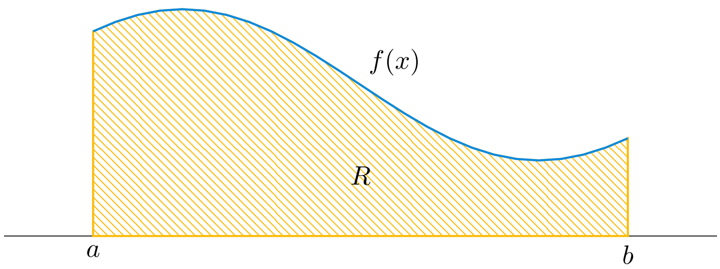
INTEGRATION - WEEK 6

1. Riemann sums and the integral

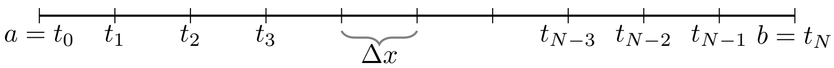
Video links

- The definition of the integral
- The integral, what does it mean?

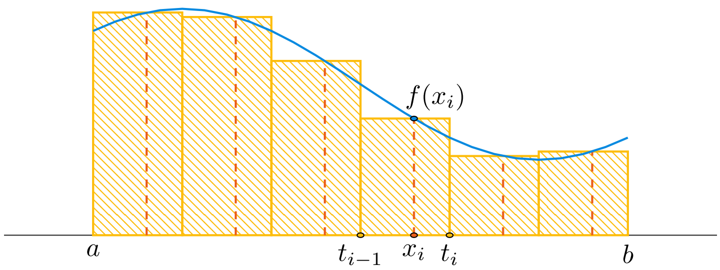
Let f be a bounded function on an interval $[a, b]$. For the moment we assume that f takes only positive values on this interval. We want to find the area of the region R between the horizontal axis and the graph of f.



If f is a constant function, then it is easy to calculate the area under the graph, since the region is a rectangle in this case. If f is not a constant function, we can *approximate* the region R with rectangles. To do this we divide the interval $[a, b]$ into N subintervals $[t_{i-1}, t_i]$ of length Δx[1]:



In each of these subintervals $[t_{i-1}, t_i]$ we choose a point x_i. Now we approximate the area between the interval $[t_{i-1}, t_i]$ and the graph of f by the area of the rectangle with height $f(x_i)$ and width Δx: this area is equal to $f(x_i)\Delta x$.

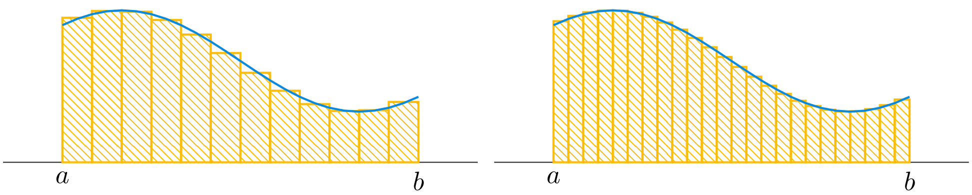


The sum of the areas of all the rectangles is an approximation of the area of the region R:

$$\text{area of } R \approx f(x_1)\Delta x + f(x_2)\Delta x + \ldots + f(x_N)\Delta x = \sum_{i=1}^{N} f(x_i)\Delta x.$$

[1] Actually, it is not necessary that all subintervals have the same length.

This sum is called a *Riemann sum* for f on the interval $[a, b]$. If we choose a larger number N of subintervals, which corresponds to choosing a smaller value for Δx, we get a better approximation of the area of the region R under the graph of f.

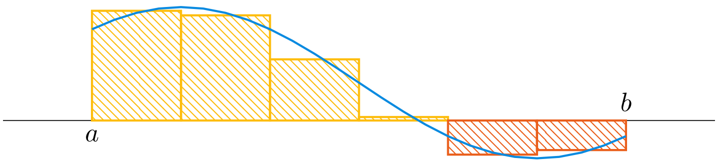


We see that we can find the area of R by letting $N \to \infty$:

$$\text{area of } R = \lim_{N\to\infty} \sum_{i=1}^{N} f(x_i)\Delta x.$$

This limit is *the integral* of f over the interval $[a, b]$, which we denote by $\displaystyle\int_a^b f(x)\, dx$.

So far we assumed that f is a positive function on $[a, b]$. Let us now consider a function f that also has negative values on $[a, b]$. Just as before, the integral of f over $[a, b]$ is defined as the limit of Riemann sums[2]:



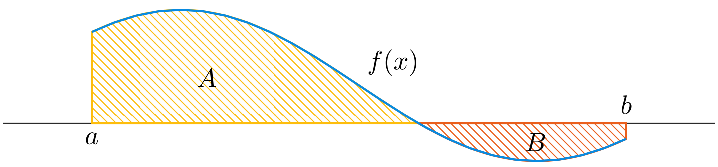
Definition of the integral

The integral of f over the interval $[a, b]$ is defined by

$$\int_a^b f(x)\, dx = \lim_{N\to\infty} \sum_{i=1}^{N} f(x_i)\Delta x.$$

In this case the integral does *not* represent the area of the region between the graph of f and the horizontal axis, but it is the *signed area*. The rectangles below the horizontal axis have a negative contribution to the Riemann sum, so the area below the horizontal axis counts negative: if A is the area of the region between the graph of f and horizontal axis that is above the axis, and B is the area of the region below the horizontal axis, then

$$\int_a^b f(x)\, dx = A - B.$$



[2]There exist bounded functions for which the limit of Riemann sums does not exist, but we will not consider such functions.

2. The fundamental theorem of calculus

Video links

- The fundamental theorem of calculus
- Proof of the fundamental theorem of calculus

Riemann sums are very useful for *approximating* integrals, but it usually very hard to calculate the exact value of an integral with the help of Riemann sums. The following theorem is extremely useful to calculate the exact value of an integral.

The fundamental theorem of calculus

If

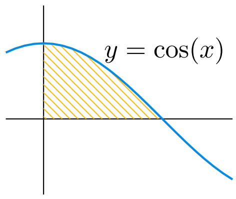
- f is a continuous function on $[a, b]$
- F is a function such that $F'(x) = f(x)$

then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

A function F such that $F'(x) = f(x)$ is called a *primitive function* or an *anti-derivative* of f.

Example 2.1. We evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \cos(x)\,dx$.



If we want to apply the fundamental theorem of calculus we need a primitive function of $\cos(x)$. The function $\sin(x)$ is such a primitive function, since $\frac{d}{dx}\sin(x) = \cos(x)$. So

$$\int_0^{\frac{\pi}{2}} \cos(x)\,dx = \Big[\sin(x)\Big]_0^{\frac{\pi}{2}} = \sin(\tfrac{\pi}{2}) - \sin(0) = 1 - 0 = 1. \qquad \blacksquare$$

Primitive functions are not unique! If F is a primitive function of f and C is a constant, then $F + C$ is also a primitive function of f, because

$$\frac{d}{dx}(F + C) = F' + 0 = f.$$

On the other hand, if F and G are both primitive functions of f, then

$$\frac{d}{dx}(F - G) = f - f = 0,$$

so that $F - G$ is constant. This means that $G = F + C$ for some constant C. Conclusion: if F is a primitive function of f, then any other primitive function of f has the form $F + C$ for some constant C. We use an indefinite integral, i.e. an integral without bounds, to denote the set of all primitive functions:

$$\int f(x)\,dx = \text{the set of all primitive functions of } f.$$

For example, the set of all primitive functions of $\cos(x)$ is

$$\int \cos(x)\,dx = \sin(x) + C,$$

where in the right-hand side C can be any constant. Note that in the right-hand side we could also write $\sin(x) + 42 + K$, since this represents the same set of functions.

In order to evaluate integrals, primitive functions of standard function are needed. Some of these can be obtained by reading the table of standard derivatives from right to left.

TABLE 1. Primitives of standard functions

$f(x)$	$F(x)$	
0	c	
x^a	$\dfrac{x^{a+1}}{a+1}$	$a \neq -1$
$\dfrac{1}{x}$	$\ln\lvert x\rvert$	
$\sin(x)$	$-\cos(x)$	
$\cos(x)$	$\sin(x)$	
e^x	e^x	

3. Properties of integrals

Video links

- Rules of calculation: Changing the integrand
- Rules of calculation: Splitting the integral
- Rules of calculation: Linear substitutions

There are several properties of integrals that can be very useful for evaluating the integrals:

- Changing the integrand:
$$\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$$
$$\int_a^b \big(f(x)+g(x)\big)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$
- Changing the integration interval:
$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$
$$\int_a^a f(x)\,dx = 0$$
$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$$
- Linear substitutions:
$$\int_a^b f(x+c)\,dx = \int_{a+c}^{b+c} f(x)\,dx$$
$$\int_a^b f(cx)\,dx = \frac{1}{c}\int_{ac}^{bc} f(x)\,dx \qquad c \neq 0$$

Using these properties it is often possible to reduce an integral of a complicated function, to integrals of standard functions.

Example 3.1. Let f be the function defined by

$$f(x) = \begin{cases} (2-3x)^4 & \text{if } x<1 \\ 12x^3+x & \text{if } x \geq 1. \end{cases}$$

We calculate the integral $I = \displaystyle\int_0^2 f(x)\,dx$.

First we split the interval of integration:

$$I = \int_0^1 (2-3x)^4\,dx + \int_1^2 (12x^3+x)\,dx.$$

For the first integral we use the rules for linear substitutions and interchange the integration bounds:

$$\begin{aligned}
\int_0^1 (2-3x)^4\,dx &= \frac{1}{-3}\int_0^{-3}(2+x)^4\,dx \\
&= -\tfrac{1}{3}\int_2^{-1} x^4\,dx \\
&= \tfrac{1}{3}\int_{-1}^{2} x^4\,dx \\
&= \tfrac{1}{3}\Big[\tfrac{1}{5}x^5\Big]_{-1}^{2} \\
&= \tfrac{1}{15}\Big(32-(-1)\Big) = 2\tfrac{1}{5}.
\end{aligned}$$

The second integral we split into two integrals:

$$\begin{aligned}
\int_1^2 \big(12x^3+x\big)\,dx &= \int_1^2 12x^3\,dx + \int_1^2 x\,dx \\
&= 12\int_1^2 x^3\,dx + \int_1^2 x\,dx \\
&= 12\Big[\tfrac{1}{4}x^4\Big]_1^2 + \Big[\tfrac{1}{2}x^2\Big]_1^2 \\
&= 3(16-1) + \tfrac{1}{2}(4-1) = 46\tfrac{1}{2}.
\end{aligned}$$

Now we find $I = 2\frac{1}{5} + 46\frac{1}{2} = 48\frac{7}{10}$. ∎

4. Advanced integration techniques

Video links

- The substitution method
- Substitution with boundaries
- Integration by parts
- Integration by parts: tips and tricks

A function H is an anti-derivative of h, if $H'(x) = h(x)$. We can write this with an indefinite integral as

$$\int h(x)\,dx = H(x) + C,$$

or

$$\int H'(x)\,dx = H(x) + C. \tag{4.1}$$

We will use this formula to find two methods which can be used to calculate integrals of 'complicated' functions.

4.1. The substitution method.

If H is the composition of two functions, $H(x) = F(g(x))$, then the derivative of H can be calculated using the chain rule:

$$H'(x) = \frac{d}{dx}F\big(g(x)\big) = F'\big(g(x)\big)g'(x).$$

Now (4.1) becomes

$$\int F'\big(g(x)\big)g'(x)\,dx = F\big(g(x)\big) + C$$

We write $F' = f$, then we have obtained the following rule for calculating indefinite integrals.

$$\boxed{\int f\big(g(x)\big)\,g'(x)\,dx = F\big(g(x)\big) + C}$$

It is often useful to introduce a new variable $u = g(x)$, i.e. we substitute u for $g(x)$. Taking the derivative then gives

$$\frac{du}{dx} = g'(x),$$

which we conveniently write as

$$du = g'(x)\,dx.$$

This gives

$$\int f\big(g(x)\big)\,g'(x)\,dx = \int f(u)\,du = F(u) + C = F\big(g(x)\big) + C,$$

which is (of course) exactly the anti-derivative of $f(g(x))g'(x)$ that we obtained above.

Example 4.1. Let us try to compute

$$\int \frac{3x^2}{x^3+1}\,dx.$$

We can write this as

$$\int \frac{1}{x^3+1}\,3x^2\,dx.$$

Now define $u = x^3 + 1$, then

$$\frac{du}{dx} = 3x^2 \qquad \Rightarrow \qquad du = 3x^2\,dx,$$

and we obtain

$$\int \frac{1}{x^3+1}\,3x^2\,dx = \int \frac{1}{u}\,du = \ln|u| + C = \ln|x^3+1| + C. \qquad \blacksquare$$

Now that we know an anti-derivative of $f(g(x))g'(x)$, we can apply the fundamental theorem of calculus to compute an integral with boundaries:

$$\int_a^b f\big(g(x)\big)\,g'(x)\,dx = F\big(g(b)\big) - F\big(g(a)\big).$$

On the other hand, again by the fundamental theorem,

$$\int_{g(a)}^{g(b)} f(u)\,du = F\big(g(b)\big) - F\big(g(a)\big).$$

Combining these expression gives the substitution rule for integrals.

Substitution rule

$$\int_a^b f\big(g(x)\big)\,g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du.$$

Example 4.2. We compute the integral

$$I = \int_0^1 (2x+2)e^{x^2+2x}\,dx$$

using the substitution rule. Define $u = x^2 + 2x$, then $du = (2x+2)dx$. For $x = 0$ we have $u = 0^2 + 0 = 0$, and for $x = 1$ we have $u = 1^2 + 1 = 2$, so

$$I = \int_0^1 e^{x^2+2x}\,(2x+2)\,dx = \int_0^2 e^u\,du = \Big[e^u\Big]_0^2 = e^2 - 1. \qquad \blacksquare$$

4.2. **Integration by parts.** We just obtained the substitution rule for integration by reversing the chain rule for differentiation. Now we try to do the same with the product rule. We use (4.1) again,

$$\int H'(x)\,dx = H(x) + C,$$

and we assume that H is the product of two functions, $H(x) = F(x)g(x)$. By the product rule

$$H'(x) = F'(x)g(x) + F(x)g'(x),$$

so that

$$\int \Big(F'(x)g(x) + F(x)g'(x)\Big)\,dx = F(x)g(x) + C.$$

We split the integral into two integrals, and move one of them to the right-hand side, to obtain

$$\int F'(x)g(x)\,dx = F(x)g(x) - \int F(x)g'(x)\,dx.$$

We may leave out the integration constant C here, because both sides contain an indefinite integral. We write $F' = f$ again, then we have obtained the following rule for rewriting an indefinite integral.

Integration by parts

$$\int f(x)g(x)\,dx = F(x)g(x) - \int F(x)g'(x)\,dx.$$

For an integral with boundaries this gives

$$\int_a^b f(x)g(x)\,dx = F(b)g(b) - F(a)g(a) - \int_a^b F(x)g'(x)\,dx.$$

Example 4.3. We compute the integral

$$\int_0^1 (x+1)e^{2x}\,dx.$$

We choose $f(x) = e^{2x}$ and $g(x) = x+1$, then $F(x) = \frac{1}{2}e^{2x}$ and $g'(x) = 1$. Note that F must be a primitive function of f, it does not matter which primitive function we choose. Now integration by parts gives

$$\begin{aligned}\int_0^1 (x+1)e^{2x}\,dx &= \left[\tfrac{1}{2}e^{2x}(x+1)\right]_0^1 - \int_0^1 \tfrac{1}{2}e^{2x}\cdot 1\,dx \\ &= \left(e^2 - \tfrac{1}{2}\right) - \left[\tfrac{1}{4}e^{2x}\right]_0^1 = \tfrac{3}{4}e^2 - \tfrac{1}{4}.\end{aligned}$$ ■