Kalman filter to solve a Linear Dynamical System

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Jupyter Notebook

1 Introduction

A state space model or SSM can be written in the following generic form:

$$z_t = g(u_t, z_{t-1}, \epsilon_t) \tag{1}$$
$$y_t = h(z_t, u_t, \delta_t) \tag{2}$$

where z_t is the hidden state, u_t is an optional input or control signal, y_t is the observation, g is the transition model, h is the observation model, e_t is the system noise at time t, and δ_t is the observation noise at time t. The primary goals in using SSMs is to recursively estimate the belief state, $p(z_t|y_{1:t}, u_{1:t}, \theta)$, where θ are the parameters of the model and is known.

An important special case of an SSM is where all the CPDs are linear-Gaussians. In other words, we assume:

$$\mu_{t,t-1} \triangleq A_t \mu_{t-1} + B_t u_t \tag{3}$$
$$\Sigma_{t,t-1} \triangleq A_t \Sigma_{t-1} A_t^T + Q_t \tag{4}$$
$$z_t = A_t z_{t-1} + B_t u_t + \epsilon_t \tag{5}$$
$$y_t = C_t z_t + D_t u_t + \delta_t \tag{6}$$
$$e_t \sim \mathcal{N}(0, Q_t) \tag{7}$$
$$\delta_t \sim \mathcal{N}(0, R_t) \tag{8}$$

This model is called a linear-Gaussian SSM (LG-SSM) or a linear dynamical system (LDS). If the parameters $\delta_t = (A_t, B_t, C_t, D_t, Q_t, R_t)$ are independent of time, the model is called stationary. In this case we can estimate the posterior mean and covariance of the hidden variable using the Kalman filter where:

$$\mu_{t,t-1} \triangleq A_t \mu_{t-1} + B_t u_t \tag{9}$$
$$\Sigma_{t,t-1} \triangleq A_t \Sigma_{t-1} A_t^T + Q_t \tag{10}$$
$$r_t \triangleq y_t - \hat{y}_t \tag{11}$$
$$\hat{y}_t \triangleq C_t \mu_{t|t-1} + D_t u_t \tag{12}$$
$$K_t \triangleq \Sigma_{t|t-1} C_t^T S_t^{-1} \tag{13}$$
$$S_t \triangleq C_t \Sigma_{t|t-1} C_t^T + R_t \tag{14}$$
$$\mu_t = \mu_{t|t-1} + K_t r_t \tag{15}$$
$$\Sigma_t = (I - K_t C_t) \Sigma_{t|t-1} \tag{16}$$

2 Implementation

Consider an object to a constant velocity in a 2D plane, let $z \in \mathbb{R}^4$ the vector with the positions and velocities and $y \in \mathbb{R}^2$ the observed location of the object. Our "best guess" about the location of the object is the posterior mean, denoted as a red cross in Figure 1. Our uncertainty associated with this is represented as an ellipse, which contains 95% of the probability mass. We see that out uncertainty fall down over time, as the effects of the initial uncertainty. We also see that the estimated trajectory has "filtered out" some of the noise.

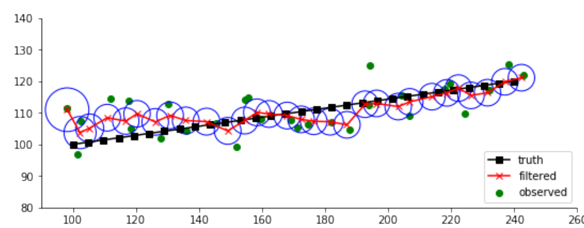

Figure 1: Illustration of Kalman filtering. Ground truth (black squares) are generated by an object moving in a 2D plane to a constant velocity. Observations (green circles) are generated by applying Gaussian noise to the ground truth. Red cross is the posterior mean, blue circles are 95% confidence ellipses derived from the posterior covariance.