátal rank of $P(n, n)$

Let P^I be the convex hull of the feasible region of an integer program, and let $P := \{\mathbf{x} \in \mathbb{R}^n | A\mathbf{x} \leq \mathbf{b}\}$ be the feasible region of its linear relaxation.



Many cut procedures exist; we looked at the Chvátal-Gomory procedure to better understand the complexity of $P(n, n)$.

Chvátal rank seems to grow: 3 for $P(3, 3)$ and 6 for $P(4, 4)$.

A first class of facet-defining inequalities

Theorem (G., 2023)

Let $A, B, C \in \mathcal{P}([n])$ be distinct sets such that $A \cup B$, $A \cup C$ and $B \cup C$ are distinct from one another and from A, B, C. Then the inequality

$$x_A + x_B + x_C \leq 1 + x_{A \cup B} + x_{A \cup C} + x_{B \cup C}$$

is valid and facet-defining for $P(n, a)$ for $a \geq 3$.

A first class of facet-defining inequalities

Theorem (G., 2023)

Let $A, B, C \in \mathcal{P}([n])$ be distinct sets such that $A \cup B$, $A \cup C$ and $B \cup C$ are distinct from one another and from A, B, C. Then the inequality

$$x_A + x_B + x_C \leq 1 + x_{A \cup B} + x_{A \cup C} + x_{B \cup C}$$

is valid and facet-defining for $P(n, a)$ for $a \geq 3$.

Example

- $A = \{1\}$, $B = \{2\}$, $C = \{3\}$:

A first class of facet-defining inequalities

Theorem (G., 2023)

Let $A, B, C \in \mathcal{P}([n])$ be distinct sets such that $A \cup B$, $A \cup C$ and $B \cup C$ are distinct from one another and from A, B, C. Then the inequality

$$x_A + x_B + x_C \leq 1 + x_{A\cup B} + x_{A\cup C} + x_{B\cup C}$$

is valid and facet-defining for $P(n, a)$ for $a \geq 3$.

Example

- $A = \{1\}$, $B = \{2\}$, $C = \{3\}$:
  $x_{\{1\}} + x_{\{2\}} + x_{\{3\}} - x_{\{1,2\}} - x_{\{1,3\}} - x_{\{2,3\}} \leq 1$
  is facet-defining for $P(n, a)$ for $n, a \geq 3$

A first class of facet-defining inequalities

Theorem (G., 2023)

Let $A, B, C \in \mathcal{P}([n])$ be distinct sets such that $A \cup B$, $A \cup C$ and $B \cup C$ are distinct from one another and from A, B, C. Then the inequality

$$x_A + x_B + x_C \leq 1 + x_{A\cup B} + x_{A\cup C} + x_{B\cup C}$$

is valid and facet-defining for $P(n, a)$ for $a \geq 3$.

Example

- $A = \{1\}$, $B = \{2\}$, $C = \{3\}$:
  $x_{\{1\}} + x_{\{2\}} + x_{\{3\}} - x_{\{1,2\}} - x_{\{1,3\}} - x_{\{2,3\}} \leq 1$
  is facet-defining for $P(n, a)$ for $n, a \geq 3$
- $A = \{1\}$, $B = \{2, 4\}$, $C = \{3, 4\}$:

A first class of facet-defining inequalities

Theorem (G., 2023)

Let $A, B, C \in \mathcal{P}([n])$ be distinct sets such that $A \cup B$, $A \cup C$ and $B \cup C$ are distinct from one another and from A, B, C. Then the inequality

$$x_A + x_B + x_C \leq 1 + x_{A\cup B} + x_{A\cup C} + x_{B\cup C}$$

is valid and facet-defining for $P(n, a)$ for $a \geq 3$.

Example

- $A = \{1\}$, $B = \{2\}$, $C = \{3\}$:
  $x_{\{1\}} + x_{\{2\}} + x_{\{3\}} - x_{\{1,2\}} - x_{\{1,3\}} - x_{\{2,3\}} \leq 1$
  is facet-defining for $P(n, a)$ for $n, a \geq 3$
- $A = \{1\}$, $B = \{2, 4\}$, $C = \{3, 4\}$:
  $x_{\{1\}} + x_{\{2,4\}} + x_{\{3,4\}} - x_{\{1,2,4\}} - x_{\{1,3,4\}} - x_{\{2,3,4\}} \leq 1$
  is facet-defining for $P(n, a)$ for $n \geq 4$, $a \geq 3$

A second class of facet-defining inequalities

Theorem (G., 2023)

Let $A, B, C \in \mathcal{P}([n])$ *be distinct sets such that* $D := A \cup B = A \cup C = B \cup C$ *and such that* D *is distinct from* A, B, C. *Then the inequality*

$$x_A + x_B + x_C \leq 1 + 2x_D$$

is valid and facet-defining for $P(n, a)$ *for* $a \geq 3$.

A second class of facet-defining inequalities

Theorem (G., 2023)

Let $A, B, C \in \mathcal{P}([n])$ be distinct sets such that $D := A \cup B = A \cup C = B \cup C$ and such that D is distinct from A, B, C. Then the inequality

$$x_A + x_B + x_C \leq 1 + 2x_D$$

is valid and facet-defining for $P(n, a)$ for $a \geq 3$.

Example

- $A = \{1, 2, 4\}$, $B = \{1, 3, 4\}$, $C = \{2, 3, 4\}$, $D = \{1, 2, 3, 4\}$:

A second class of facet-defining inequalities

Theorem (G., 2023)

Let $A, B, C \in \mathcal{P}([n])$ be distinct sets such that $D := A \cup B = A \cup C = B \cup C$ and such that D is distinct from A, B, C. Then the inequality

$$x_A + x_B + x_C \leq 1 + 2x_D$$

is valid and facet-defining for $P(n, a)$ for $a \geq 3$.

Example

- $A = \{1, 2, 4\}$, $B = \{1, 3, 4\}$, $C = \{2, 3, 4\}$, $D = \{1, 2, 3, 4\}$:
  $x_{\{1,2,4\}} + x_{\{1,3,4\}} + x_{\{2,3,4\}} - 2x_{\{1,2,3,4\}} \leq 1$
  is facet-defining for $P(n, a)$ for $n \geq 4$, $a \geq 3$

A second class of facet-defining inequalities

Theorem (G., 2023)

Let $A, B, C \in \mathcal{P}([n])$ be distinct sets such that $D := A \cup B = A \cup C = B \cup C$ and such that D is distinct from A, B, C. Then the inequality

$$x_A + x_B + x_C \leq 1 + 2x_D$$

is valid and facet-defining for $P(n, a)$ for $a \geq 3$.

Example

- $A = \{1, 2, 4\}$, $B = \{1, 3, 4\}$, $C = \{2, 3, 4\}$, $D = \{1, 2, 3, 4\}$:
  $x_{\{1,2,4\}} + x_{\{1,3,4\}} + x_{\{2,3,4\}} - 2x_{\{1,2,3,4\}} \leq 1$
  is facet-defining for $P(n, a)$ for $n \geq 4$, $a \geq 3$
- $A = \{3, 4\}$, $B = \{1, 2, 3\}$, $C = \{1, 2, 4\}$, $D = \{1, 2, 3, 4\}$:

A second class of facet-defining inequalities

Theorem (G., 2023)

Let $A, B, C \in \mathcal{P}([n])$ be distinct sets such that $D := A \cup B = A \cup C = B \cup C$ and such that D is distinct from A, B, C. Then the inequality

$$x_A + x_B + x_C \leq 1 + 2x_D$$

is valid and facet-defining for $P(n, a)$ for $a \geq 3$.

Example

- $A = \{1,2,4\}$, $B = \{1,3,4\}$, $C = \{2,3,4\}$, $D = \{1,2,3,4\}$:
  $x_{\{1,2,4\}} + x_{\{1,3,4\}} + x_{\{2,3,4\}} - 2x_{\{1,2,3,4\}} \leq 1$
  is facet-defining for $P(n, a)$ for $n \geq 4$, $a \geq 3$
- $A = \{3,4\}$, $B = \{1,2,3\}$, $C = \{1,2,4\}$, $D = \{1,2,3,4\}$:
  $x_{\{3,4\}} + x_{\{1,2,3\}} + x_{\{1,2,4\}} - 2x_{\{1,2,3,4\}} \leq 1$
  is facet-defining for $P(n, a)$ for $n \geq 4$, $a \geq 3$

A third class of facet-defining inequalities

Theorem (G., 2023)

The inequality

$$x_{A\setminus S} + x_{B_1} + x_{B_2} \leq 1 + x_A + x_{B_1 \cup B_2}$$

is facet-defining for $P(n, a)$ *for* $a \geq 4$ *where* $B_1, B_2 \subset A \in \mathcal{P}([n])$, $S \subset B_1, B_2$, *and* $A \setminus S, B_1, B_2, B_1 \cup B_2, A$ *are distinct.*

A third class of facet-defining inequalities

Theorem (G., 2023)

The inequality

$$x_{A \setminus S} + x_{B_1} + x_{B_2} \leq 1 + x_A + x_{B_1 \cup B_2}$$

is facet-defining for $P(n, a)$ *for* $a \geq 4$ *where* $B_1, B_2 \subset A \in \mathcal{P}([n])$, $S \subset B_1, B_2$, *and* $A \setminus S, B_1, B_2, B_1 \cup B_2, A$ *are distinct.*

Example

$A = \{1, 2, 3, 4, 5\}$, $S = \{4, 5\}$, $B_1 = \{1, 4, 5\}$ and $B_2 = \{3, 4, 5\}$:

A third class of facet-defining inequalities

Theorem (G., 2023)

The inequality

$$x_{A \setminus S} + x_{B_1} + x_{B_2} \leq 1 + x_A + x_{B_1 \cup B_2}$$

is facet-defining for $P(n, a)$ *for* $a \geq 4$ *where* $B_1, B_2 \subset A \in \mathcal{P}([n])$, $S \subset B_1, B_2$, *and* $A \setminus S, B_1, B_2, B_1 \cup B_2, A$ *are distinct.*

Example

$A = \{1, 2, 3, 4, 5\}$, $S = \{4, 5\}$, $B_1 = \{1, 4, 5\}$ and $B_2 = \{3, 4, 5\}$:

$$x_{\{1,2,3\}} + x_{\{1,4,5\}} + x_{\{3,4,5\}} \leq 1 + x_{\{1,2,3,4,5\}} + x_{\{1,3,4,5\}}$$

is facet-defining for $P(n, a)$ for any $n, a \geq 5$.

Proof Sketch

For $B_1, B_2 \subset A \in \mathcal{P}([n])$, $S \subset B_1, B_2$, and $A \setminus S, B_1, B_2, B_1 \cup B_2, A$, show $x_{A \setminus S} + x_{B_1} + x_{B_2} \leq 1 + x_A + x_{B_1 \cup B_2}$ is facet-defining.

Proof Sketch

For $B_1, B_2 \subset A \in \mathcal{P}([n])$, $S \subset B_1, B_2$, and $A \setminus S, B_1, B_2, B_1 \cup B_2, A$, show $x_{A\setminus S} + x_{B_1} + x_{B_2} \leq 1 + x_A + x_{B_1 \cup B_2}$ is facet-defining.

1. Show the validity of the inequality
2. Show that the face induced by this inequality has dimension $2^n - 1$

Proof Sketch

For $B_1, B_2 \subset A \in \mathcal{P}([n])$, $S \subset B_1, B_2$, and $A \setminus S, B_1, B_2, B_1 \cup B_2, A$, show $x_{A \setminus S} + x_{B_1} + x_{B_2} \leq 1 + x_A + x_{B_1 \cup B_2}$ is facet-defining.

1. Show the validity of the inequality
2. Show that the face induced by this inequality has dimension $2^n - 1$

$$\begin{array}{c}
\begin{matrix} \mathcal{S}_0 & \mathcal{S}_1 & \mathcal{S}_2 & \mathcal{S}_3 & \mathcal{S}_4 & \mathcal{S}_5 \end{matrix} \\
\begin{pmatrix}
J & 0 & 0 & 0 & 0 & 0 \\
M_1 & I & 0 & 0 & 0 & 0 \\
M_2 & 0 & I & 0 & 0 & 0 \\
M_4 & 0 & 0 & I & 0 & 0 \\
M_7 & 0 & 0 & 0 & I & 0 \\
M_{11} & M_{12} & M_{13} & M_{14} & M_{15} & I
\end{pmatrix}
\end{array}$$

where $\mathcal{P}([n]) = \mathcal{S}_0 \cup \mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3 \cup \mathcal{S}_4 \cup \mathcal{S}_5$

Proof Sketch

For $B_1, B_2 \subset A \in \mathcal{P}([n])$, $S \subset B_1, B_2$, and $A \setminus S, B_1, B_2, B_1 \cup B_2, A$, show $x_{A\setminus S} + x_{B_1} + x_{B_2} \leq 1 + x_A + x_{B_1 \cup B_2}$ is facet-defining.

1. Show the validity of the inequality
2. Show that the face induced by this inequality has dimension $2^n - 1$

$$
\begin{array}{cccccc}
 & A \setminus S & B_1 & B_2 & A & B_1 \cup B_2 \\
J := & \multicolumn{5}{c}{\left(\begin{array}{ccccc}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
1 & 1 & 0 & 1 & 0 \\
0 & 1 & 1 & 0 & 1
\end{array}\right)}
\end{array}
\begin{array}{c}
\\
\{A \setminus S\} \\
\{B_1\} \\
\{B_2\} \\
\{A \setminus S, B_1, A\} \\
\{B_1, B_2, B_1 \cup B_2\}
\end{array}
$$

A fourth class of facet-defining inequalities

Theorem (G., 2023)

The inequality

$$\sum_{i=1}^{n} x_{[n]\setminus\{i\}} - (n-1)x_{[n]} \leq 1$$

is valid and facet-defining for $P(n,n)$.

A fourth class of facet-defining inequalities

Theorem (G., 2023)

The inequality

$$\sum_{i=1}^{n} x_{[n]\setminus\{i\}} - (n-1)x_{[n]} \leq 1$$

is valid and facet-defining for $P(n, n)$.

Example

$x_{\{1,2,3\}} + x_{\{1,2,4\}} + x_{\{1,3,4\}} + x_{\{2,3,4\}} - 3x_{\{1,2,3,4\}} \leq 1$
is facet-defining for $P(4, 4)$

Conjectured inequalities

Conjecture (G., 2023)

Fix $i \in \mathbb{N}$*. If* $n = \sum_{k=i-1}^{m} \binom{m}{k}$ *for* $m \in \mathbb{N}$ *and* $n \geq i+1$*, then* $\sum_{\substack{S \in \mathcal{P}([n]): \\ |S|=i}} x_S \leq \binom{m+1}{i}$ *is a tight valid inequality for* $P(n,n)$*.*

Furthermore, if $\sum_{k=i-1}^{m-1} \binom{m-1}{k} < n < \sum_{k=i-1}^{m} \binom{m}{k}$ *for* $m \in \mathbb{N}$ *and* $n \geq i+1$*, then* $\sum_{\substack{S \in \mathcal{P}([n]): \\ |S|=i}} x_S \leq \binom{m+1}{i} - 1$ *is a valid inequality for* $P(n,n)$ *though maybe not tight.*

Conjectured inequalities

Conjecture (G., 2023)

Fix $i \in \mathbb{N}$. If $n = \sum_{k=i-1}^{m} \binom{m}{k}$ for $m \in \mathbb{N}$ and $n \geq i+1$, then $\sum_{\substack{S \in \mathcal{P}([n]):\\ |S|=i}} x_S \leq \binom{m+1}{i}$ is a tight valid inequality for $P(n,n)$.
Furthermore, if $\sum_{k=i-1}^{m-1} \binom{m-1}{k} < n < \sum_{k=i-1}^{m} \binom{m}{k}$ for $m \in \mathbb{N}$ and $n \geq i+1$, then $\sum_{\substack{S \in \mathcal{P}([n]):\\ |S|=i}} x_S \leq \binom{m+1}{i} - 1$ is a valid inequality for $P(n,n)$ though maybe not tight.

n \ i	1	2	3	4	5	6	7	8	9
2	2/2								
3	2/2	3/3							
4	3/3	3/5	4/4						
5	3/3	3/5	4/9	5/5					
6	3/3	4/5	4/9	5/14	6/6				
7	3/3	6/6	4/9	5/14	6/20	7/7			
8	4/4	6/9	5/9	6/14	6/20	7/27	8/8		
9	4/4	6/9	6/9	7/14	6/20	7/27	8/35	9/9	
10	4/4	6/9	8/9	7/14	7/20	8/27	8/35	9/44	10/10
11	4/4	6/9	10/10	8/14	7/20	10/27	8/35	9/44	10/54

Some context for the conjecture

Though not facet-defining, these inequalities help to decrease the integrality gap between $f(n, n)$ and its linear relaxation.

n	$f(n, n)$	relaxation with cuts	relaxation without cuts
3	5	6.167	6.5
4	8	9.25	9.6
5	9	12	13.558
6	10	15.25	18.482
7	12	19.4	24.188
8	16	23.8	30.636
9	17	27.8	37.931
10	18	32.4	46.068

Some evidence for the conjectured inequalities

Theorem (G., 2023)

The inequality $\sum_{\substack{S\in\mathcal{P}([n]):\\ |S|=n-1}} x_S \leq n$ *is a valid and tight for* $P(n,n)$.

Theorem (G., 2023)

The inequality $\sum_{\substack{S\in\mathcal{P}([n])\\ |S|=1}} x_S \leq \binom{\lfloor \log_2 n\rfloor+1}{1}$ *is valid and tight for* $P(n,n)$.

Theorem (G., 2023)

The inequality $\sum_{\substack{S\in\mathcal{P}([n]):\\ |S|=n-2}} x_S \leq n-1$ *is valid and tight for* $P(n,n)$.

Conclusion

Summary:

- We computed many new classes of facets for $P(n, n)$ and some of their Chvátal ranks for $4 \leq n \leq 6$
- We proved some new classes of facet-defining inequalities for $P(n, n)$ for general n
- We conjectured some new inequalities that are helpful to compute $f(n, n)$ and proved them in certain cases

Conclusion

Summary:

- We computed many new classes of facets for $P(n, n)$ and some of their Chvátal ranks for $4 \leq n \leq 6$
- We proved some new classes of facet-defining inequalities for $P(n, n)$ for general n
- We conjectured some new inequalities that are helpful to compute $f(n, n)$ and proved them in certain cases

Future work:

- Find a more refined conjecture for the maximum of $\sum_{\substack{S \in \mathcal{P}([n]): \\ |S|=i}} x_S$
- Find the maximum (or a conjecture) for $\sum_{\substack{S \in \mathcal{P}([n]): \\ |S|=i \text{ or } j}} x_S$ for some i, j
- Use the data of $P(4, 4)$, $P(5, 5)$, $P(6, 6)$ to prove more (facet-defining) inequalities that are especially helpful to compute $f(n, n)$
- Prove some bounds on how fast the Chvátal rank grows for $P(n, n)$
- Prove some bounds for the Chvátal rank of $\mathbf{1}^\top \mathbf{x} \leq f(n, n)$

Thank you!