EQUATIONS - WEEK 4

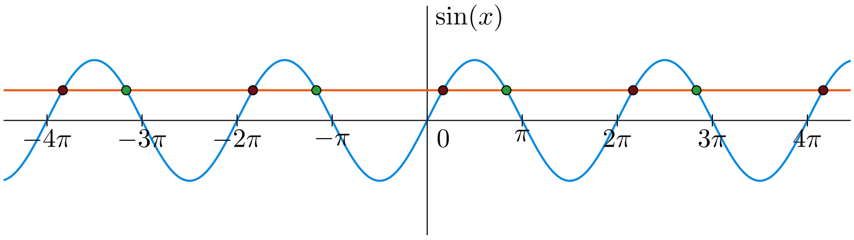
1. Equations involving trigonometric functions

Video links

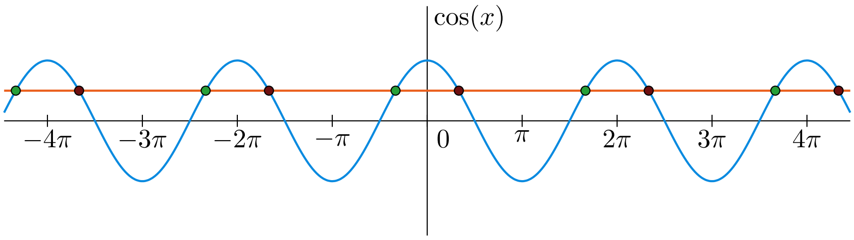
- Equations involving trigonometric functions

We can solve an equation involving trigonometric functions if we can simplify it to one of the following cases:

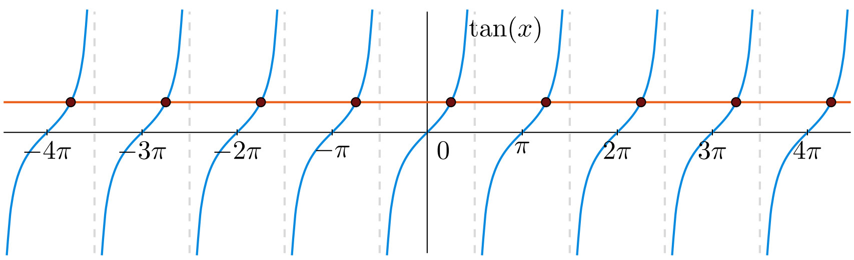
- **$\sin(x) = c$**:
  if $x = \alpha$ is a solution, then $x = \alpha + 2k\pi$ and $x = \pi - \alpha + 2k\pi$ are also solutions for all integers k.



- **$\cos(x) = c$**:
  if $x = \beta$ is a solution, then $x = \beta + 2k\pi$ and $x = -\beta + 2k\pi$ are also solutions for all integers k.



- **$\tan(x) = c$**:
  if $x = \gamma$ is a solution, then $x = \gamma + k\pi$ are also solutions for all integers k.



Example 1.1. Solve

$$\sin(x) = \frac{1}{2}.$$

From the table we know that $x = \frac{1}{6}\pi$ is a solution. Then all solutions on the domain $\mathbb{R}$ are

$$x = \tfrac{1}{6}\pi + 2k\pi \quad \text{and} \quad x = \pi - \tfrac{1}{6}\pi + 2k\pi \quad \text{with } k \text{ an integer.} \qquad \blacksquare$$

Example 1.2. Solve

$$\cos(x) = \frac{1}{2}.$$

From the table we know that $x = \frac{1}{3}\pi$ is a solution. Then all solutions on the domain $\mathbb{R}$ are

$$x = \tfrac{1}{3}\pi + 2k\pi \quad \text{and} \quad x = -\tfrac{1}{3}\pi + 2k\pi \quad \text{with } k \text{ an integer.} \qquad \blacksquare$$

Example 1.3. Solve

$$\tan(x) = 1.$$

From the table we know that $x = \frac{1}{4}\pi$ is a solution. Then all solutions on the domain $\mathbb{R}$ are

$$x = \tfrac{1}{4}\pi + k\pi \quad \text{with } k \text{ an integer.} \qquad \blacksquare$$

2. Equations involving exponentials and logarithms

Video links

- Equations involving exponentials and logarithms

To solve an equation involving exponential functions or logarithms, we try to simplify to an equation of the form $b^A = b^B$ or $\log_b(A) = \log_b(B)$. This leads to the equation $A = B$, since both b^x and $\log_b(x)$ are injective, i.e.

Exponentials and logarithms

- $b^A = b^B \quad \Longrightarrow \quad A = B$
- $\log_b(A) = \log_b(B) \quad \Longrightarrow \quad A = B$

Here A and B are expressions wich can still involve the unknown x.

Example 2.1. Solve

$$e^{x^2} = e^{5x+6}.$$

This equation is already in the form $e^A = e^B$, so we obtain the quadratic equation

$$x^2 = 5x + 6.$$

We can solve this equation by factorizing:

$$x^2 - 5x - 6 = 0 \iff (x-6)(x+1) = 0.$$

So we have: $x = -1$ or $x = 6$. ∎

Example 2.2. Solve

$$\log_2(x) + 1 = \log_2(x^2).$$

Before we actually solve this equation, we can already say that a solution x must satisfy $x > 0$ (and $x^2 > 0$), since $\log_2(x)$ is only defined for $x > 0$.

Using rules of calculation for logarithms, the left-hand side of the equation can be written as

$$\log_2(x) + \log_2(2) = \log_2(2x).$$

So the equation can be simplified to

$$\log_2(2x) = \log_2(x^2) \iff 2x = x^2,$$

which leads to $x = 0$ or $x = 2$. Checking both possible solutions we conclude that $x = 2$ is the only valid solution. ∎

Example 2.3. Solve

$$\ln(x) + 1 = \ln(x^2).$$

The left-hand side can be written as

$$\ln(x) + \ln(e) = \ln(e\,x).$$

So the equation simplifies to

$$\ln(ex) = \ln(x^2) \iff e\,x = x^2,$$

which leads to $x = 0$ or $x = e$. Checking both possible solutions we conclude that $x = e$ is the only valid solution. ∎

3. Inequalities

Video links

- Inequalities

Strategy: first solve the corresponding equation. Also check for values of the unknown for which the inequality has no meaning. These values (solutions and singularities) divide the real line into several intervals. Check the inequality in each of the different intervals.

Example 3.1. Solve

$$\frac{1}{x} \geq x.$$

First we solve the equation

$$\frac{1}{x} = x$$

Multiplying both sides by x leads to $x^2 = 1$, so $x = \pm 1$. Furthermore, the left-hand side of our equation does not exists for $x = 0$, so this is a singulary. Now we divide the real line into the intervals

$$(-\infty, -1), \qquad (-1, 0), \qquad (0, 1) \qquad \text{and} \qquad (1, \infty).$$

In each interval we only need to check one value for x. Moreover, we need to check the boundary points $x = 0$ and $x = \pm 1$ separately.

Checking $x = -2$, $x = -\frac{1}{2}$, $x = \frac{1}{2}$ and $x = 2$ we conclude that the inequality holds for x on $(-\infty, -1)$ and $(0, 1)$. Checking the boundary points we conclude that the inequality also holds for $x = \pm 1$.

Conclusion: the inequality holds for x in $(-\infty, -1]$ or $(0, 1]$. ■

Example 3.2. Solve

$$\sqrt{x-1} < x - 7.$$

The left-hand side only exists for $x \geq 1$. Squaring both sides of the equation $\sqrt{x-1} = x - 7$ leads to $x - 1 = (x-7)^2$ or $x^2 - 15x + 50 = 0$. This can be factored as $(x-5)(x-10) = 0$ which leads to $x = 5$ or $x = 10$. Note that both are valid solutions. Now we divide the (natural) domain $[1, \infty)$ into the intervals $(1, 5)$, $(5, 10)$ and $(10, \infty)$ and check the boundary points separately.

For x on $[1, 5)$ the inequality clearly does *not* hold, since the right-hand side is negative.

For x on $(5, 10)$ the inequality also does *not* hold.

For x on $(10, \infty)$ the inequality holds and for the boundary point $x = 10$ it does *not* hold.

Conclusion: the inequality holds for x in $(10, \infty)$. ■

4. Systems of equations

Video links

- Systems of equations

We use two methods to solve a system of equations: the *substitution method* and the *elimination method*.

- **substitution**: try to solve one of the (two) equations for one of the (two) unknowns and *substitute* this into the other equation(s).

- **elimination**: try to find a combination of the (two) equations such that one of the (two) unknowns is *eliminated*. Solve the simpler equation(s) for the other unknown(s) and substitute this into the other equation(s).

Example 4.1 (Substitution method)**.** Solve

$$\begin{cases} 2x - y = 0 \\ 3x + 2y = 7 \end{cases}$$

$$\begin{cases} 2x - y = 0 \\ 3x + 2y = 7 \end{cases}$$

First solve the first equation for y: $y = 2x$. Then substitute this into the second equation:

$$\begin{cases} y = 2x \\ 3x + 2 \cdot (2x) = 7 \end{cases} \implies \begin{cases} y = 2x \\ 7x = 7 \end{cases} \implies \begin{cases} y = 2 \\ x = 1. \end{cases} \qquad \blacksquare$$

Example 4.2 (Elimination method)**.** Solve

$$\begin{cases} 2x - y = 7 \\ 3x + 2y = 14 \end{cases}$$

Eliminate y by adding the first equation twice to the second equation:

$$4x - 2y = 14$$

$$\begin{cases} 2x - y = 7 \\ 7x = 28 \end{cases} \implies \begin{cases} 2x - y = 7 \\ x = 4 \end{cases} \implies \begin{cases} 2 \cdot 4 - y = 7 \\ x = 4 \end{cases} \implies \begin{cases} y = 1 \\ x = 4. \end{cases}$$

$\blacksquare$