EQUATIONS - WEEK 3

1. Equations involving polynomials

Video links

- Linear and quadratic equations
- Polynomials of degree 3 and higher

An equation of degree 2 written in standard form

$$ax^2 + bx + c = 0 \tag{1.1}$$

can be solved using the quadratic formula. We assume here that $a \neq 0$, otherwise the equation would be of degree 1.

Quadratic formula

The *discriminant* D of the 2nd degree equation (1.1) is defined by

$$D = b^2 - 4ac.$$

- If $D < 0$, then equation (1.1) has no solutions.
- If $D = 0$, then equation (1.1) has one solution:

$$x = -\frac{b}{2a}.$$

- If $D > 0$, then equation (1.1) has two solutions:

$$x = \frac{-b + \sqrt{D}}{2a} \qquad \text{and} \qquad x = \frac{-b - \sqrt{D}}{2a}.$$

The quadratic formula can be derived as follows. First multiply equation (1.1) by 4a:

$$4a^2x^2 + 4abx + 4ac = 0$$

Now bring the constant term to the right-hand side, and complete the square,

$$\begin{aligned} 4a^2x^2 + 4abx = -4ac \qquad &\Leftrightarrow \qquad (2ax + b)^2 - b^2 = -4ac \\ &\Leftrightarrow \qquad (2ax + b)^2 = b^2 - 4ac = D. \end{aligned}$$

The left-hand side is a square, which can never be negative. So if $D < 0$, there are no solutions. If $D \geq 0$ we can take square roots to obtain

$$2ax + b = \sqrt{D} \qquad \text{or} \qquad 2ax + b = -\sqrt{D}.$$

In both equations bring b to the right-hand side,

$$2ax = -b + \sqrt{D} \qquad \text{or} \qquad 2ax = -b - \sqrt{D},$$

and divide by a,

$$x = \frac{-b + \sqrt{D}}{2a} \qquad \text{or} \qquad x = \frac{-b - \sqrt{D}}{2a}.$$

Note that if $D = 0$ these two solutions are the same. So in this case there is only one solution.

2. Equations involving rational functions and square roots

Video links

- Equations involving rational functions
- Equations involving square roots

The strategie for solving equations involving **rational functions** is:
- Multiply both sides of the equation by the common denominators of both the left-hand and right-hand side of the equation.
- Solve the simpler equation involving polynomials.
- Check all possible solutions in the original equation.

This can be summarized as:

Simplify, Solve and Check!

Example 2.1. We solve the equation

$$\frac{1}{x} + \frac{2}{1+x} = 2.$$

Multiply both sides of the equation by the common denominator $x(1+x)$:

$$1 + x + 2x = 2x(1+x) \quad \Longleftrightarrow \quad 2x^2 - x - 1 = 0.$$

Factor this to $(2x+1)(x-1) = 0$ which leads to the solutions $x = -\frac{1}{2}$ and $x = 1$. Checking both in the original equation shows that both are valid solutions. ■

Equations involving **square roots** (or other radicals):
- First try to isolate the square root (or other radical).
- Take the square (or another appropriate power) on both sides of the equation.
- Solve the simpler equation involving polynomials.
- Check all possible solutions in the original equation.

So again:

Simplify, Solve and Check!

Example 2.2. We solve the equation

$$\sqrt{x-1} + 3 = x.$$

Isolating the square root gives: $\sqrt{x-1} = x - 3$. Squaring both sides of this equation leads to $x - 1 = (x-3)^2$ which simplifies to $x^2 - 7x + 10 = 0$. Factor this to $(x-2)(x-5) = 0$ which leads to the possible solutions $x = 2$ or $x = 5$. Checking both in the original equation shows that $x = 2$ is *not* a valid solution, while $x = 5$ is. So, $x = 5$ is the only solution. ■

3. Overview of strategies

Video links

- Strategies for equation solving

General techniques

- $A\,B = 0 \quad \Longrightarrow \quad A = 0 \quad \text{or} \quad B = 0$
- $A\,B = A\,C \quad \Longrightarrow \quad A = 0 \quad \text{or} \quad B = C$
- $A^2 = B^2 \quad \Longrightarrow \quad A = B \quad \text{or} \quad A = -B$
- $|A| = |B| \quad \Longrightarrow \quad \pm A = \pm B$

Example 3.1.

$$(x+2)(x-6) = 0.$$

This holds whenever $x+2=0$ or $x=6=0$. So, the solutions are $x=-2$ and $x=6$. ■

Example 3.2.

$$(x+2)^2 = (2x+1)^2.$$

This holds whenever $x+2 = 2x+1$ or $x+2 = -(2x+1)$. So, the solutions are $x=-1$ and $x=1$. ■

Example 3.3.

$$|x| = |2x-1|.$$

Distinguish between the following *four* possibilities:

1. $x = 2x-1$ gives $x=1$
2. $x = -(2x-1)$ gives $x=\frac{1}{3}$
3. $-x = 2x-1$ gives $x=\frac{1}{3}$
4. $-x = -(2x-1)$ gives $x=1$

Checking both possible solutions $x=\frac{1}{3}$ and $x=1$ shows that both are valid solutions.

Alternatively, we may use that

$$|x| = \begin{cases} x & x \geq 0 \\ -x & x \leq 0 \end{cases} \qquad \text{and} \qquad |2x-1| = \begin{cases} 2x-1 & x \geq \frac{1}{2} \\ -(2x-1) & x \leq \frac{1}{2} \end{cases}$$

and distinguish between the following *three* possibilities:

(1) For $x \leq 0$ we have: $-x = -(2x-1)$. This leads to $x=1$ which is *not* on the interval $(-\infty, 0]$.
(2) For $0 \leq x \leq \frac{1}{2}$ we have: $x = -(2x-1)$. This leads to $x=\frac{1}{3}$ which is on the interval $\left[0, \frac{1}{2}\right]$.
(3) For $x \geq \frac{1}{2}$ we have: $x = 2x-1$. This leads to $x=1$ which is on the interval $\left[\frac{1}{2}, \infty\right)$.

Again, we conclude that $x=\frac{1}{3}$ and $x=1$ are the only solutions. ■