Assignment 1

Biomedical Imaging (BIOENG 1340) - Fall 2020

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Instructions: Please show your solutions to each problem in full, explaining your approach clearly and including plots (as needed) to justify your answers. For computer programs, please remember to turn in your code through the course's Blackboard website, in addition to your report containing plots / figures which are requested. If you have collaborated with another student on solving this homework assignment please state so (e.g. "I helped John with question 1").

This assignment is due on **Thursday, 4 Sept 2020** via Canvas, including an MS Word Document "report" or scanned PDFs of hand-written ones for the written explanations associated with each question in the assignment, as well as any associated code and result files which are required to be submitted.

NOTE: As discussed in class, only Q3 in this assignment is Scored whereas it is recommended to ensure that you can solve Q1 and Q2 based on what we discussed in Class.

LEARNING GOALS:

- Working with Vectors & Matrices in Matlab.
- Building and solving a basic set of linear equations in Matlab to estimate parameters of a linear system operating on a 1D signal.

BASIC LINEAR ALGEBRA & APPLIED PROGRAMING

1. **(SELF ASSESSMENT)** Consider the arbitrary signal f[x], with k = 1 to N observations, and a linear signal model v[x] = αx + β, with α & β being constants to be determined so that v[x] fits f[x] in the least squares sense i.e. min | (v[x] – f[x]) ^{2} | = 0.

   a. Please draw up what you understand is the Linear Time Invariant Systems Block-diagram Model view of this problem, wherein the System is defined as the v[x] = αx + β linear signal model.
   b. Write down the algebraic matrix formulation of this fitting problem, starting with a system of equations defining the fitting problem, given

the linear signal model and the N observations. Note that this Linear Algebra / Matrix form of the system is the form which will help you solve for the parameters of the System i.e. α & β, and should something like A k = v, where $k = [\alpha, \beta]^T$, elaborating on the contents of each term, assuming "N" discrete observations / recordings i.e. (x_k, $f[x_k]$) for k = 1 to N.

c. Write a Matlab program to compute the least squares fit of the model to the data stored in the i.e. '**HW1_Q1.mat**' contains a signal (data stored in the variable '**f**' such that the data constitutes, (x_k, $f[x_k]$) data-points for fitting), assuming that A\b in Matlab solves the least-squares fitting problem. Plot the raw data as well as the best fit line, $v[x] = \alpha x + \beta$ found (overlaid on top of each other). Remember to upload your code through Blackboard.

d. Compare your solution to the fitting problem using A\b in Matlab to the solution for $x = (A^TA)^{-1} A^T b$. Note that here "$^{-1}$" indicates the inverse operator and "^{T}" indicates the matrix transpose. Plot the best fit line, $v[x] = \alpha x + \beta$, and compare it against the solution obtained in part (a). Note that as per rules of matrix multiplication, $(A^TA)^{-1} A^T b$ must be evaluated by multiplying matrices from "right to left" i.e. first compute $b_1 = A^T b$ and then multiply the result with $(A^TA)^{-1} b_1$.

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2. (SELF ASSESSMENT) Repeat problem 1a to 1c but now instead considering a quadratic model $v[x] = \alpha x^2 + \beta x + \gamma$, with α, β and γ being constants to be determined so that v[x] fits f[x] in the least squares sense i.e. min $| (v[x] - f[x])^2 | = 0$. You may just use the "x = A\b" concept in Matlab to solve this system and submit your code and a screenshot of the final fitted model plotted over the scatter plot of the given (x_k, $f[x_k]$) data.

NOTE: You may use the code provided in class this week, as starter code. Please pay special attention to the Matlab usage of the "\" operator to divide a Matrix by a vector and solve an Ax=b type linear system as well as the dimensions of A, x and b in order for them to constitute a linear system which is compatible for matrix multiplication. **Please submit snapshots of any plots which you generate in addition to your code.**

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3. **(50 points)** For a linear fitting problem, show that ($A^T A$) x = A^T b has an algebraic equivalent which can derived from the sum of least-squares minimization problem, given N data points of (x_k, f[x_k]) i.e. k = 1 to N, to minimize a functional Q[x] to zero, where Q[x] is given by the following:

$$Q[x] = \sum_{k=0}^{n} (v[x_k] - (\alpha\, x_k + \beta))^2$$

As discussed in class, please create a matrix 2x2 matrix system to solve for x_k starting with $\frac{\partial Q}{\partial \alpha}$=0 and $\frac{\partial Q}{\partial \beta}$=0. Prove that this algebraic equivalent turns out to be the following, showing all steps:

$$\begin{pmatrix} \sum_{k=1}^{N} {x_k}^2 & \sum_{k=1}^{N} x_k \\ \sum_{k=1}^{N} x_k & \sum_{k=1}^{N} 1 \end{pmatrix} \binom{\alpha}{\beta} = \begin{pmatrix} \sum_{k=1}^{N} x_k\, v[x_k] \\ \sum_{k=1}^{N} v[x_k] \end{pmatrix}$$

where, $\begin{pmatrix} \sum_{k=1}^{N} {x_k}^2 & \sum_{k=1}^{N} x_k \\ \sum_{k=1}^{N} x_k & \sum_{k=1}^{N} 1 \end{pmatrix}$ is the same as ($A^T A$) and $\begin{pmatrix} \sum_{k=1}^{N} x_k \\ \sum_{k=1}^{N} x_k \end{pmatrix}$ is the same as A^T b.