DIFFERENTIATION - WEEK 5

1. The derivative and the tangent line

Video links

- Explaining the concept
- The tangent line
- Non-differentiable functions

Let $A = \big(a, f(a)\big)$ be a point on the graph of a function f. We want to find the slope s of the tangent line at A.

We start by choosing another point $B = \big(b, f(b)\big)$ on the graph of f. If we choose B close to the point A, then the straight line L through A and B is almost the tangent line. We set $\Delta x = b - a$ and $\Delta y = f(b) - f(a)$, the distance (with a sign!) between the x-coordinates of A and B, respectively the y-coordinates. Then the slope of the line L is $\frac{\Delta y}{\Delta x}$. Since $b = a + \Delta x$ we have $f(b) = f(a + \Delta x)$. So we find the following expression for the slope of the line L:

$$\frac{\Delta y}{\Delta x} = \frac{f(a + \Delta x) - f(a)}{\Delta x}.$$

If we take a very small value for Δx, i.e. we take B very close to A, we get a good approximation of the slope s of the tangent line T, so

$$s \approx \frac{f(a + \Delta x) - f(a)}{\Delta x}.$$

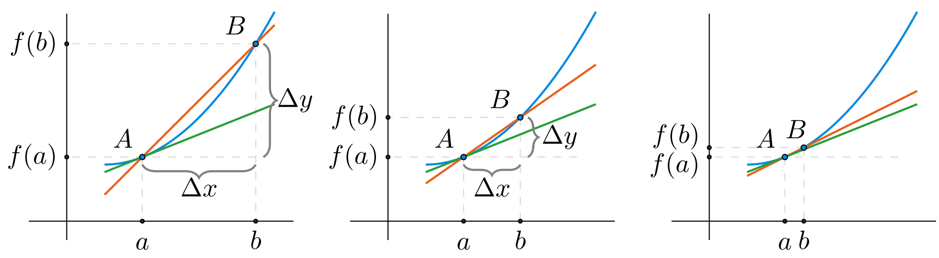


Figure 1. The line through A and B (orange) approximates the tangent line (green) as Δx tends to 0.

We find the slope s by letting Δx tend to 0:

$$s = \lim_{\Delta x \to 0} \frac{f(a + \Delta x) - f(a)}{\Delta x}.$$

This number s is called *the derivative of f at the point a*. We usually denote this number by $f'(a)$ or $\frac{df}{dx}(a)$. So we have the following definition for $f'(a)$, where a is a point in the domain of the function f.

Definition of derivative

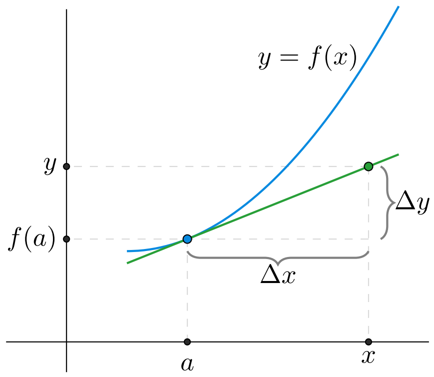
The derivative of f at the point a is defined by

$$f'(a) = \lim_{\Delta x \to 0} \frac{f(a+\Delta x) - f(a)}{\Delta x}.$$

The above limit *might not exist*! If the limit exists, we say that f is *differentiable at a*.

If a function f is differentiable in every point a in an interval (c, d), we can make a function $f' : (c, d) \to \mathbb{R}$ mapping each a in (c, d) to $f'(a)$. This function f' is called *the derivative function* of f. We also use the notation $\frac{df}{dx}$ for the derivative function of f.

If we know the derivative of a function f at a, we can immediately write down an equation of the tangent line at $(a, f'(a))$. The tangent line is a straight line, with slope equal to $f'(a)$. On the other hand, if we take an arbitrary point (x, y) on the tangent line, then the slope is equal to $\frac{\Delta y}{\Delta x}$, see figure.



So we get the identity

$$f'(a) = \frac{\Delta y}{\Delta x} = \frac{y - f(a)}{x - a}.$$

Solving for y gives us the following equation of the tangent line.

Equation of tangent line

The equation of the tangent line to the graph of f at $\big(a, f(a)\big)$ is

$$y = f(a) + f'(a)(x - a).$$

2. Derivatives of the standard functions and rules for differentiation

Video links

- Rules of calculation: background
- Rules of calculation part 1
- Rules of calculation part 2
- Derivatives of power functions and polynomials
- Standard derivatives: the sine
- Standard derivatives: a^x and x^p

In order to differentiate functions that are composed of standard functions, e.g. $f(x) = x^2e^{2x+1} + \sin(x)$, we need to know derivatives of standard functions and rules for differentiation. These are given in Tables 1 and 2.

Table 1. Derivatives of standard functions

$f(x)$	$f'(x)$	
c	0	
x^a	ax^{a-1}	$a \neq 0$
$\sin(x)$	$\cos(x)$	
$\cos(x)$	$-\sin(x)$	
$\tan(x)$	$\frac{1}{\cos^2(x)} = 1 + \tan^2(x)$	
e^x	e^x	
a^x	$a^x \ln(a)$	$a > 0$
$\ln(x)$	$\frac{1}{x}$	
$\log_a(x)$	$\frac{1}{x\ln(a)}$	$a > 0, a \neq 1$

Example 2.1. We calculate the derivative function of $f(x) = x^2e^{2x+1} + \sin(x)$ using Tables 1 and 2:

$$
\begin{aligned}
f'(x) &= \frac{d}{dx}\Big(x^2e^{2x+1}\Big) + \frac{d}{dx}\Big(\sin(x)\Big) && \text{sum rule} \\
&= \frac{d}{dx}\left(x^2\right) \cdot e^{2x+1} + x^2 \cdot \frac{d}{dx}\Big(e^{2x+1}\Big) + \cos(x) && \text{product rule} \\
&= 2x \cdot e^{2x+1} + x^2 \cdot \Big(e^{2x+1} \cdot \frac{d}{dx}(2x+1)\Big) + \cos(x) && \text{chain rule} \\
&= 2xe^{2x+1} + x^2 \cdot \Big(e^{2x+1} \cdot 2\Big) + \cos(x) \\
&= (2x + 2x^2)e^{2x+1} + \cos(x).
\end{aligned}
$$

■

TABLE 2. Rules for differentiation

$F(x)$	$F'(x)$	
$cf(x)$	$cf'(x)$	
$f(x)+g(x)$	$f'(x)+g'(x)$	sum rule
$f(x)g(x)$	$f'(x)g(x)+f(x)g'(x)$	product rule
$\dfrac{f(x)}{g(x)}$	$\dfrac{f'(x)g(x)-f(x)g'(x)}{g(x)^2}$	quotient rule
$f\big(g(x)\big)$	$f'\big(g(x)\big)g'(x)$	chain rule

3. Extreme values

Video links

- Finding minima and maxima

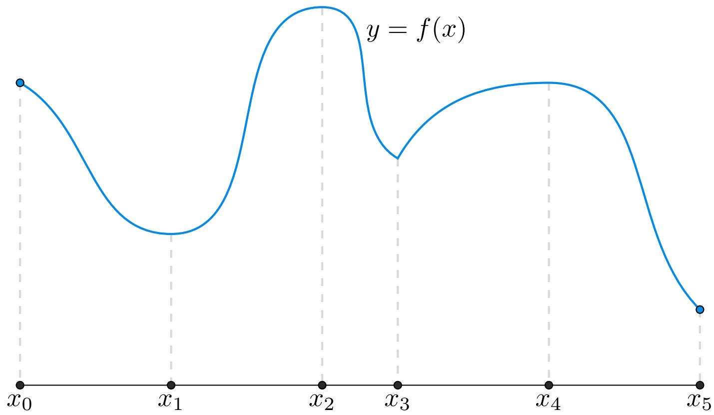
The derivative of a function f is very useful for finding extreme values (minima and maxima) of f. We distinguish between two types of extreme values of function: *global* and *local* extreme values. Suppose f has domain D, then

- f has a *global maximum* in $x = a$ if $f(a) \geq f(x)$ for all $x \in D$
- f has a *global minimum* in $x = b$ if $f(b) \leq f(x)$ for all $x \in D$

- f has a *local maximum* at $x = c$ if there exists an open interval I such that $c \in I$ and $f(c) \geq f(x)$ for all $x \in I \cap D$
- f has a *local minimum* at $x = d$ if there exists an open interval J such that $d \in J$ and $f(d) \leq f(x)$ for all $x \in J \cap D$.

Note that a global extreme value is also a local extreme value.

Example 3.1. Here you see the graph of a function f.



f has local maxima at x_0, x_2 and x_4, and f has local minima at x_1, x_3 and x_5. Furthermore, f has a global maximum at x_2 and a global minimum at x_5. ■

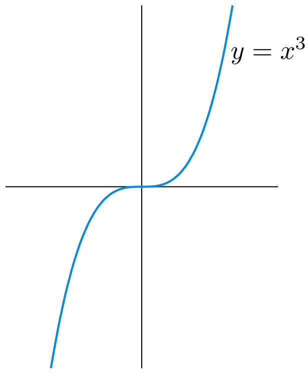
If a is not a boundary point of the domain of f and $f'(a) \neq 0$, then on a small interval around a the function f is either increasing ($f'(a) > 0$), or decreasing ($f'(a) < 0$), so f does not have an extreme value at $x = a$. This implies the following theorem.

Local extreme values

If f has a local extreme value at $x = a$, then one of the following statements holds:

- a is a critical point of f, i.e. $f'(a) = 0$
- a is a singular point of f, i.e. $f'(a)$ does not exist
- a is a boundary point of the domain of f

To find the extreme values of a functions f we first determine all the critical points, singular points or boundary points. But it is *not* guaranteed that f has an extreme value at one of these points. For example, $f(x) = x^3$ has a critical point at $x = 0$, because $f'(x) = 3x^2$ is equal to 0 at $x = 0$, but f has no extreme value at $x = 0$.



If a computer or calculator is not available to draw that graph of the function, a *sign chart* of f' is very useful to determine whether a critical point, singular point or boundary point is a local extreme value. This is illustrated in the following example.

Example 3.2. We determine all the extreme values of $f(x) = x^4 - \frac{4}{3}x^3$ on the interal $[-1, 2]$.

First we determine the critical points, so we need the derivative of f:

$$f'(x) = 4x^3 - 4x^2 = 4x^2(x-1).$$

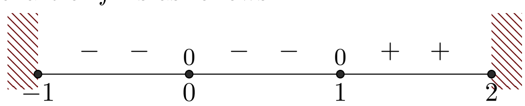
The solutions of the equation $f'(x) = 0$ are $x = 0$ and $x = 1$, so these are the critical points of f. Now f *may* attain extreme values at

- the critical points: $x = 0$ and $x = 1$
- the singular points: there are none
- the boundary points: $x = -1$ and $x = 2$.

Next we make a sign chart of f' in which we indicate the critical points, the singular points (there are none), and the boundary points. Between such points $f'(x)$ is either positive or negative, because f' can only change sign at a critical point or at a singular point. In this case we have

$$f'(-\tfrac{1}{2}) = 4 \cdot \tfrac{1}{4} \cdot (-\tfrac{1}{2} - 1) = -\tfrac{3}{2} < 0,$$
$$f'(\tfrac{1}{2}) = 4 \cdot \tfrac{1}{4} \cdot (\tfrac{1}{2} - 1) = -\tfrac{1}{2} < 0,$$
$$f'(\tfrac{3}{2}) = 4 \cdot \tfrac{9}{4} \cdot (\tfrac{3}{2} - 1) = \tfrac{9}{2} > 0,$$

and then the sign chart of f' is as follows:



Now we see that f' is negative on $(-1, 0)$, so f is decreasing on this interval. This implies that f has a local maximum at the boundary point $x = -1$. On $(0, 1)$ f' is also negative, so f has no extreme value at the critical point $x = 0$. On the interval $(1, 2)$ f' is positive, so f is increasing. This means that at the critical point $x = 1$ the functions f goes from decreasing to increasing, meaning that f has local minimum at $x = 1$. Finally, at the boundary point $x = 2$ f has a local maximum. ■