

# Aplicação de PINNs no Problema de Kovasznay

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# Sumário

① Motivação

② Escoamento de Kovasznay

③ PINNs

④ Resultados

# Rio Guamá

Fonte: Canal Agricultura & Natureza

# Problemas do Milênio

## Birch and Swinnerton-Dyer Conjecture

Supported by much experimental evidence, this conjecture relates the number of points on an elliptic curve mod  $p$  to the rank of the group of rational points. Elliptic curves, defined by cubic equations in two variables, are fundamental mathematical objects that arise in many areas: Wiles' proof of the Fermat Conjecture, factorization of



## Hodge Conjecture

The answer to this conjecture determines how much of the topology of the solution set of a system of algebraic equations can be defined in terms of further algebraic equations. The Hodge conjecture is known in certain special cases, e.g., when the solution set has dimension less than four. But in dimension four it is unknown.



## Navier-Stokes Equation

This is the equation which governs the flow of fluids such as water and air. However, there is no proof for the most basic questions one can ask: do solutions exist, and are they unique? Why ask for a proof? Because a proof gives not only certitude, but also understanding.



## P vs NP

If it is easy to check that a solution to a problem is correct, is it also easy to solve the problem? This is the essence of the "P vs NP" question. Typical of the NP problems is that of the Hamiltonian Path Problem: given  $N$  cities to visit, how can one do this without visiting a city twice? If you give me a solution, I can easily check that it is correct. But I cannot so easily find a solution.



## Riemann Hypothesis

The prime number theorem determines the average distribution of the primes. The Riemann hypothesis tells us about the deviation from the average. Formulated in Riemann's 1859 paper, it asserts that all the "non-obvious" zeros of the zeta function are complex numbers with real part  $1/2$ .



## Yang-Mills & the Mass Gap

Experiment and computer simulations suggest the existence of a "mass gap" in the solution to the quantum version of the Yang-Mills equations. But no proof of this property is known.

Fonte: Clay Mathematics Institute

# Equações de Navier–Stokes

As equações de Navier–Stokes descrevem o comportamento de escoamentos viscosos e constituem o modelo fundamental da dinâmica dos fluidos.

$$\rho \left( \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = -\nabla p + \nu \Delta \mathbf{u} + \mathbf{f}, \quad (1)$$
$$\nabla \cdot \mathbf{u} = 0.$$

Essas equações governam uma ampla classe de problemas em engenharia e ciências aplicadas.

# Problema de Kovasznay

Em regime incompressível, estacionário e bidimensional, sob hipóteses de fluido newtoniano e viscosidade constante, as Equações de Navier–Stokes se reduzem à

$$\begin{aligned} \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0 \\ \frac{\partial u^2}{\partial x} + \frac{\partial(uv)}{\partial y} &= -\frac{1}{\rho} \frac{\partial p}{\partial x} + \frac{1}{Re} \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \\ \frac{\partial v^2}{\partial y} + \frac{\partial(uv)}{\partial x} &= -\frac{1}{\rho} \frac{\partial p}{\partial y} + \frac{1}{Re} \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) \end{aligned} \quad (2)$$

Em que

$$Re = \frac{V_0 L_0}{\nu}$$

# Solução

Esse problema admite uma solução analítica fechada, obtida por Kovasznay em 1948.

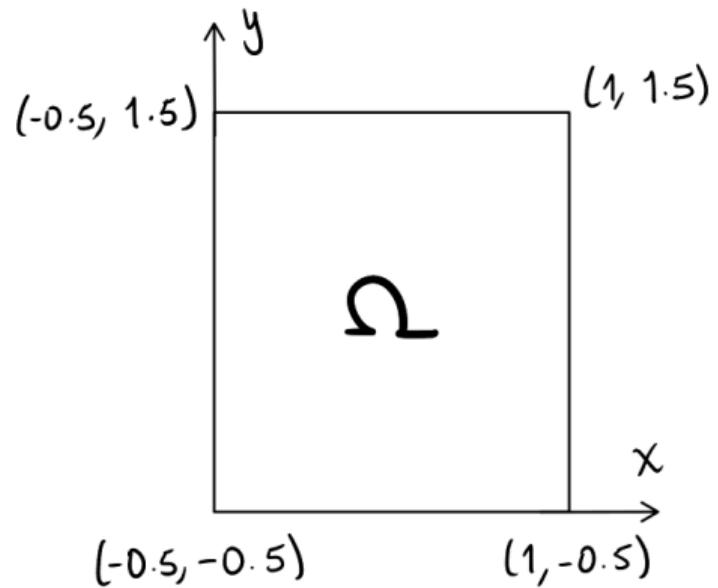
$$\begin{aligned}
 u(x, y) &= 1 - e^{\lambda x} \cos(2\pi y), \\
 v(x, y) &= \frac{\lambda}{2\pi} e^{\lambda x} \sin(2\pi y), \\
 p(x, y) &= \frac{1}{2} \left(1 - e^{2\lambda x}\right),
 \end{aligned} \tag{3}$$

Com

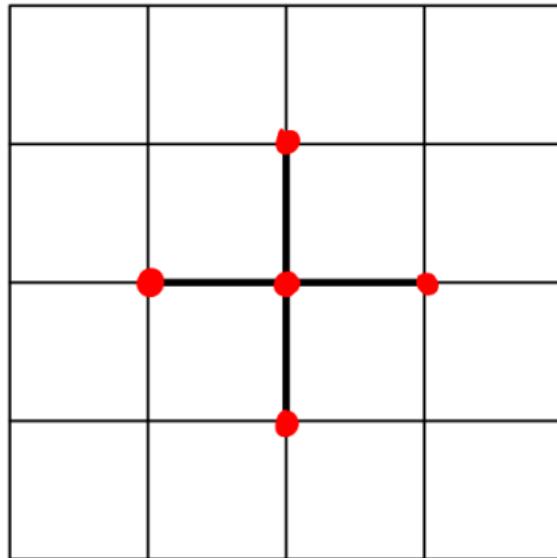
$$\lambda = \frac{Re}{2} - \sqrt{\left(\frac{Re}{2}\right)^2 + 4\pi^2}.$$

# Domínio e Condições de Contorno

- Domínio:  $\Omega = [-0.5, 1.0] \times [-0.5, 1.5]$ ;
- $\mathbf{u}(x, y) = \mathbf{u}_D(x, y), \quad (x, y) \in \partial\Omega$ ;
- $p(x, y) = p_D(x, y), \quad (x, y) \in \Gamma_{\text{out}}$ .



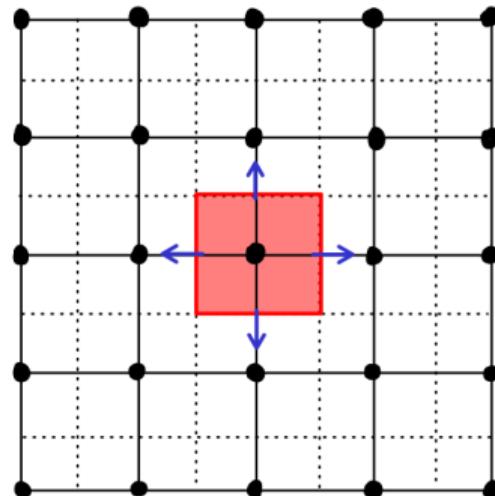
# Método das Diferenças Finitas (MDF)



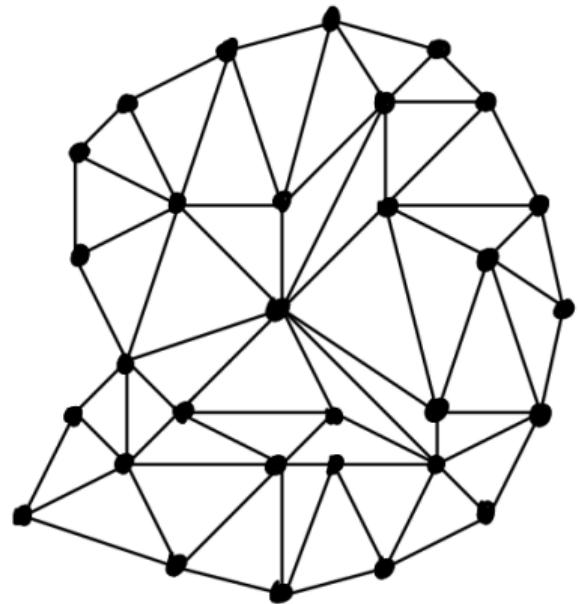
- Domínio discretizado em uma malha regular;
- Derivadas aproximadas por diferenças finitas;
- Fácil implementação em domínios simples.

# Método dos Volumes Finitos (FVM)

- Integração das equações em volumes de controle;
- Requer tratamento cuidadoso dos fluxos nas faces.



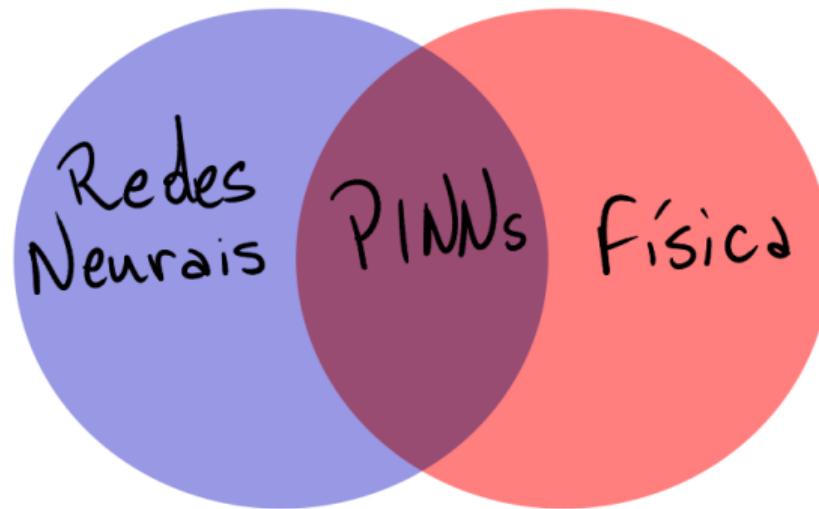
# Método dos Elementos Finitos (FEM)



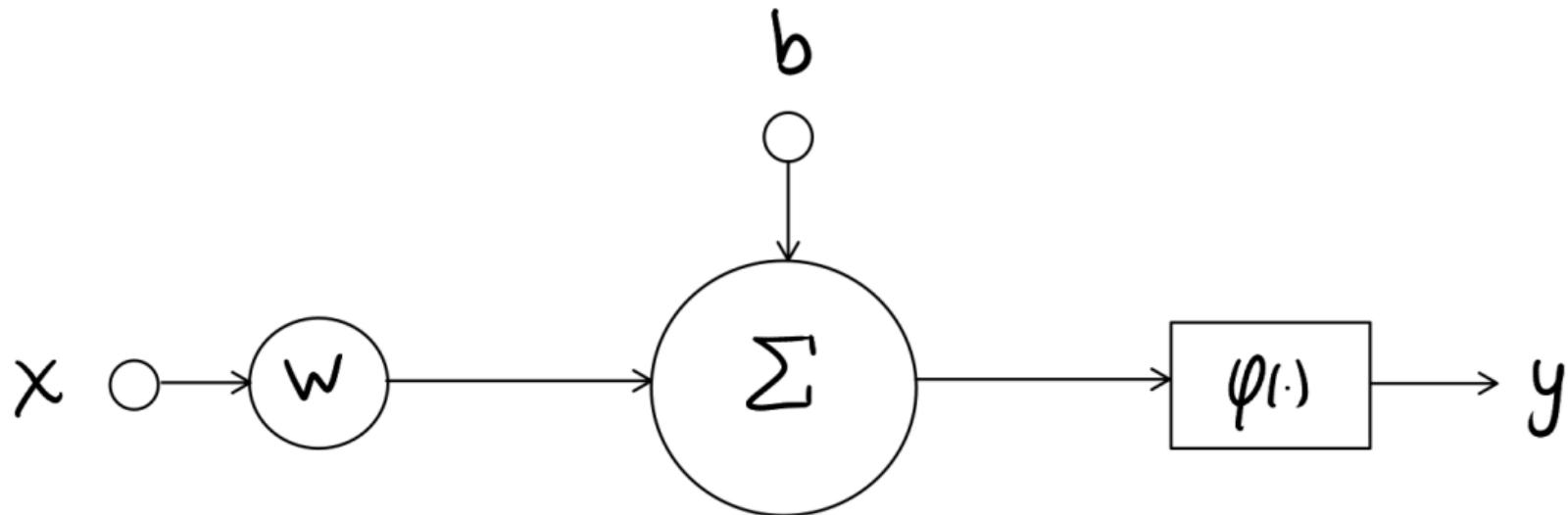
- Aproximação da solução por funções de base;
- Alta flexibilidade para geometrias complexas.

# O que são PINNs?

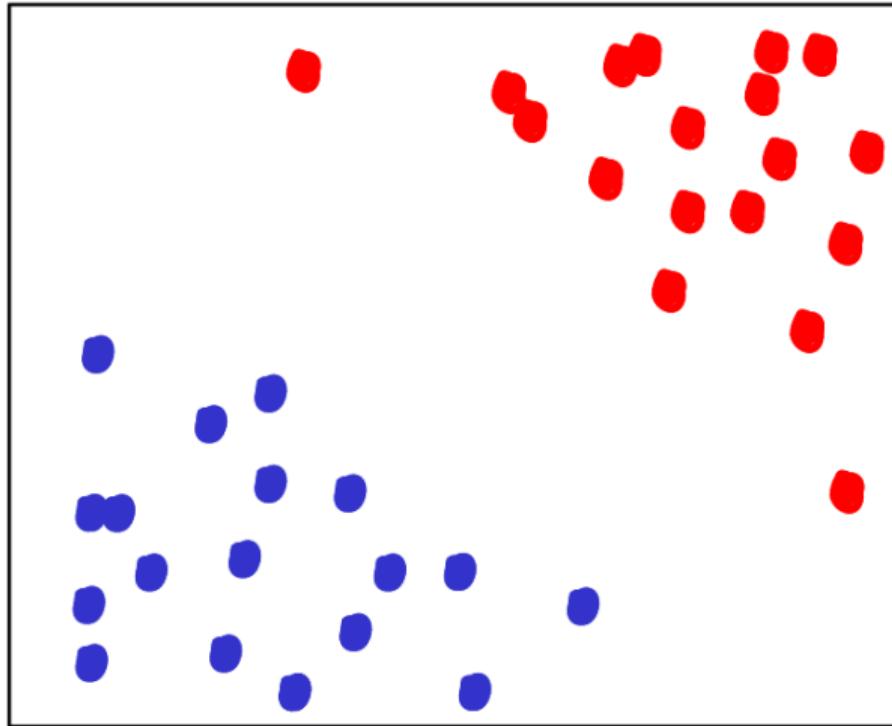
As Physics-Informed Neural Networks (PINNs) transformam um problema de EDP em um problema de otimização, usando uma rede neural que é penalizada por violar as equações físicas e as condições de contorno.



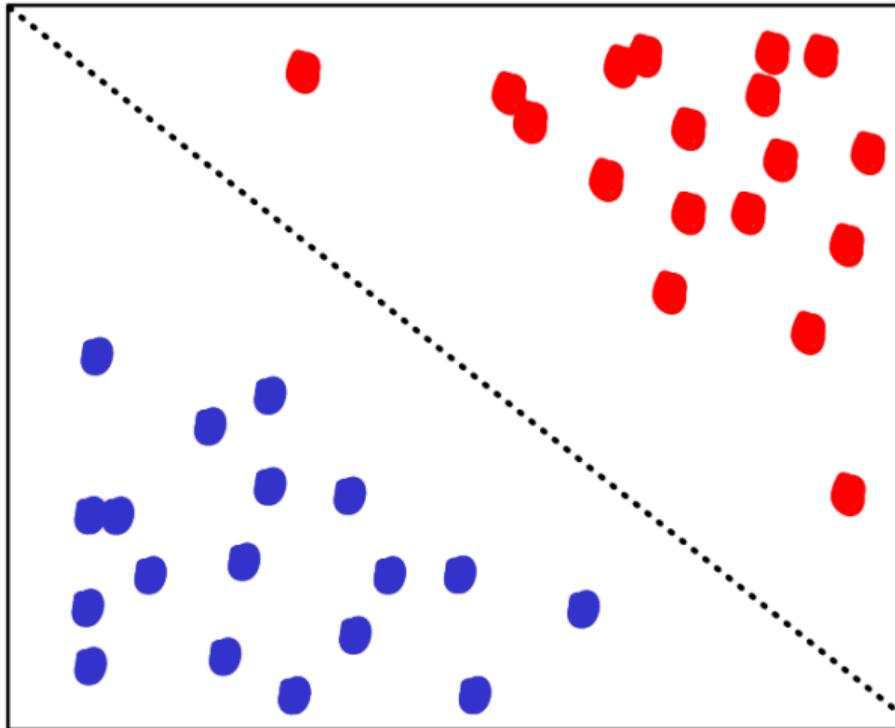
Beleza, mas o que são redes neurais?



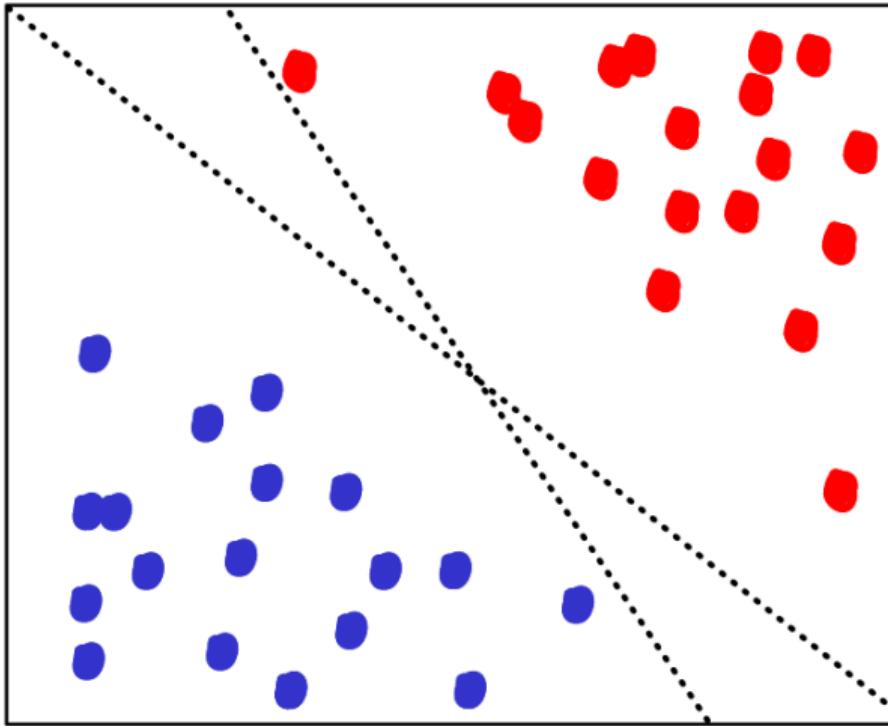
# Problema de Classificação



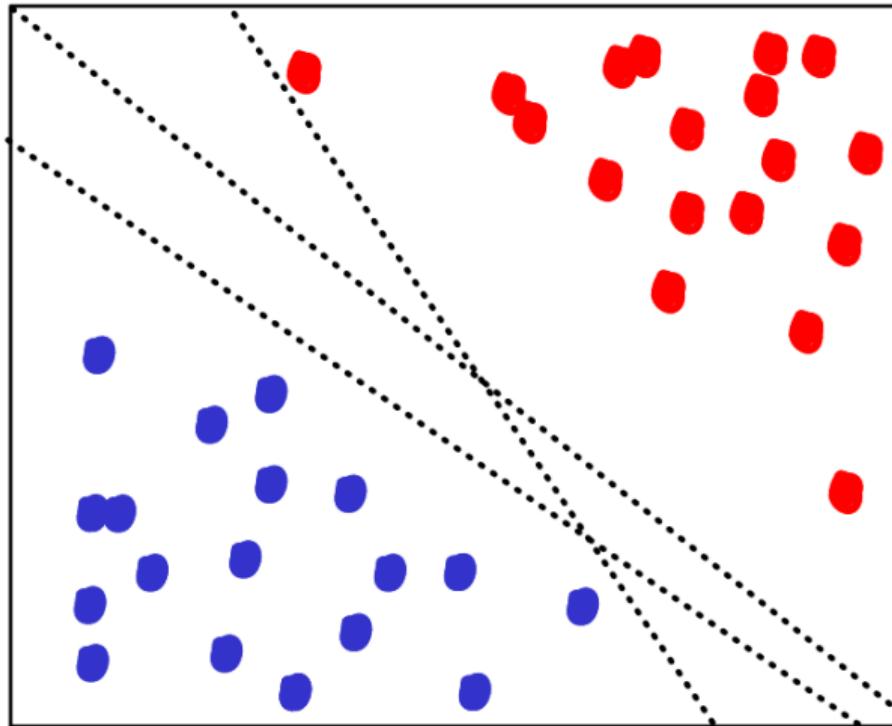
# Problema de Classificação



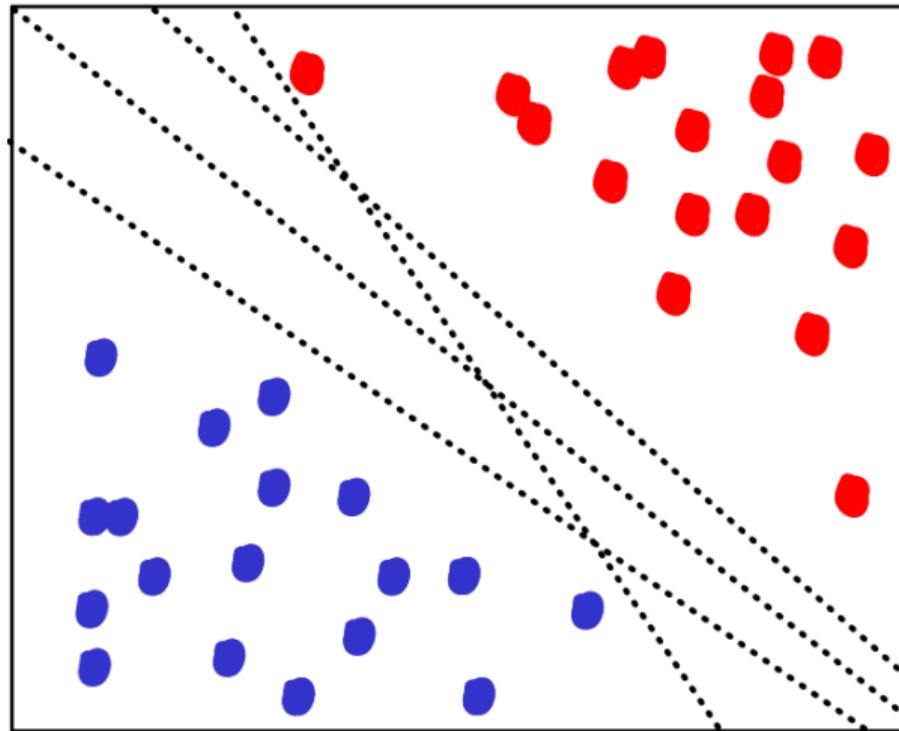
# Problema de Classificação



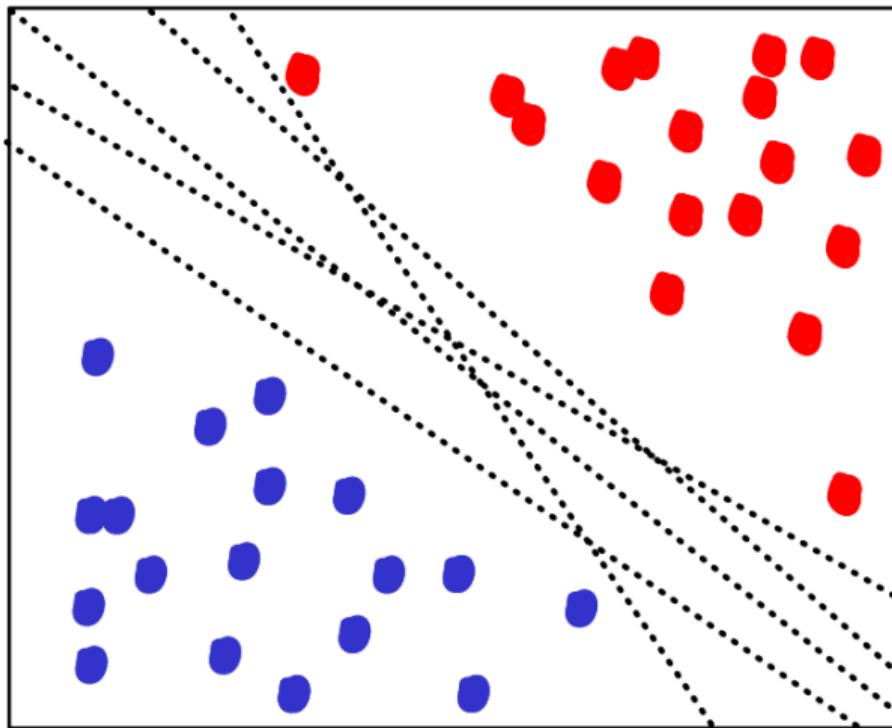
# Problema de Classificação



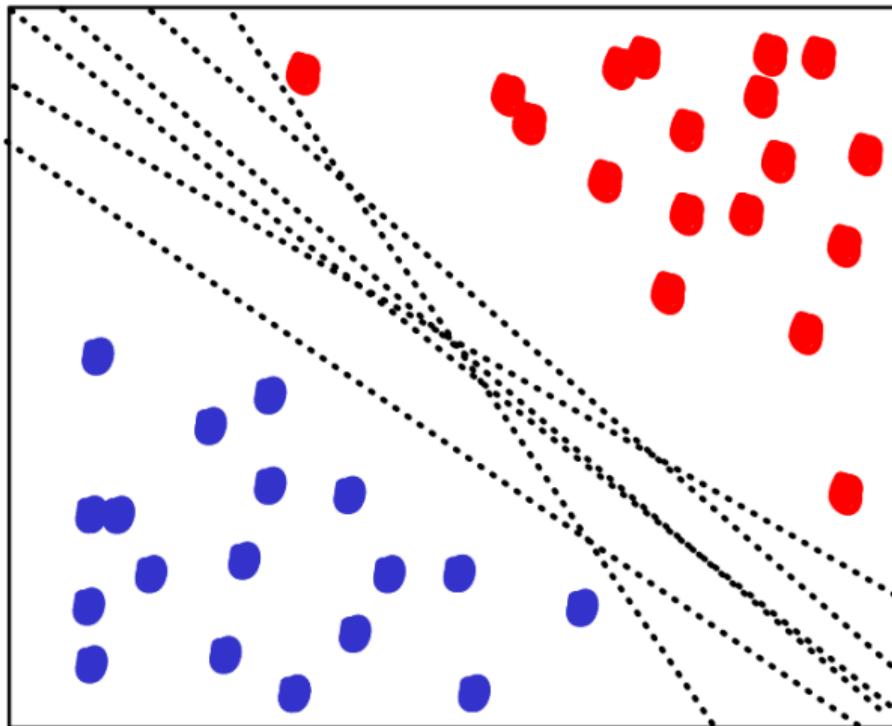
# Problema de Classificação



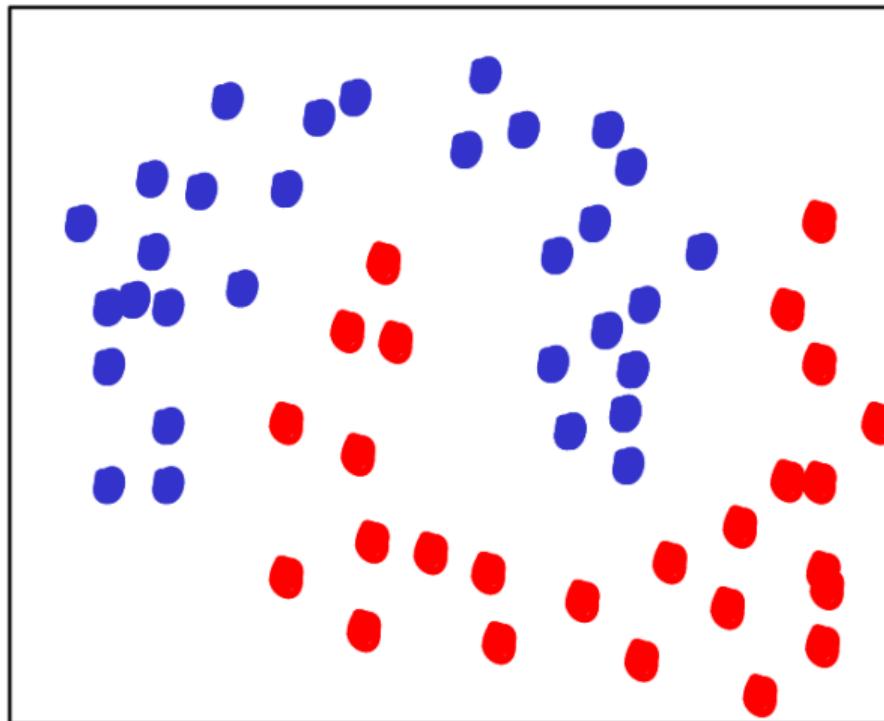
# Problema de Classificação



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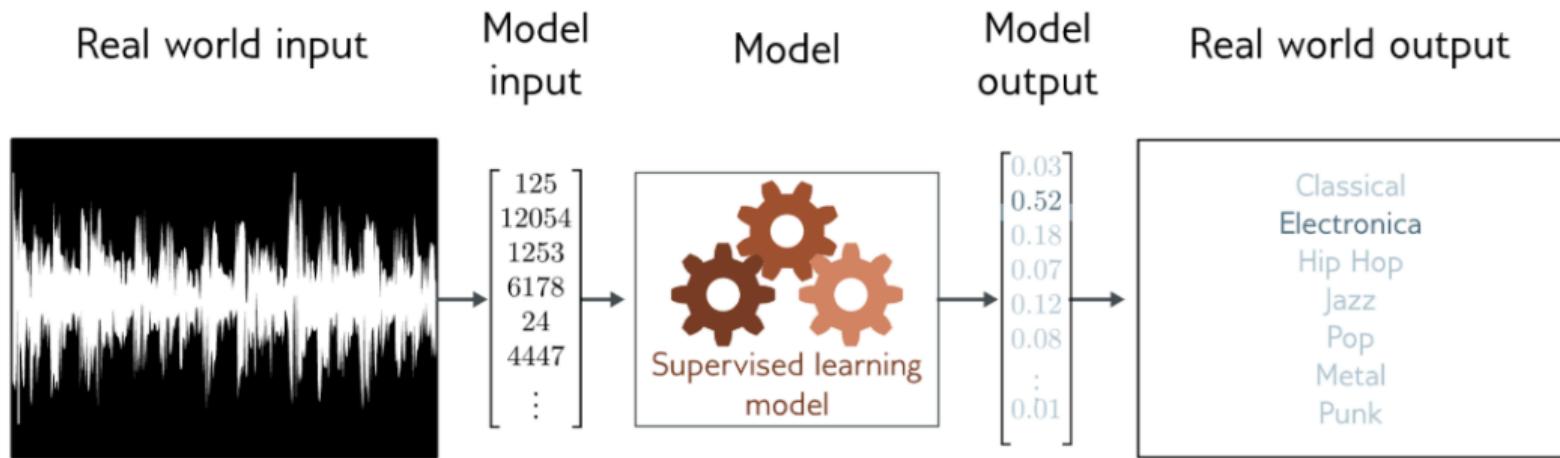
# Problema de Classificação



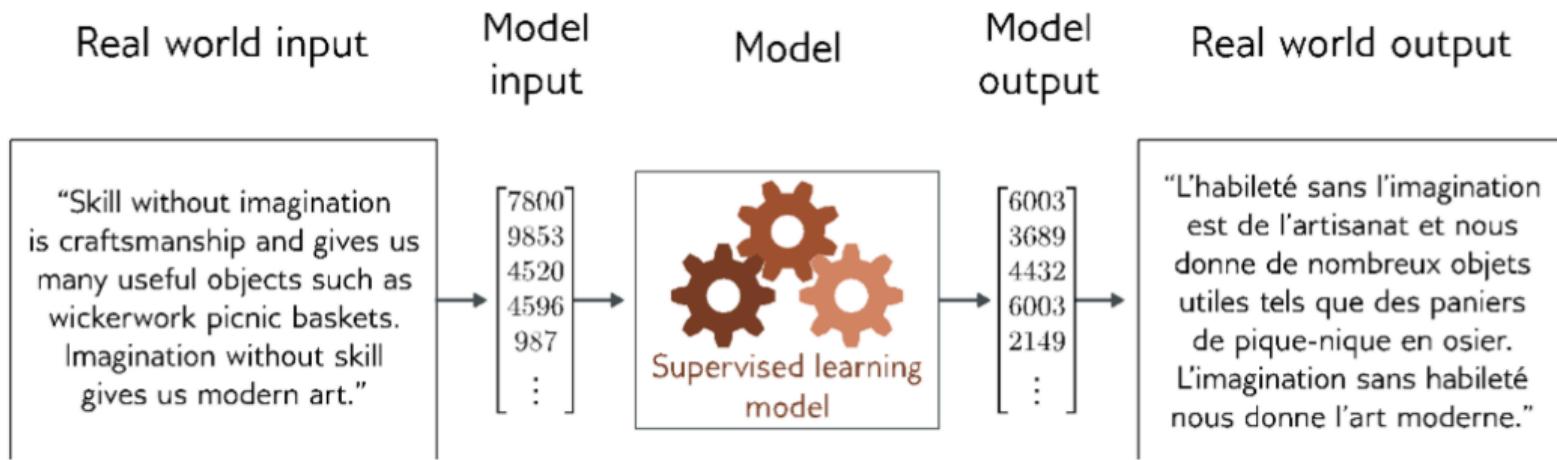
# Disclaimer

Os próximos slides são honestamente roubados do livro  
Understanding Deep Learning, de Simon Prince.

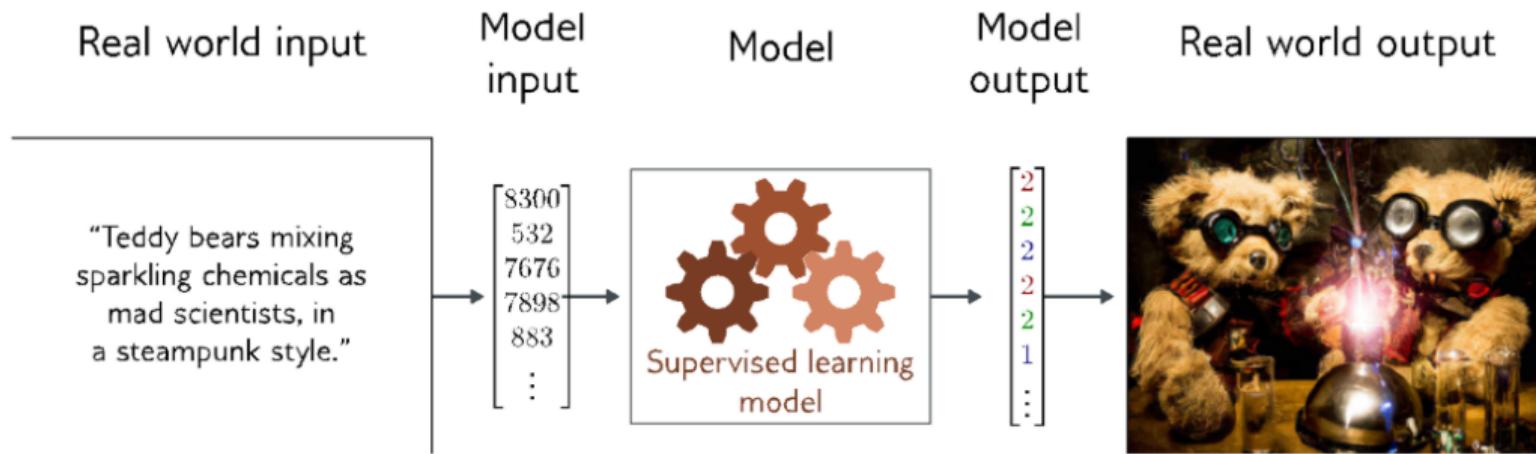
# Classificação de gênero de música



# Tradução



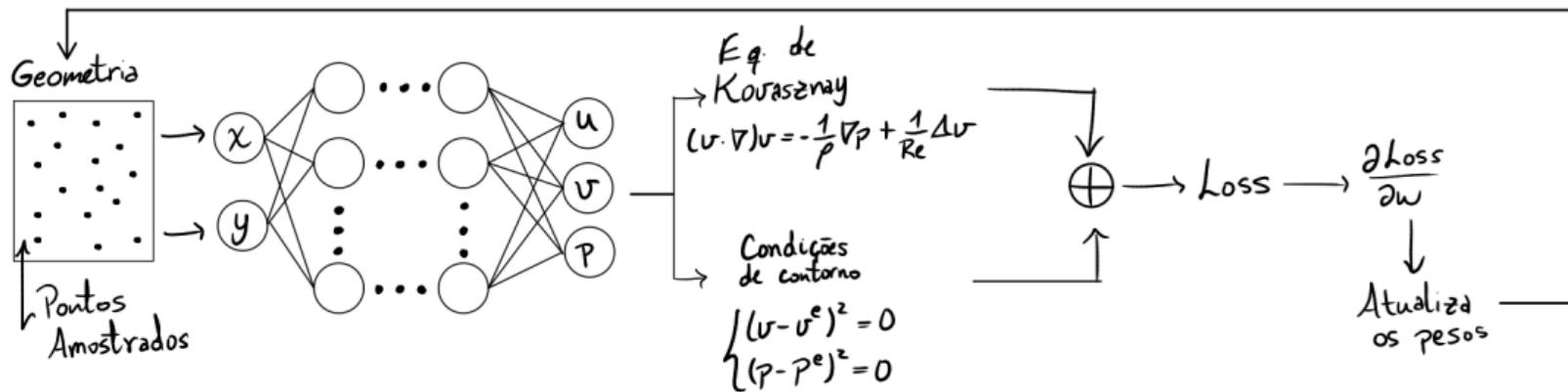
# Geração de imagens



# Uma PINNzinha aprendendo

Fonte: Ian Henderson, 2022.

# Arquitetura da nossa PINN



# Resíduos da EDPs

Resíduos das equações ficam da seguinte forma:

$$r_x = u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + \frac{\partial p}{\partial x} - \frac{1}{Re} \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right),$$

$$r_y = u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + \frac{\partial p}{\partial y} - \frac{1}{Re} \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right).$$

$$r_c = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}.$$

# Resíduos em $\partial\Omega$

Já os resíduos das condições de contorno ficam:

$$r_u = u_{RN} - u_D,$$

$$r_v = v_{RN} - v_D,$$

$$r_p = p_{RN} - p_D.$$

## Loss function

A loss function da PINN é definida como a soma dos erros quadráticos médios (MSE) dos resíduos das EDPs e das condições de contorno:

$$\mathcal{L} = \mathcal{L}_{\text{EDP}} + \mathcal{L}_{\text{CC}},$$

onde

$$\mathcal{L}_{\text{EDP}} = \frac{1}{n} \sum_{i=1}^n (r_x^2 + r_y^2 + r_c^2), \quad \mathcal{L}_{\text{CC}} = \frac{1}{n} \sum_{i=1}^n (r_u^2 + r_v^2 + r_p^2).$$

# Treinamento

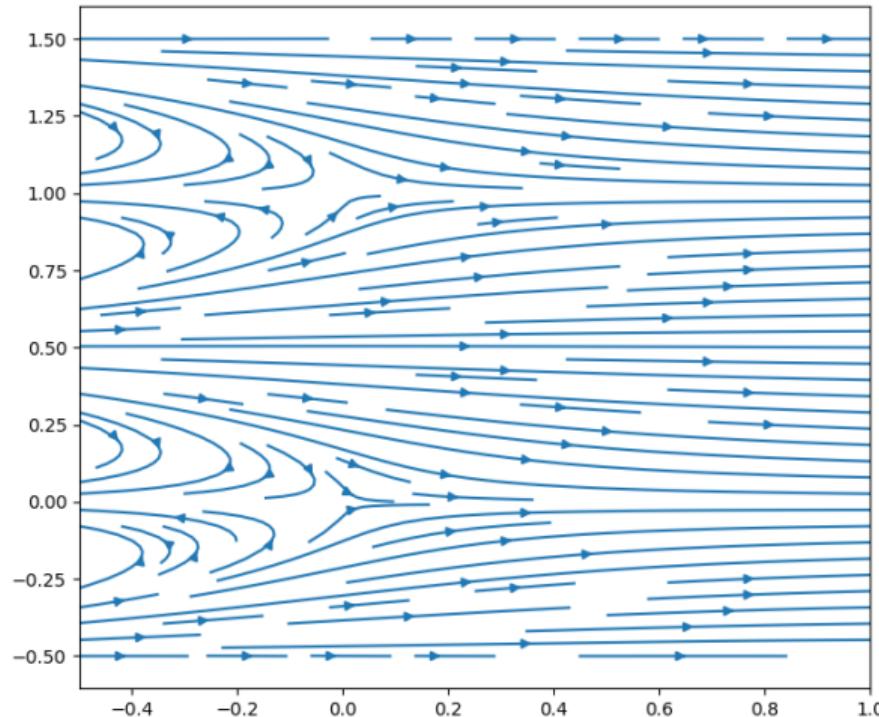
No treinamento, foram usados os otimizadores ADAM e L-BFGS.

Fonte: Kaivalya Tota, 2020.

# Configurações de Treinamento

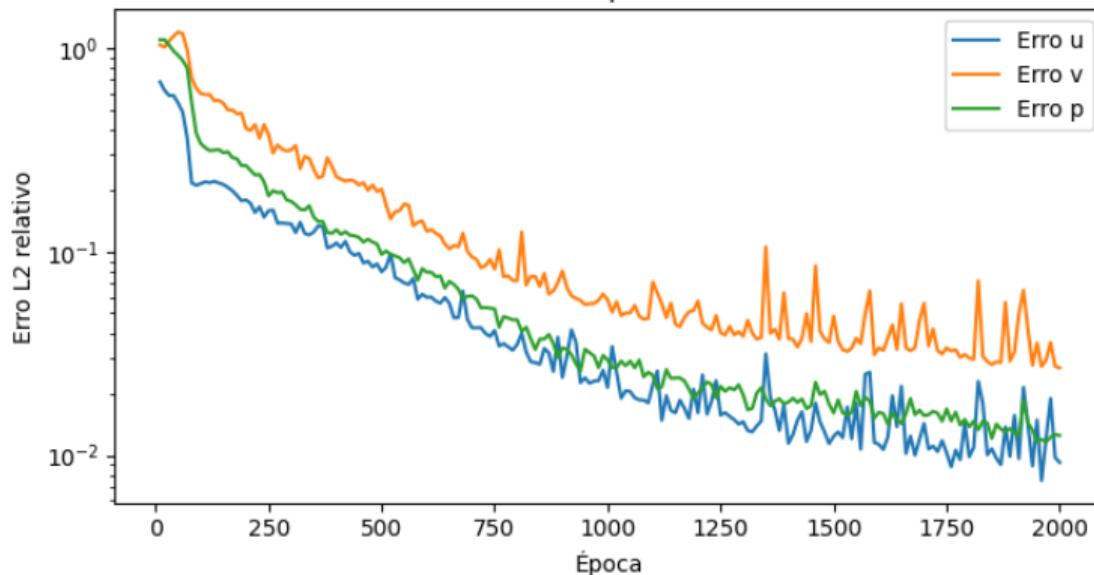
- Domínio:  $\Omega = [-0.5, 1.0] \times [-0.5, 1.5]$ ;
- Pontos internos:  $n_\Omega = 2601$ ;
- Pontos de contorno:  $n_{\partial\Omega} = 400$ ;
- Pontos de teste:  $n_{\text{test}} = 10000$ ;
- Arquitetura:  $[2, 50, 50, 50, 50, 3]$ ;
- Função de ativação:  $\tanh$ ;
- Otimizadores:
  - Adam ( $\eta = 10^{-3}$ , com 2000 iterações);
  - L-BFGS.

# Streamplot do Escoamento

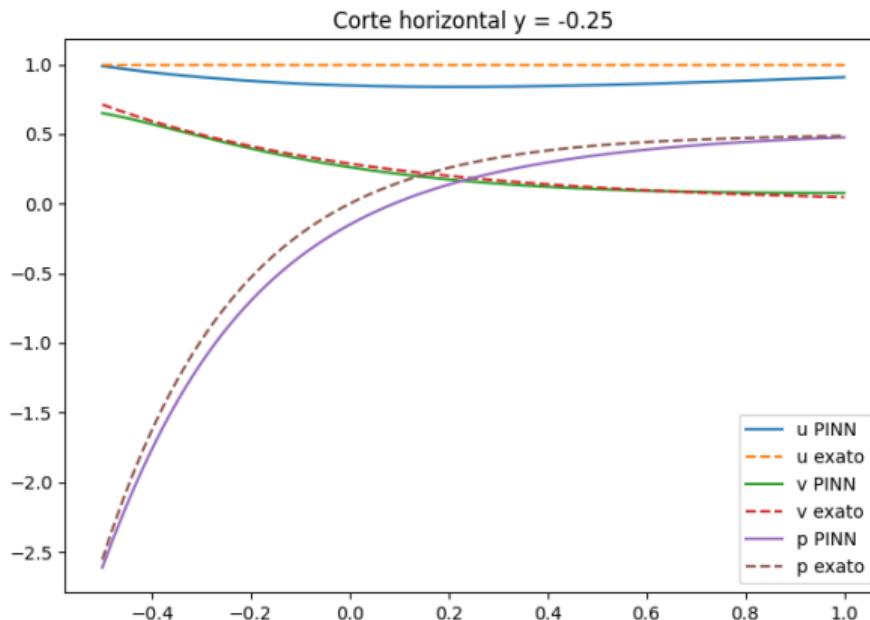


# Erro relativo

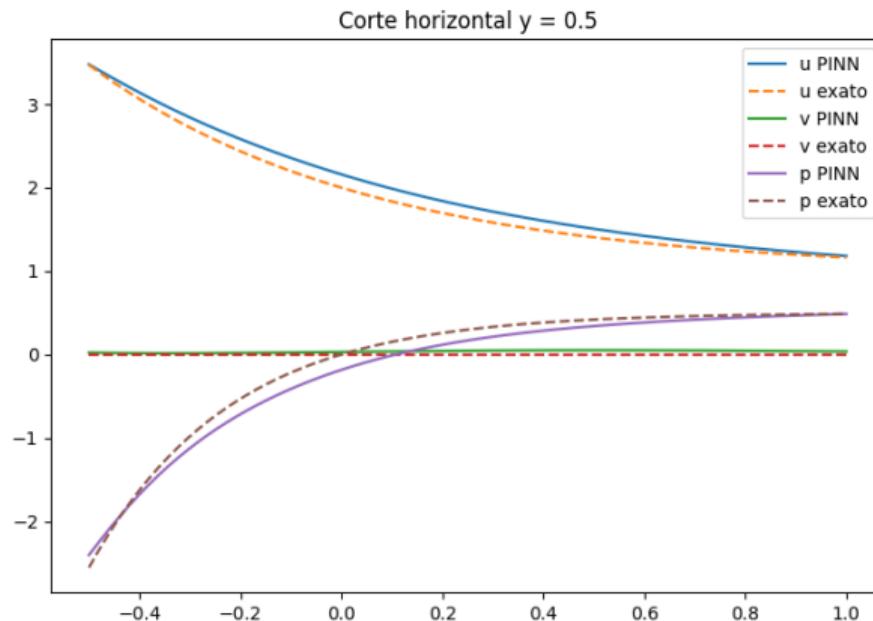
Curva de aprendizado

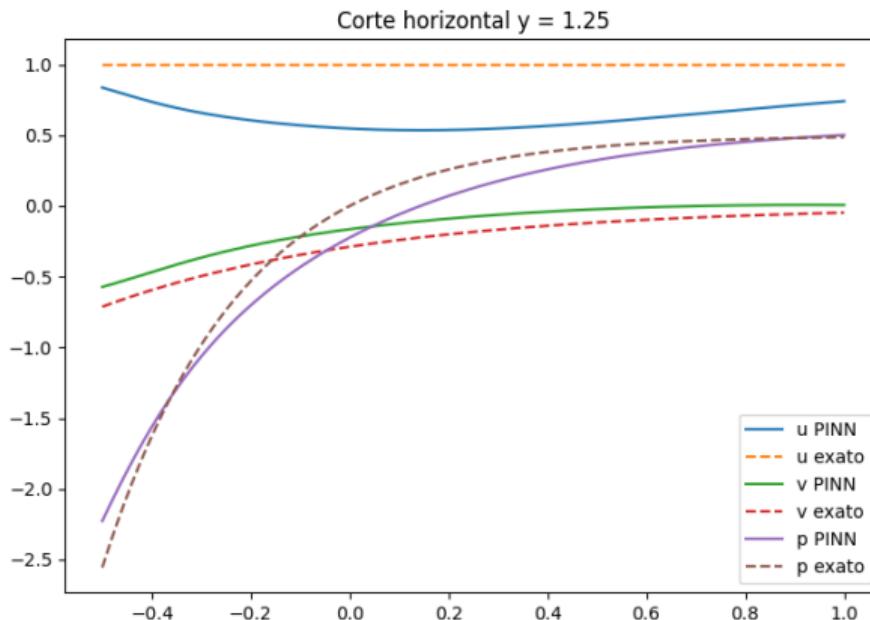


Corte em  $y = -0.25$

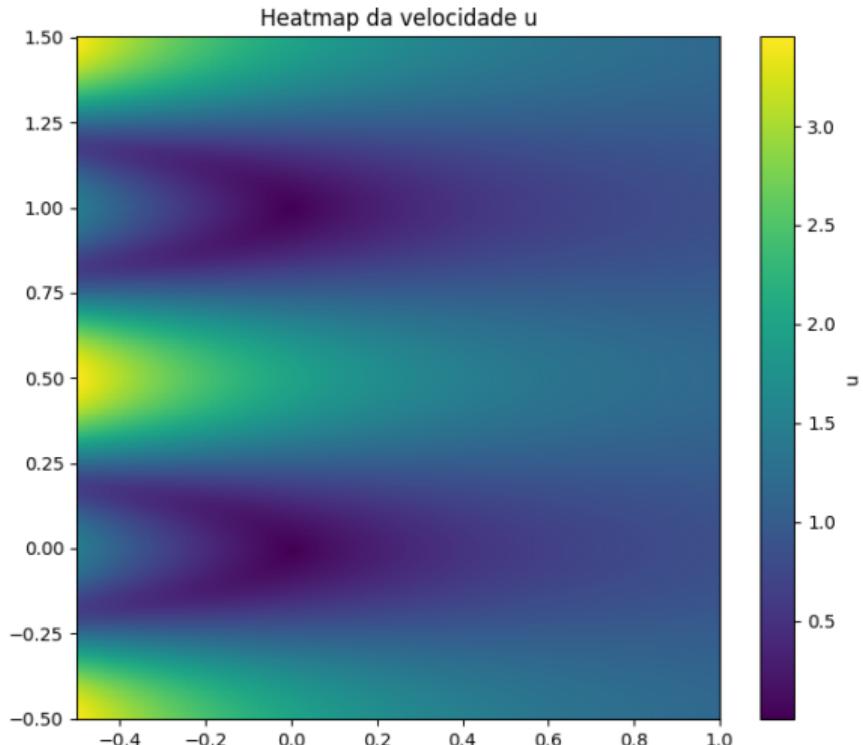


# Corte em $y = 0.5$

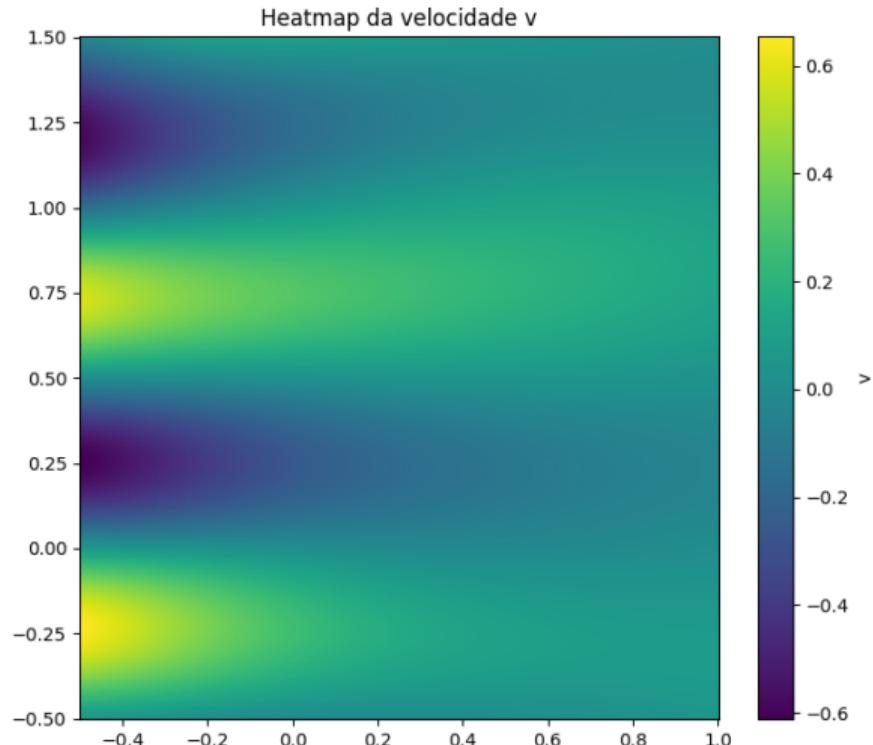


Corte em  $y = 1.25$ 

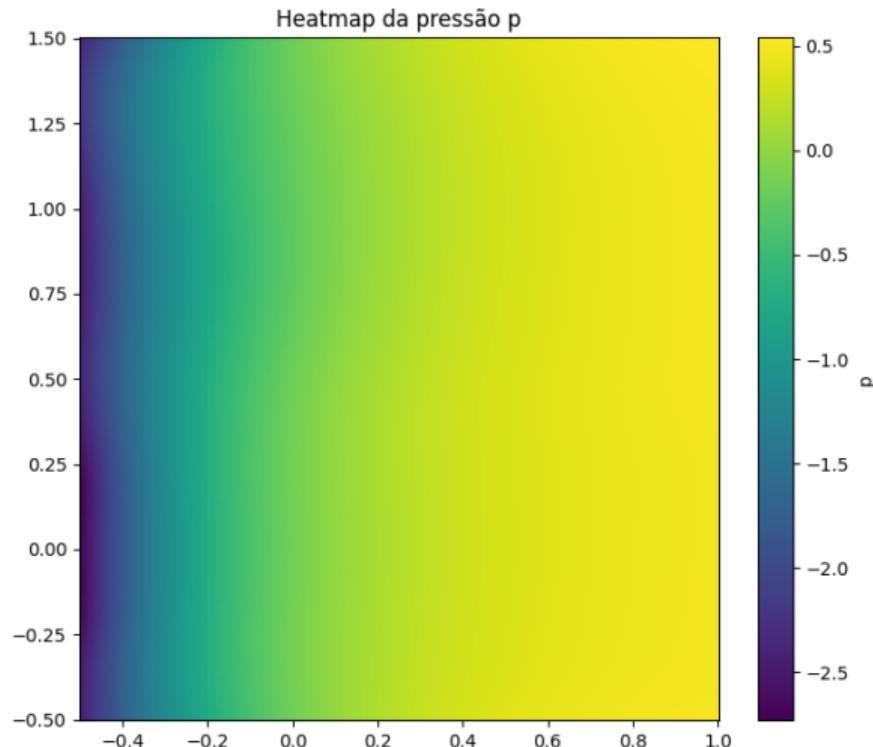
# Mapa de calor de $u$



# Mapa de calor de v



# Mapa de calor de p



# PINN aprendendo o streamplot do escoamento

# Bolinhas se mexendo

:)

*Obrigado!*

# Algumas referências

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