FUNCTIONS - WEEK 2

1. Trigonometric functions

Video links

- Trigonometric functions: definitions
- Trigonometric functions: rules of calculation

When working with the trigonometric functions $\sin(x)$, $\cos(x)$ and $\tan(x)$, it is often useful to simplify them using (one of the many) identities for these functions. Two fundamental identities are

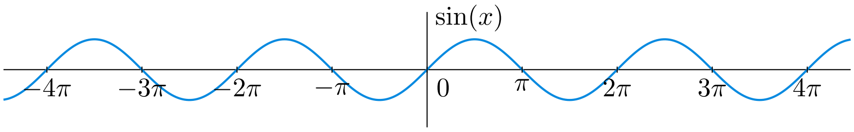
$$\sin^2(x) + \cos^2(x) = 1$$

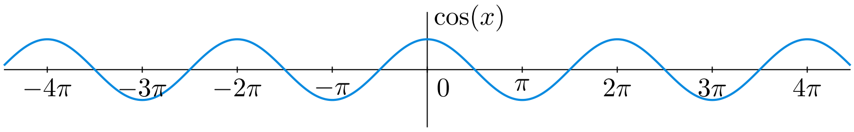
which follows from the Pythagorean Theorem, and

$$\tan(x) = \frac{\sin(x)}{\cos(x)}$$

which is the definition of the tangent function.

For several identities, especially for $\sin(x)$ and $\cos(x)$, one should just keep the graphs in mind to remember them.





Periodicity

For any integer k,

- $\sin(x) = \sin(x + 2k\pi)$
- $\cos(x) = \cos(x + 2k\pi)$
- $\tan(x) = \tan(x + k\pi)$

Note that the sine and cosine have period 2π, but the tangent has period π.

Reflections and translations

- $\sin(-x) = -\sin(x)$
- $\cos(-x) = \cos(x)$
- $\sin(\frac{\pi}{2} - x) = \cos(x)$
- $\cos(\frac{\pi}{2} - x) = \sin(x)$
- $\sin(\pi - x) = \sin(x)$
- $\cos(\pi - x) = -\cos(x)$

The following identities are also very useful for simplifying expressions involving sines and cosines.

Double angle formulas

- $\sin(2x) = 2\sin(x)\cos(x)$
- $$\begin{aligned}\cos(2x) &= \cos^2(x) - \sin^2(x)\\ &= 2\cos^2(x) - 1\\ &= 1 - 2\sin^2(x)\end{aligned}$$

Addition formulas

- $\sin(x + y) = \cos(x)\sin(y) + \sin(x)\cos(y)$
- $\cos(x + y) = \cos(x)\cos(y) - \sin(x)\sin(y)$

Note that the double angle formulas are actually special cases of the addition formulas:

$$\sin(2x) = \sin(x + x) = \cos(x)\sin(x) + \sin(x)\cos(x) = 2\sin(x)\cos(x),$$

and

$$\cos(2x) = \cos(x + x) = \cos(x)\cos(x) - \sin(x)\sin(x) = \cos^2(x) - \sin^2(x).$$

The other two formulas for $\cos(2x)$ can be obtained by using $\sin^2(x) + \cos^2(x) = 1$.

2. Compositions

Video links

- Composing functions
- Scaling and translation

From two functions f and g we can make a new function by composing them, that is, by applying the two functions after each other: $f\big(g(x)\big)$ or $g\big(f(x)\big)$.

Example 2.1. With $f(x) = \sin(x)$ and $g(x) = x^2$ we have

$$f\big(g(x)\big) = \sin(x^2) \qquad \text{and} \qquad g\big(f(x)\big) = \big(\sin(x)\big)^2. \qquad \blacksquare$$

As you can see from the example, the order in which the functions are applied matters. So, in general, $f\big(g(x)\big) \neq g\big(f(x)\big)$!

If we compose with a linear function we can say what the graph of the composed functions looks like. This is described in the Table 1 below. In general, it is very hard to say how the graph of a composed function $f\big(g(x)\big)$ is obtained from the graphs of f and g.

Table 1. Composing with linear functions

$f(x)+a$	Vertical shift by a upwards
$f(x+a)$	Horizontal shift by a to the left
$af(x)$	Vertical scaling by factor a
$f(ax)$	Horizontal scaling by factor $\frac{1}{a}$

3. Exponential functions

Video links

- Exponential functions

A function $f(x)$ is an **exponential function** if it can be written as $f(x) = Ab^x$, where $A \neq 0$ and $b > 0$. The number b is called the **base** of the exponential function.

When working with exponential functions, the following properties are often very useful. They follow directly from the rules of calculation for power functions.

Rules of calculation

- $b^{x+y} = b^x b^y$
- $b^{x-y} = \dfrac{b^x}{b^y}$
- $(b^x)^y = b^{xy}$
- $b^x c^x = (bc)^x$

We assume here that $b, c > 0$, and x and y can be any real number.

4. Logarithms

Video links

- Inverse functions
- Logarithms
- Logarithms - Rules of calculation

Let $b > 1$. The **logarithm** in base b is defined as the inverse function of the exponential function b^x:

Definition of the logarithm

Let $y \in \mathbb{R}$ and $x \in (0, \infty)$, then

$$y = \log_b(x) \qquad \Leftrightarrow \qquad x = b^y.$$

Example 4.1. Let's take base $b = 2$. Then $\log_2(16) = 4$, since $2^4 = 16$, and $\log_2(\frac{1}{8}) = -3$, since $2^{-3} = \frac{1}{8}$. ■

When dealing with equations involving logarithms the following properties can be very useful.

Rules of calculation

1. $\log_b(AB) = \log_b(A) + \log_b(B)$
2. $\log_b(\frac{A}{B}) = \log_b(A) - \log_b(B)$
3. $\log_b(A^K) = K \log_b(A)$
4. $\log_b(A) = \dfrac{\log_c(A)}{\log_c(b)}$ (change of base formula)

We assume here that $A, B > 0$ and $b, c > 1$. The number K can be any real number.

The rules of calculation for logarithms follow from the rules of calculation for exponential functions and the definition of the logarithm. Since the range of the exponential function b^x consists of all positive numbers, there exist numbers x and y such that $A = b^x$ and $B = b^y$. The definition of the logarithm then says that $x = \log_b(A)$ and $y = \log_b(B)$.

Let's consider the first rule:

$$\begin{aligned}\log_b(AB) &= \log_b(b^x b^y)\\ &= \log_b(b^{x+y})\\ &= x + y\\ &= \log_b(A) + \log_b(B).\end{aligned}$$

The second rule is a consequence of the first rule. Replace A by $\frac{A}{B}$:

$$\log_b\left(\tfrac{A}{B}B\right) = \log_b\left(\tfrac{A}{B}\right) + \log_b(B) \qquad \Rightarrow \qquad \log_b\left(\tfrac{A}{B}\right) = \log_b(A) - \log_b(B).$$

Let's now consider the third rule:

$$\begin{aligned}\log_b(A^K) &= \log_b\left((b^x)^K\right)\\ &= \log_b(b^{Kx})\\ &= Kx\\ &= K\log_b(A)\end{aligned}\quad.$$

Finally we look at the change of base formula:

$$\begin{aligned}\log_c(A) &= \log_c(b^x)\\ &= x\log_c(b)\\ &= \log_b(A)\log_c(b).\end{aligned}$$

Dividing both sides by $\log_c(b)$ gives the change of base formula as stated above.

Example 4.2. We can simplify the expression $\log_2\left(\sqrt{2}x^3(\frac{1}{2})^y\right)$ using the rules of calculation and the definition of the logarithm:

$$\begin{aligned}\log_2\left(\sqrt{2}x^3(\tfrac{1}{2})^y\right) &= \log_2\left(\sqrt{2}x^3\right)+\log_2\left((\tfrac{1}{2})^y\right) && \textsf{rule 1}\\ &= \log_2(2^{\frac{1}{2}})+\log_2(x^3)+\log_2(2^{-y}) && \textsf{rule 1}\\ &= \tfrac{1}{2}+3\log_2(x)-y && \textsf{rule 3.}\end{aligned}$$

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