FUNCTIONS - WEEK 1

1. Functions

Video links

- What is a function?
- How to describe a function
- Domain and range

A function describes a relation between two sets. It can be seen as a 'machine' that takes inputs from one set, and given an input, produces some output in another set. More precisely, you need the following data to describe a function:

- A set of inputs, called the **domain**;
- A set in which the outputs end up, called the **codomain**;
- A rule that tells you how to associate to each element of the domain *precisely one* element of the codomain.

The set of all possible outputs is called **the range** of the function. The range is always contained in the codomain, but the codomain may be larger.

In this course, we only consider functions that send real numbers to real numbers. In particular, for every function we consider, the domain, codomain and range are sets of real numbers.

Example 1.1. Consider the function $f(x) = 3 + \sqrt{x}$. Then we have the following:

- The maximal possible domain for this function is the set of $x \geq 0$, since the square root function is only defined for non-negative x.
- For any input in the domain, this function produces a real number. So we can choose the codomain simply to be $\mathbb{R}$, the set of all real numbers.
- If we choose the domain to be as large as possible (all $x \geq 0$), then the function can attain all values greater than or equal to 3. This is the range.
- The rule that describes the function is given by the formula. In words: "take a number as input, take its square root and add 3". ■

2. Polynomial functions

Video links

- What is a polynomial?
- Graphs of polynomial functions

A function $f(x)$ is called a **polynomial function** if it can be written in the form:

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0, \qquad a_n \neq 0$$

for some constants $a_0, a_1, \ldots, a_n$ and some non-negative integer n.

The number n, the highest occurring power of x, is called the **degree** of the polynomial function. It can be zero, in that case the function is simply constant. The constants $a_0, a_1, \ldots, a_n$ are called the **coefficients**. If you write a polynomial function in the form above, we say that it is in **standard form**.

Example 2.1. Examples of polynomial functions are

- $f(x) = 5$ (degree 0)
- $g(x) = -3x + 7$ (degree 1)
- $h(x) = \frac{1}{2}x^6 - 20x^3 + 3x^2$ (degree 6) ■

Alternate forms. Besides the standard form, there are other useful ways to write a polynomial function:

- **Factorized form** (or **factored form**): write a polynomial function as a product of lower degree polynomial functions. For example: some degree 2 polynomial functions can be written as $f(x) = a(x-p)(x-q)$ for some constants a, p and q. This can be useful for finding zeros of a polynomial function. Unfortunately, it is not always possible.
- Any degree 2 polynomial function can be written as $a(x-r)^2 + s$, for certain constants a, r and s. This too is very convenient for finding the zeros of such a function.

3. Rational functions

Video links

- Rational functions

A **rational function** is a function that can be written in the form
$$\frac{P(x)}{Q(x)},$$
where P and Q are polynomials, and Q not identically equal to zero.

Example 3.1. Examples of rational functions are

- $f(x) = \dfrac{1}{x}$
- $g(x) = \dfrac{x+3}{x^4+6x+1}$

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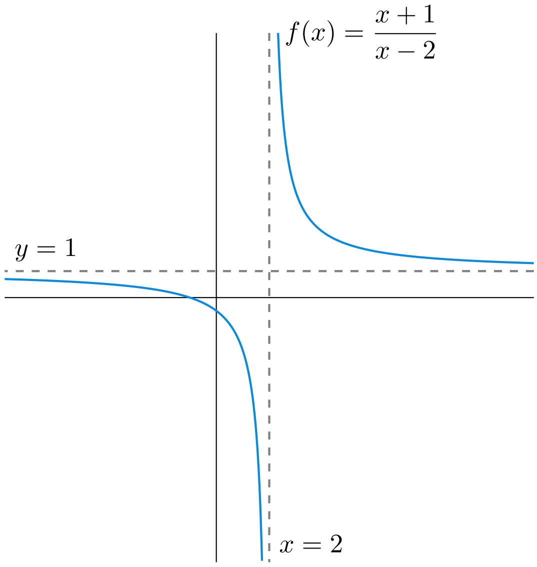
An **asymptote** of a function is a straight line which is approached by the graph of the function. A rational function $P(x)/Q(x)$ has **vertical asymptotes** only where the denominator $Q(x) = 0$. This is certainly true when the numerator $P(x) \neq 0$.

A **horizontal asymptote** exists if degree(P)$\leq$ degree(Q) and can be calculated by dividing numerator and denominator by the highest power of x occurring in the denominator and subsequently noting that $\frac{1}{x^n} \to 0$ as x becomes very large.

Example 3.2. The rational function $f(x) = \frac{x+1}{x-2}$ has a vertical asymptote at $x = 2$, because the denominator is equal to 0 at $x = 2$, and the numerator is not equal to 0. The function has a horizontal asymptote $y = 1$, since for $x \neq 0$
$$f(x) = \frac{x+1}{x-2} = \frac{1+\frac{1}{x}}{1-\frac{2}{x}} \quad \to \quad \frac{1+0}{1-0} = 1,$$
as x becomes very large.



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4. Power functions

Video links

- Power functions: integer exponents
- Power functions: noninteger exponents

A function $f(x)$ is called a **power function** if it can be written as $f(x) = x^a$, for some real number a. The number a is called the **exponent** of the power function.

Power functions obey the following rules of calculation.

Rules of calculation

For any exponents a and b and for any real numbers $x > 0$ and $y > 0$ we have:

- $x^a x^b = x^{a+b}$
- $\dfrac{x^a}{x^b} = x^{a-b}$
- $(x^a)^b = x^{ab}$
- $(xy)^a = x^a y^a$

As long as all exponents are integer, these rules also hold for $x < 0$ and/or $y < 0$. For non-integer exponents, you have to be very careful when x or y becomes negative, as the rules may no longer be true!