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AN AGGREGATION THEOREM FOR SECURITIES MARKETS*

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Alternative sets of sufficient conditions are developed under which equilibrium security rates of return are determined as if there exist only identical individuals whose resources, beliefs, and tastes are a composite of the actual individuals in the economy. These conditions include as special cases all those previously examined in the literature (including conditions sufficient to produce the two-parameter mean-variance model), as well as others. Whenever such a composite individual exists it is shown that (1) valuation equations take a specific form and contain only exogenous parameters of the economy; (2) market exchange arrangements are Pareto-optimal; and (3) competitive value-maximizing firms make completely specified Pareto-optimal production decisions both over dates and states. These results rely on the observation that under popular homogeneity assumptions regarding beliefs and tastes, even though the securities market may be incomplete, equilibrium rates of return are determined as if there were an otherwise similar Arrow–Debreu economy.

1. Introduction

The chief difficulty befouling the analysis of securities market equilibrium is the problem of aggregation. When individual decisions are aggregated to form equilibrium relationships, individual-specific choice information must be eliminated and replaced by exogenous parameters of the economy. Solution of the aggregation problem then leads to closed-form valuation equations and theorems on the optimality of market exchange arrangements and production decisions.

This paper demonstrates that valuation equations and optimality theorems are quite easy to derive if the aggregation problem can be solved first. In particular, it is shown that whenever an individual can be constructed whose resources, beliefs, and tastes are a composite of the actual individuals in the economy, then (1) valuation equations take a specific form and contain only exogenous parameters of the economy (section 4); (2) market exchange arrangements are Pareto-optimal (section 5); and (3) competitive value-maximizing firms make completely specified Pareto-optimal production decisions both over dates and states (section 6). Section 2 describes the economy, and section 3 develops alternative sets of sufficient conditions under which a composite individual may be constructed. These include as special cases almost all published

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conditions for which the aggregation problem has been solved, as well as others.[1]

The aggregation theorem relies on the observation that under certain conditions, even though the securities market may be incomplete, the equilibrium rates of return will be determined as if there were an otherwise similar complete (Arrow–Debreu) market. For example, these conditions include those sufficient to produce the two-parameter mean-variance equilibrium model. Therefore, in a fundamental sense, these incomplete market models are a special case of an Arrow–Debreu economy. When viewed in this light, it is not surprising that the optimality features of the Arrow–Debreu economy carry over to an appropriately specified incomplete market.

2. The economy

2.1. General markets

Each individual in a perfect and competitive economy allocates his present wealth W_0 among present consumption C_0 and the remainder $W_0 - C_0$ to risk-free and many risky securities. α denotes the proportion of $W_0 - C_0$ he allocates to risk-free securities, which earn rate of return r_F. β_j denotes the proportions of the remainder $(W_0 - C_0)(1-\alpha)$ he allocates to each of J ($j = 1, 2, \ldots, J$) risky securities so that $\sum_j \beta_j = 1$. Each security j earns rate of return r_{je} if each of E ($e = 1, 2, \ldots, E$) states occurs. Consequently, if state e occurs his future wealth

$$W_{1e} = (W_0 - C_0)\,(1 + \alpha r_F + (1-\alpha) \sum_j \beta_j r_{je}).$$

Each individual is assumed to obey the Savage axioms of rational choice: He has beliefs and tastes representable by probabilities and a utility function so that he maximizes his expected utility. Let π_e denote the subjective probability he attaches to state e and let $U(C_0) + \rho V(W_{1e})$ denote his additive utility function over present consumption and future wealth, where ρ is a constant denoting his rate of patience and U and V are functions. Assume $U' > 0$, $U'' < 0$, $\rho > 0$, $V' > 0$, and $V'' < 0$.

In brief, the individual solves the following programming problem:

$$\max_{C_0,\, \alpha,\, \{\beta_j\}} U(C_0) + \rho \sum_e \pi_e V[(W_0 - C_0)(1 + \alpha r_F + (1-\alpha) \sum_j \beta_j r_{je})] - \lambda[\sum_j \beta_j - 1],$$

[1]For example, Mossin (1973, pp. 68–73) has solved the aggregation problem under conditions of homogeneous beliefs and quadratic utility and Lintner (1969) under conditions of exponential utility and normal probability assessments. Wilson's (1968) classic paper 'The theory of syndicates' derives these as well as other solutions causing section 3 of this paper to be somewhat redundant. However, by way of contrast, the aggregation theorem of this paper is cast within an explicit securities market equilibrium context, considers choice of both present consumption and future wealth, and is demonstrated by more transparent proofs. See also Wilson (1967) for a multiperiod analysis of the aggregation problem under conditions of time-additive exponential utility. Hakansson (1970) also has developed results that overlap to some extent those of sections 3 and 5 of this paper.

where λ is a Lagrangian multiplier. Differentiating partially with respect to each choice variable, the necessary and sufficient conditions for an optimum are

$$U'(C_0) = (1+r_F)\rho[\sum_e \pi_e V'(W_{1e})], \tag{1}$$

$$\sum_e \pi_e V'(W_{1e})(r_{je}-r_F) = 0, \qquad \text{(all } j\text{)}, \tag{2}$$

$$W_{1e} = (W_0-C_0)(1+\alpha r_F+(1-\alpha)\sum_j \beta_j r_{je}), \qquad \text{where } \sum_j \beta_j = 1, \quad \text{(all } e\text{)}. \tag{3}$$

To provide for the endogenous determination of security rates of return, we append closure conditions that effectively take production decisions across dates and states as given. Let C_0^M and $\{S_{j1e}\}$ be respectively the aggregate present consumption and the aggregate future dollar return for each of the J risky securities if state e occurs. Consequently, if the population of the economy is I ($i = 1, 2, \ldots, I$), then

$$\sum_i C_0^i = C_0^M, \tag{4}$$

$$\sum_i (W_0^i - C_0^i)(1+r_F)\alpha^i = 0, \tag{5}$$

$$\sum_i (W_0^i - C_0^i)(1+r_{je})(1-\alpha^i)\beta_j^i = S_{j1e}, \qquad \text{(all } j \text{ and } e\text{)}. \tag{6}$$

In equilibrium, these three closure conditions must hold for the optimum choices of $(C_0^i, \alpha^i, \{\beta_j^i\})$ for all i, along with conditions (1)–(3) which must hold for each individual i.

2.2. Separation

In general, each decision variable at the individual level will depend on all parameters of the model (W_0, r_F, $\{r_{je}\}$, $\{\pi_e\}$, U, ρ, V). Consequently, the consumption C_0, leverage α and diversification $\{\beta_j\}$ decisions must be made simultaneously. However, there is a class of utility functions $V(W_{1e})$ for which these decisions are to some extent separable. In particular, consider the class of utility functions satisfying the differential equation

$$-V'(W_{1e})/V''(W_{1e}) = A + BW_{1e}, \tag{7}$$

where A and B are fixed parameters. This equation has three solutions[2] depending on the value of B,

(a) $\quad V(W_{1e}) \sim -A\exp(-W_{1e}/A), \qquad (B = 0),$

(b) $\quad V(W_{1e}) \sim \ln(A + W_{1e}), \qquad (B = 1),$

(c) $\quad V(W_{1e}) \sim \dfrac{b}{1-b}(A+BW_{1e})^{1-b}, \qquad (B \neq 0, 1),$

[2]The solutions listed are those possibly consistent with $V' > 0$ and $V'' < 0$. Required for this consistency in all cases $A \geqq 0$. Moreover, for functions (a), $A > 0$, and for functions (c), if $A = 0$, then $B > 0$ and if $B = -1, -\frac{1}{2}, -\frac{1}{3}, \ldots$, then $-A/B > W_{1e}$.

where $b \equiv B^{-1}$ and $\sim$ means 'is equivalent up to an increasing linear transformation to'. This class of utility functions is quite rich, containing as special cases: the constant absolute risk aversion function ($B = 0$), the constant proportional risk aversion function ($A = 0$), quadratic utility ($B = -1$), and a cubic utility function ($B = -\frac{1}{2}$).

Although other separation results for this class of utility functions can be derived,[3] we shall only need the following theorem:

Theorem (Universal portfolio separation). *If the economy is composed of individuals with the same beliefs and taste parameters B, then the optimal portfolio of risky securities $\{\beta_j\}$ will be the same for all individuals.*

Proof. Since elements of this proof have appeared in Pye (1967), only an outline will be given. When $B \neq 0$, $V'(W_{1e}) = (A+BW_{1e})^{-b}$ so that from conditions (2) and (3)

$$\sum_e \pi_e[A+B(W_0-C_0)(1+\alpha r_F+(1-\alpha)\sum_j \beta_j r_{je})]^{-b}(r_{je}-r_F) = 0, \quad (\text{all } j).$$

For a second individual (represented by primed variables) faced with the same opportunities $(r_F, \{r_{je}\})$, suppose that $\{\pi'_e = \pi_e\}$ and $B' = B$. Then similarly,

$$\sum_e \pi_e[A'+B(W'_0-C'_0)(1+\alpha' r_F+(1-\alpha')\sum_j \beta'_j r_{je})]^{-b}(r_{je}-r_F) = 0, \quad (\text{all } j).$$

The portfolio decisions of these two individuals are related by

$$(W'_0-C'_0)(1-\alpha')\beta'_j = \left[\frac{A'+B(W'_0-C'_0)(1+r_F)}{A+B(W_0-C_0)(1+r_F)}\right] \times (W_0-C_0)(1-\alpha)\beta_j, \quad (\text{all } j),$$

since this equation satisfies the above first-order conditions. Summing this equation over all j and since $\sum_j\beta'_j = \sum_j\beta_j = 1$,

$$(W'_0-C'_0)(1-\alpha') = \left[\frac{A'+B(W'_0-C'_0)(1+r_F)}{A+B(W_0-C_0)(1+r_F)}\right](W_0-C_0)(1-\alpha).$$

Dividing this into the prior equation, $\beta'_j = \beta_j$ for all j. When $B = 0$, then $V'(W_{1e}) = \exp(-W_{1e}/A)$. A similar analysis but using

[3]For example, it can be shown that: (1) If the economy is composed of individuals with the same beliefs and taste parameters $A = 0$ and B, then the optimal leverage α and diversification $\{\beta_j\}$ decisions will be the same for all individuals (even though they have different resources, rates of patience, and utility functions U). (2) If, for any individual, $A = 0$ and $B = 1$ (logarithmic utility), then his optimal consumption decision C_0 is independent of his opportunities $(r_F, \{r_{je}\})$ and beliefs.

$$(W_0'-C_0')(1-\alpha')\beta_j' = [A'/A](W_0-C_0)(1-\alpha)\beta_j$$

will also yield the result $\beta_j' = \beta_j$ for all j. Q.E.D.

Let $r_{Me} \equiv \sum_j \beta_j r_{je}$ denote the rate of return on this common portfolio of risky securities. Since all risky securities must be held, M can be interpreted as the market portfolio of all securities. All individuals will hold this same portfolio of risky securities even though they have different parameters (W_0, U, ρ and A). Observe that this conclusion is independent of the number J of risky securities available in the market. In particular, whether the number of securities is less than the number of states ($J+1 < E$) or equal to the number of states ($J+1 = E$), the theorem holds.

2.3. Complete markets

In general, $J+1 < E$. However, it will be useful to consider the simplification in conditions (1)–(6) that results under a complete securities market ($J+1 = E$). In this case, it is well known that choices will be made as if there exist a full set of 'state contingent claims' in place of the actual securities. That is, individuals behave as if for each state e there exists a security for which

$$\begin{aligned} r_e &> -1, \qquad \text{if } e \text{ occurs,} \\ r_e &= -1, \qquad \text{otherwise.} \end{aligned}$$

With a one-to-one correspondence between securities and states, securities may be indexed simply by subscript e.

In this case the wealth constraint reduces to $W = (W_0-C_0)(1+r_e)\beta_e$ (all e), so that the programming problem becomes

$$\max_{C_0,\,\{\beta_e\}} U(C_0)+\rho\sum_e \pi_e V[(W_0-C_0)(1+r_e)\beta_e]-\lambda[\sum_e \beta_e - 1],$$

with necessary and sufficient conditions for an optimum

$$\begin{aligned} U'(C_0) &= (1+r_e)\rho\pi_e V'(W_{1e}), && \text{(all } e\text{)}, \\ W_{1e} &= (W_0-C_0)(1+r_e)\beta_e, && \text{where } \sum_e \beta_e = 1, \qquad \text{(all } e\text{)}. \end{aligned}$$

However, since given choice of C_0 there is a one-to-one correspondence between choice of β_e and W_{1e}, the problem can be reformulated as one of choosing $\{W_{1e}\}$ directly. Summing the wealth constraint over states e and since $\sum_e \beta_e = 1$,

$$W_0 = C_0 + \sum_e W_{1e}(1+r_e)^{-1}.$$

The reformulated programming problem becomes

$$\max_{C_0,\,\{W_{1e}\}} U(C_0)+\rho\sum_e \pi_e V(W_{1e})-\lambda[C_0+\sum_e W_{1e}(1+r_e)^{-1}-W_0],$$

with necessary and sufficient conditions for an optimum

$$U'(C_0) = (1+r_e)\rho\pi_e V'(W_{1e}), \qquad \text{(all } e) \tag{8}$$

$$W_0 = C_0 + \sum_e W_{1e}(1+r_e)^{-1}. \tag{9}$$

Closure conditions replacing (4)–(6) are simply

$$C_0^M = \sum_i C_0^i, \tag{10}$$

$$W_{1e}^M \equiv \sum_j S_{j1e} = \sum_i W_{1e}^i, \qquad \text{(all } e). \tag{11}$$

Since they permit incomplete markets, economies with universal portfolio separation would appear to be a generalization of economies with complete markets. However, this is deceiving in the following sense: *in an economy with universal portfolio separation individuals reach the same optimal allocation in terms of final outcomes* ($C_0^i, \{W_{1e}^i\}$) *as in the corresponding complete markets economy with the same homogeneity assumptions.* For example, if all individuals have homogeneous beliefs and quadratic utility ($B = -1$), then the same optimal allocations ($C_0^i, \{W_{1e}^i\}$) are reached in an otherwise similar economy but with complete markets. An immediate consequence of this is that *the rates of return on securities in an economy with universal portfolio separation but with incomplete markets are determined as if there were complete markets.* (See appendix 2.) This result will prove useful throughout this paper.

3. Homogeneity and aggregation

With this preparation, we are ready to develop the main theorem. For the economy described in section 2, the aggregation problem is the derivation of rates of return ($r_F, \{r_{je}\}$) in terms of the exogenous specifications[4] of the economy: beliefs $\{\pi_e^i\}$, tastes $\{\rho_i, U_i, V_i\}$, production ($C_0^M, \{S_{j1e}\}$), and population I. To solve the aggregation problem, we usually try to introduce production variables into conditions (1)–(3) by summing them, respectively, over all individuals:

$$\sum_i U_i'(C_0^i) = (1+r_F)[\sum_i \rho_i \sum_e \pi_e^i V_i'(W_{1e}^i)], \tag{1'}$$

$$\sum_i \sum_e \pi_e^i V_i'(W_{1e}^i)(r_{je}-r_F) = 0, \qquad \text{(all } j), \tag{2'}$$

$$W_{1e}^M = \sum_j S_{j1e}, \qquad \text{(all } e). \tag{3'}$$

[4]Generally, these exogenous specifications should also include the initial endowment of each individual's claims to present consumption and future wealth. However, as a property of all the solutions to the aggregation problem developed in this paper, equilibrium rates of return are insensitive to redistribution of initial wealth. Consequently, individual endowments can be ignored.

Eq. (3′) follows from (5) and (6). Unfortunately, (1′) and (2′) have remained complex expressions, containing personal information that is the result of individual choices (C_0^i, $\{W_{1e}^i\}$). If there were some way to eliminate this individual-specific choice information from conditions (1′) and (2′), the aggregation problem could be solved.

We will solve the problem by assuming individuals have partially similar economic characteristics. In the trivial case of identical individuals, since at the optimum $C_0^i = C_0^M/I \equiv C_0$ and $W_{1e}^i = W_{1e}^M/I \equiv W_{1e}$ for all i and e, (1′) and (2′) reduce to

$$U'(C_0) = (1+r_F)\rho[\sum_e \pi_e V'(W_{1e})],$$

$$\sum_e \pi_e V'(W_{1e})(r_{je}-r_F) = 0, \qquad (\text{all } j),$$

where the i differentiation can be omitted. Since together with (3′) these simplified conditions determine solutions to equilibrium rates of return (r_F, $\{r_{je}\}$), and since the conditions are specified in terms of exogenous parameters only, the aggregation problem has been solved. Moreover, we can trivially construct a composite individual with resources $W_0 \equiv W_0^M/I \equiv \sum_i W_0^i/I$, beliefs $\{\pi_e\}$ and tastes ρ, U and V so that equilibrium rates of return are determined as if there exist only these composite individuals.

Less trivial solutions to the aggregation problem can be obtained by restricting tastes to satisfy eq. (7). Moreover, since the equilibrium risk-free rate of return must also be determined, utility functions U and V will be assumed of the same form so that

$$-U'(C_0)/U''(C_0) = A + BC_0.$$

Even though individuals are allowed to differ in certain respects, the aggregation problem can be solved. For example, as long as individuals have the same beliefs, rates of patience, and taste parameters $B \neq 0$, they can have different resources and taste parameters A_i. Moreover, when $B = 0$ (exponential utility), even the rates of patience may differ. When $A = 0$ and $B = 1$ (logarithmic utility), the rates of patience may also differ provided individuals have the same resources and beliefs. The chief objection to these solutions is the required homogeneity of beliefs. However, when $B = 0$ and a complete market exists, then the aggregation problem may be solved even though individuals have different resources, beliefs, rates of patience and taste parameters A_i. Also, when $A = 0$, $B = 1$ and a complete market exists, then the aggregation problem may be solved even though individuals have different beliefs provided they have the same resources and rates of patience.

The following theorem summarizes these remarks. A proof is provided in appendix 1.

Theorem (Aggregation). *Consider the following sets of homogeneity conditions:*[5]

(i) *All individuals have the same resources W_0, beliefs $\{\pi_e\}$, and tastes ρ, U and V.*

(ii) *All individuals have the same beliefs $\{\pi_e\}$, rates of patience ρ, and taste parameters $B \neq 0$.*

(iii) *All individuals have the same beliefs $\{\pi_e\}$ and taste parameters $B = 0$.*

(iv) *All individuals have the same resources W_0, beliefs $\{\pi_e\}$, and taste parameters $A = 0$ and $B = 1$.*

(v) *A complete market exists and all individuals have the same taste parameter $B = 0$.*

(vi) *A complete market exists and all individuals have the same resources W_0 and tastes ρ, $A = 0$ and $B = 1$.*

Equilibrium rates of return are determined in case (i) as if there exist only composite individuals each with resources W_0, beliefs $\{\pi_e\}$, and tastes ρ, U and V; and equilibrium rates of return are determined in cases (ii)-(vi) as if there exist only composite individuals each with the following economic characteristics:[6]

Resources $\quad W_0 \equiv \sum_i W_0^i/I;$

Beliefs $\quad \pi_e \equiv \prod_i \pi_e^{i\, A_i/\Sigma_i A_i}, \quad (A \neq 0),$

or

$\pi_e \equiv \sum_i \pi_e^i/I, \qquad (A = 0), \quad (\text{all } e);$

[5]The sufficient conditions for the aggregation theorem permit some weakening. In cases (ii)-(vi), utility functions U and V can differ so that they satisfy

$$-U'(C_0)/U''(C_0) = A_0 + BC_0, \text{ and}$$

$$-V'(W_{1e})/V''(W_{1e}) = A_1 + BW_{1e},$$

where generally $A_0 \neq A_1$. The composite parameters will then be $A_0 = \Sigma_i A_{0i}/I$, $A_1 = \Sigma_i A_{1i}/I$ and B. However, when $B = 0$ there must exist a constant k, the same for all individuals, such that $A_{1i} = kA_{0i}$ for all i. These generalizations permit a multiperiod extension of the theorem to the scenario in which individuals maximize

$$E[\sum_{t=0}^{T} \rho_t U(C_t)],$$

where E is an expectation operator and T $(t = 0, 1, 2, \ldots, T)$ is an individual's lifetime. See appendix 2 for weakening the complete market assumption of cases (v) and (vi).

[6]When $A \neq 0$ and beliefs are heterogeneous, one caveat is in order. The composite beliefs

$$\pi_e \equiv \prod_i \pi_e^{i\, A_i/\Sigma_i A_i}, \quad (\text{all } e),$$

a geometric average of individual beliefs, does not fulfill all the properties of a probability measure. Although π_e is non-negative for all e, $\Sigma_e \pi_e \neq 1$, even though $\Sigma_e \pi_e^i = 1$ for all i.

Tastes $\rho \equiv \prod_i \rho_i^{A_i/\Sigma_i A_i}, \qquad (A \neq 0),$

or

$$(1+\rho)^{-1} = \sum_i (1+\rho_i)^{-1}/I, \quad (A = 0),$$

$$A \equiv \sum_i A_i/I,\ B.$$

In brief, whenever the conditions of the aggregation theorem are met, equilibrium rates of return are determined as if there exist only composite individuals, each of whom solves the following programming problem:

$$\max_{C_0,\, \alpha,\, \{\beta_j\}} U(C_0) + \rho \sum_e \pi_e V[(W_0 - C_0)(1 + \alpha r_F + (1-\alpha)\sum_j \beta_j r_{je})] - \lambda[\sum_j \beta_j - 1],$$

with necessary and sufficient conditions for an optimum

$$U'(C_0) = (1+r_F)\rho[\sum_e \pi_e V'(W_{1e})], \tag{12}$$

$$\sum_e \pi_e V'(W_{1e})(r_{je} - r_F) = 0, \qquad (\text{all } j). \tag{13}$$

Here $C_0 \equiv C_0^M/I$, $W_{1e} \equiv W_{1e}^M/I = \sum_j S_{j1e}/I$ (all e), $W_0 \equiv W_0^M/I$ and $\{\pi_e\}$, U, V and ρ are appropriately defined composites.

Since equilibrium rates of return are uniquely determined by (12) and (13), it follows trivially that:

Corollary (Resource distribution irrelevancy). *Whenever a composite individual can be constructed, in equilibrium, rates of return are insensitive to the distribution of resources among individuals.*

Consequently, if the allocation of resources is Pareto-optimal (as will be shown in sections 5 and 6) and if every Pareto-optimal allocation of resources can be achieved by an appropriate redistribution of resources, then *any* interference by a central planner that alters rates of return must *ipso facto* lead to a nonoptimal allocation of resources.

It also follows trivially from (12) and (13) that:

Corollary (Population irrelevancy). *Whenever a composite individual can be constructed, in equilibrium, rates of return are insensitive to the population as long as the economic characteristics of the composite individual remain unchanged.*

For example, in an economy of identical individuals, rates of return are insensitive to the population. More generally, whenever a composite individual exists, rates of return are insensitive to decreases or increases in the number of those same composite individuals.

The aggregation theorem provides only sets of sufficient conditions, not also necessary conditions, for the construction of a composite individual. More generally, we will define a composite individual by the following traits:

(1) his initial wealth, optimal present consumption, and optimal future wealth for every state are arithmetic averages of their corresponding aggregate values;
(2) any homogeneous economic characteristic shared by all actual individuals is also an economic characteristic of the composite individual;
(3) his beliefs for each state are a function of the beliefs of all actual individuals for the corresponding state and at most also depend on the tastes of all actual individuals;
(4) his rate of patience and taste parameters are, respectively, a function of the rates of patience and taste parameters of all actual individuals and at most also depend on the beliefs of all actual individuals;
(5) equilibrium rates of return are determined as if there exist only composite individuals.

(1) imposes the requirement that $W_0 \equiv W_0^M/I$, $C_0 \equiv C_0^M/I$ and $W_{1e} \equiv W_{1e}^M/I$. (2), for example, requires that if π_e^i is the same for all i, then the composite belief $\pi_e \equiv \pi_e^i$. Observe that this is satisfied by

$$\pi_e \equiv \prod_i \pi_e^{i\,A_i/\Sigma_i A_i}.$$

(3) and (4) exclude the construction of composite beliefs and tastes as functions of initial wealth $\{W_0^i\}$ and aggregate production variables $(C_0^M, \{S_{j1e}\})$. When production choice is introduced (section 6), it will be shown that production choices $(C_0^M, \{S_{j1e}\})$ by competitive value-maximizing firms, as well as equilibrium rates of return $(r_F, \{r_{je}\})$, are determined as if there exist only composite individuals. In brief, a composite (average) individual must be arithmetically average with respect to endogenously determined variables, share commonly held traits, have traits corresponding to exogenous traits of actual individuals that depend only on those exogenous traits and are non-dictatorial, and make the same decisions with regard to commonly shared endogenously determined variables as reached jointly by actual individuals.[7]

[7]The composite beliefs of this paper should not be confused with the 'consensus beliefs' defined in Rubinstein (1973d). Consensus beliefs are those beliefs that if held by all individuals in an otherwise similar economy would generate the same equilibrium rates of return. For example, if a complete market exists and all individuals have the same tastes ρ, $A = 0$ and $B = 1$, then consensus beliefs

$$\pi_e^M \equiv \sum_i \frac{W_0^i}{W_0^M}\,\pi_e^i, \quad \text{(all } e\text{)}.$$

Since π_e^M is a function not only of individual exogenous parameters $\{\pi_e^i\}$ but also $\{W_0^i\}$, which are functions of the endogenously determined security rates of return, π_e^M is not a composite. Therefore, composite beliefs are always consensus beliefs but not vice versa. This definition also excludes other sets of homogeneity conditions, consistent with heterogeneous normal probability assessments, from which Lintner (1969, 1972) has constructed complex 'composites'. These 'composite' parameters, to the extent they do not overlap cases (i)-(vi), are all functions of aggregate production variables $\{S_{j1e}\}$.

The remainder of this paper will work directly with (12) and (13) presuming sufficient conditions for the construction of composite parameters are met.

4. Valuation

With the aggregation problem solved, it remains to derive closed-form solutions to the equilibrium rates of return.

Theorem (valuation[8]). *Whenever composite individual can be constructed, in equilibrium*

$$1+r_F = \frac{U'(C_0)}{\rho E[V'(W_1)]}, \tag{14}$$

$$E(r_j) = r_F + \lambda\kappa(r_j, -V'(W_1)) \operatorname{Std} r_j, \qquad (\text{all } j), \tag{15}$$

where

$$\lambda \equiv \operatorname{Std}[V'(W_1)]/E[V'(W_1)] > 0.$$

Proof. Eq. (14) follows immediately from (12). Solving (13) for the equilibrium expected rate of return on any risky security j,

$$E(r_j - r_F)E[V'(W_1)] + \operatorname{Cov}[r_j - r_F, V'(W_1)] = 0,$$

so that

$$E(r_j) = r_F + E[V'(W_1)]^{-1} \operatorname{Cov}(r_j, -V'(W_1)).$$

Eq. (15) follows immediately from the definition of the correlation coefficient. Since $V'(W_1) > 0$, then $\lambda > 0$. Q.E.D.

Generalizing the discussion in Rubinstein (1973b, pp. 168-171), $\kappa(r_j, -V'(W_1))$ can be interpreted as a measure of the *non-diversifiable risk* of security j. It follows from (15) that the risk premium $E(r_j) - r_F$ will be positive if and only if $\kappa(r_j, -V'(W_1))$ is positive. Moreover, despite the fact that sufficient conditions for the two-parameter (mean-variance) model are not required for (15), nonetheless $|E(r_j) - r_F|$ varies directly with $\operatorname{Std} r_j$ [provided $\kappa(r_j, -V'(W_1)) \neq 0$]. By multiplying (15) by any portfolio proportion β_j and then summing over j, it is also easy to see that (15) must hold for any portfolio of risky securities, including the market portfolio of risky securities as a special case.

[8] E is an expectation operator, Std a standard deviation operator, κ the correlation coefficient of r_j, and $-V'(W_1)$, and λ is a constant as defined and is not to be confused with a Lagrangian multiplier. In the proof below, Cov is a covariance operator. Beja (1971) has derived a valuation equation similar to (15), except that it is not clearly linked to the construction of a composite individual. Moreover, in place of $-V'(W_1)$, Beja inserts an undefined 'market factor'. In this case, as Stephen Ross has mentioned to me, Beja's valuation equation is a tautology since there must exist some random variable in place of $-V'(W_1)$ so that (15) is true for all j.

Generalizing from the discussion in Rubinstein (1973a, pp. 65/6), when no risk-free security exists,[9] (15) will generalize to

$$E(r_j) = E(r_{\bar{p}}) + \lambda\kappa(r_j, -V'(W_1))\text{Std } r_j, \qquad (\text{all } j),$$

where $\bar{p}$ is any portfolio with zero non-diversifiable risk; that is, $\kappa(r_{\bar{p}}, -V'(W_1)) = 0$.

The popular two-parameter valuation equation is a special case of (15).

Corollary (Two-parameter valuation). *Whenever a composite individual can be constructed, if either $B = -1$ (quadratic utility) or the composite individual makes normal probability assessments, then in equilibrium*

$$E(r_j) = r_F + \lambda\kappa(r_j, r_M)\text{Std } r_j, \qquad (\text{all } j), \tag{16}$$

where M is the market portfolio of all securities and

$$\lambda = \frac{-E[V''(W_1)]\text{Std } W_1}{E[V'(W_1)]}\,.$$

Proof. If $B = -1$, then $V'(W_{1e}) = A - W_{1e}$. Defining the rate of return on the market portfolio of all securities $r_{Me} \equiv W_{1e}/(W_0 - C_0)$, then $V'(W_{1e}) = A - (W_0 - C_0)r_{Me}$. Eq. (16) follows upon substitution of this into (15). On the other hand, if the composite individual makes normal probability assessments, then as shown in Rubinstein (1973c, pp. 613/4) $\text{Cov}(r_j, -V'(W_1)) = -E[V''(W_1)]\,\text{Cov}(r_j, W_1)$. Eq. (16) follows from this. Q.E.D.

More generally, suppose sufficient conditions to construct a composite individual are not satisfied but all individuals make homogeneous normal probability assessments. As shown in Rubinstein (1973c), valuation equation (16) will still hold except with the redefinition

$$\lambda = \left\{ \sum_i \frac{-E[V_i'(W_1^i)]}{E[V_i''(W_1^i)]} \Big/ I \right\}^{-1} \text{Std } W_1\,.$$

However, in this case λ is stated in terms of individual-specific choice information and the aggregation problem remains unsolved. Although this does not prevent (16) from yielding refutable empirical hypotheses as in Jensen (1972), they are not as strong as they would otherwise be had the aggregation problem been solved. Without its solution, little can be said about the size and determinants of λ, except that $\lambda > 0$. This contrasts with the comparative statics results available in Rubinstein (1973c) when the aggregation problem can be solved.

[9] When no risk-free security exists, we must check that a composite individual still exists since the proof in appendix 1 relies on the existence of a risk-free security. It can be shown that a composite individual exists, even in the absence of a risk-free security, in cases (i) and (iv) and in case (ii) provided $A = 0$. See appendix 2.

When special restrictions, other than sufficient conditions to construct a composite individual, are not placed on beliefs and tastes, the two-parameter case can be generalized to a multiparameter valuation equation.

Corollary (Multiparameter valuation). *Whenever a composite individual can be constructed, in equilibrium*

$$E(r_j) = r_F + \sum_{n=2}^{\infty} \theta_n \sigma_n(r_j, r_M), \qquad (\text{all } j), \tag{17}$$

where M is the market portfolio of all securities,

$$\theta_n \equiv \frac{-V^{(n)}(W_0 - C_0)^{n-1}}{(n-1)!E[V'(W_1)]}, \qquad (\text{all } n \geqq 2),$$

where $V^{(n)}$ is the nth derivative of V evaluated at $E(W_1)$, and

$$\sigma_n(r_j, r_M) \equiv E[(r_j - Er_j)(r_M - Er_M)^{n-1}], \qquad (\text{all } j \text{ and } n).$$

Proof. With the prior construction of a composite individual, the proof follows immediately from Rubinstein (1973a, pp. 62–64). Q.E.D.

Interpreting valuation equation (17), the equilibrium expected rate of return of any security equals the risk-free rate plus a risk premium equal to the weighted sum of the joint moments $\{\sigma_n\}$ of the rate of return of the security with the market rate of return, where each joint moment is weighted by an appropriately normalized derivative of V evaluated at $E(W_1)$. Eq. (17) indicates that the joint moments are the appropriate *measures of security risk* because they reflect the contribution of a marginal increase in the holdings of a security to the corresponding central moments of future composite wealth, which are the appropriate *measures of portfolio risk* in a multiparameter model. Each joint moment is weighted by the ratio θ_n reflecting the corresponding *market measure of risk aversion.*

In brief, all discrete time valuation models developed in the literature and for which the aggregation problem has been solved are special cases of (15). Valuation equation (16) holds with composite quadratic utility or composite normal probability assessments, and (17) is merely an alternative form of (15).

5. Exchange-efficiency

Exchange-efficiency is defined as circumstances under which individuals are not motivated to create exchange arrangements outside the market. Since the economies satisfying the aggregation theorem of section 3 reach the same allocation of present consumption and future wealth as would have emerged with a complete market, and since a complete market is always exchange-efficient, then all economies satisfying the aggregation theorem must also be exchange-efficient.

A related theorem[10] will now be proven:

Theorem (Exchange-efficiency). *Whenever a composite individual can be constructed, in equilibrium, the present value of the sum of the returns on a set of securities is equal to the sum of the present values of the returns on those securities.*

Proof. Define the aggregate present value of any risky security $S_{j0} \equiv S_{j1e}/(1+r_{je})$. Substituting this into (12) and (13)

$$S_{j0} = \frac{\rho E[V'(W_1)S_{j1}]}{U'(C_0)}, \qquad (\text{all } j).$$

Consider any two risky securities j and k with summed returns or future values $S_{j1}+S_{k1}$. Clearly, since

$$\frac{\rho E[V'(W_1)(S_{j1}+S_{k1})]}{U'(C_0)} = \frac{\rho E[V'(W_1)S_{j1}]}{U'(C_0)} + \frac{\rho E[V'(W_1)S_{k1}]}{U'(C_0)},$$

the present value of $\{S_{j1e}+S_{k1e}\}$ equals $S_{j0}+S_{k0}$. This analysis is easily extended to combinations of any number of securities. Q.E.D.

As one would expect, whenever a composite individual can be constructed the packaging of securities makes no difference, and, in particular, corporate capital structure (even if bankruptcy is possible) and non-synergistic mergers (even if bankruptcy is possible) are irrelevant.

6. Production-efficiency

Production-efficiency is defined as circumstances under which value-maximizing firms make Pareto-optimal production decisions. Since the economies satisfying the aggregation theorem of section 3 reach the same allocation of present consumption and future wealth as would have emerged with a complete market, and since under appropriate competitive conditions a complete market is always production-efficient, then all economies satisfying the aggregation theorem and the competitive conditions must also be production-efficient.
A related theorem will now be proven:

Theorem (Production-efficiency). *Whenever a composite invididual can be constructed, in equilibrium, competitive value-maximizing firms make the same production decisions as would the composite individual.*

[10]See Stiglitz (1969, p. 790) for a verbal discussion of this theorem, and see Mossin (1973, p. 87) for a similar, although incorrect, version. One caveat is in order. When individuals may have different beliefs [cases (v) and (vi)], we must check that eliminating the option of purchasing two securities separately does not undermine the complete markets condition for the construction of a composite individual. See appendix 2.

Proof. For any firm j let f_j denote its production function mapping present investment H_{j0} and state e into future value S_{j1e} so that $S_{j1e} = f_j(H_{j0}, e)$ where $f'_j > 0$ and $f''_j < 0$. This embodies the competitive assumptions of *no production externalities* since f_j is assumed independent of the production decisions of other firms and *decreasing returns to scale* since f_j is strictly increasing and concave in H_{j0}. From the previous section, in equilibrium, the present value S_{j0} of firm j is determined such that

$$S_{j0} = \frac{\rho E[V'(W_1)S_{j1}]}{U'(C_0)} = \frac{\rho E[V'(W_1)f_j(H_{j0})]}{U'(C_0)}.$$

A value-maximizing firm will increase H_{j0} until $\mathrm{d}S_{j0}/\mathrm{d}H_{j0} = 1$ or alternatively until

$$\frac{\rho E[V'(W_1)f'_j(H^*_{j0})]}{U'(C_0)} = 1. \tag{18}$$

This embodies the assumption that firm j is competitive in the sense that it *acts as if it cannot affect the aggregate amounts of production* (C_0^M, $\{W_{1e}^M\}$) *both over dates and across states*. Since f_j is strictly concave, eq. (18) determines a unique optimal level of investment H^*_{j0}.

If instead of delegating production decisions to competitive value-maximizing firms, the composite individual were to select the optimal investment levels $\{H^*_{j0}\}$, then he would solve the following programming problem:

$$\max_{C_0,\,\{H_{j0}\}} U(C_0) + \rho \sum_e \pi_e V(W_{1e}) \quad \text{s.t.} \quad W_0 = C_0 + \sum_j H_{j0}/I,$$

where $W_{1e} = \sum_j f_j(H_{j0}, e)/I$. Substituting in the constraint the problem becomes

$$\max_{\{H_{j0}\}} U(W_0 - \sum_j H_{j0}/I) + \rho \sum_e \pi_e V[\sum_j f_j(H_{j0}, e)/I].$$

Differentiating partially by H_{j0} the necessary and sufficient condition for an optimum (given C_0) is

$$-U'(C_0) + \rho E[V'(W_1)f'_j(H^*_{j0})] = 0.$$

However, this condition, which uniquely determines H^*_{j0}, is identical to eq. (18). Consequently, a composite individual will make the same production decisions as competitive value-maximizing firms.[11] Q.E.D.

Other papers – Fama (1972) and Stiglitz (1972) – have floundered on an inappropriate definition of competition among firms. The definition used here, which is consistent with Merton and Subrahmanyam (1973), requires absence

[11] This theorem holds even if each firm produces many commodities. Simply redefine the production function as $S_{j1e} = f_j(H_{j0}{}^1, H_{j0}{}^2, \ldots, H_{j0}{}^c, \ldots, H_{j0}{}^C; e)$, where $C(c = 1, 2, \ldots, C)$ is the number of commodities produced by firm j, and differentiate partially with respect to $\{H_{j0}{}^c\}$. Observe it is unnecessary to require that $\partial^2 f_j(\cdot)/\partial H_{j0}{}^c \partial H_{j0}{}^{c'} = 0$, where c and c' are different commodities.

of production externalities, decreasing returns to scale, and that firms act as if they cannot influence the aggregate amounts of production over dates and across states.

Appendix 1

Proof of Aggregation Theorem

(i) Proved in the text.

(ii) In light of section 2.3, rates of return of actual securities are determined as if they were set in a complete securities market. Consequently, in place of conditions (1)–(6), we can work instead with (8)–(11). When $B \neq 0$, $U_i'(C_0^i) = (A_i + BC_0^i)^{-b}$ and $V_i'(W_{1e}^i) = (A_i + BW_{1e}^i)^{-b}$, so that from (8)

$$(A_i + BC_0^i)^{-b} = (1+r_e)\rho\pi_e(A_i + BW_{1e}^i)^{-b},$$

therefore

$$A_i + BC_0^i = [(1+r_e)\rho\pi_e]^{-B}(A_i + BW_{1e}^i).$$

Summing over all individuals

$$\sum_i A_i + B\sum_i C_0^i = [(1+r_e)\rho\pi_e]^{-B}(\sum_i A_i + B\sum_i W_{1e}^i).$$

Using closure conditions (10) and (11) and dividing by I,

(a) $$(A + BC_0)^{-b} = (1+r_e)\rho\pi_e(A + BW_{1e})^{-b}, \qquad \text{(all } e\text{)},$$

where $C_0 \equiv C_0^M/I$, $W_{1e} \equiv W_{1e}^M/I$ all e, and $A \equiv \sum_i A_i/I$. Also using closure conditions (10) and (11), from (9)

(b) $$W_0 = C_0 + \sum_e W_{1e}(1+r_e)^{-1},$$

where $W_0 \equiv W_0^M/I \equiv \sum_i W_0^i/I$. Observe that (a) and (b) are the necessary and sufficient conditions for an optimum that would have been derived had there existed an individual with the specified composite resources, beliefs, and tastes. Since rates of return are determined as if there were a complete market, conditions (1)–(3) applying to the actual economy must hold for a composite individual.

(iii) Again rates of return on actual securities are determined as if there were a complete market. The proof follows immediately from the proof for (v).

(iv) In light of section 2.3, rates of return of actual securities are determined as if they were set in a complete market. Consequently, we can work directly with conditions (8)–(11). When $A = 0$ and $B = 1$, $U_i'(C_0^i) = (C_0^i)^{-1}$ and $V_i'(W_{1e}^i) = (W_{1e}^i)^{-1}$, so that from (8)

$$W_{1e}^i = (1+r_e)\rho_i\pi_e C_0^i.$$

Summing this over e and using (9), $W_0 - C_0^i = \rho_i C_0^i$ so that $C_0^i = W_0(1+\rho_i)^{-1}$.

Substituting this into the above equation and summing over all individuals

$$\sum_i W_{1e}^i = (1+r_e)\pi_e W_0 \sum_i \rho_i(1+\rho_i)^{-1},$$

using closure condition (11) and dividing by I,

$$W_{1e} = (1+r_e)\pi_e W_0[\sum_i \rho_i(1+\rho_i)^{-1}/I],$$

where $W_{1e} \equiv W_{1e}^M/I$. By defining composite rate of patience,

$$\rho \equiv \{1 - [\sum_i (1+\rho_i)^{-1}/I]\}/[\sum_i (1+\rho_i)^{-1}/I],$$

then $\sum_i \rho_i(1+\rho_i)^{-1}/I = \rho(1+\rho)^{-1}$ so that

$$W_{1e} = (1+r_e)\rho\pi_e W_0(1+\rho)^{-1}.$$

Since $C_0^i = W_0(1+\rho_i)^{-1}$ and using closure condition (10), $C_0 = W_0 \sum_i (1+\rho_i)^{-1}$, where $C_0 \equiv C_0^M/I$. From the above definition of $\rho, (1+\rho)^{-1} = \sum_i(1+\rho_i)^{-1}/I$, so that $C_0 = W_0(1+\rho)^{-1}$, and

(a′) $\qquad W_{1e} = (1+r_e)\rho\pi_e C_0, \qquad$ (all e).

Also using closure conditions (10) and (11), from (9)

(b′) $\qquad W_0 = C_0 + \sum_e W_{1e}(1+r_e)^{-1}.$

From this point the argument is identical to (ii).

(v) When $B = 0$, $U_i'(C_0^i) = \exp(-C_0^i/A_i)$ and $V_i'(W_{1e}^i) = \exp(-W_{1e}^i/A_i)$ so that from (8)

$$\exp(-C_0^i/A_i) = (1+r_e)\rho_i\pi_e^i \exp(-W_{1e}^i/A_i),$$

therefore

$$-C_0^i = A_i\ln(1+r_e) + A_i\ln\rho_i + A_i\ln\pi_e^i - W_{1e}^i.$$

Summing over all individuals

$$-\sum_i C_0^i = \sum_i A_i\ln(1+r_e) + \ln(\prod_i \rho_i^{A_i}) + \ln(\prod_i \pi_e^{iA_i}) - \sum_i W_{1e}^i.$$

Using closure conditions (10) and (11) and dividing by I,

$$-C_0 = A\ln(1+r_e) + \ln(\prod_i \rho_i^{A_i/I}) + \ln(\prod_i \pi_e^{iA_i/I}) - W_{1e},$$

where $C_0 \equiv C_0^M/I$, $W_{1e} \equiv W_{1e}^M/I$ and $A \equiv \sum_i A_i/I$. Dividing by A,

$$-C_0/A = \ln(1+r_e) + \ln\rho + \ln\pi_e - W_{1e}/A,$$

where $\rho \equiv \prod_i \rho_i^{A_i/\Sigma_i A_i}$ and $\pi_e \equiv \prod_i \pi_e^{iA_i/\Sigma_i A_i}$. Consequently,

(a″) $\qquad \exp(-C_0/A) = (1+r_e)\rho\pi_e \exp(-W_{1e}/A), \qquad$ (all e).

Also using closure conditions (10) and (11), from (9)

$$W_0 = C_0 + \sum_e W_{1e}(1+r_e)^{-1}, \tag{b''}$$

where $W_0 \equiv W_0^M/I$. Observe that (a″) and (b″) are the necessary and sufficient conditions for an optimum that would have been derived had there existed an individual with the specified composite resources, beliefs, and tastes.

(vi) When $A = 0$ and $B = 1$, from (8)

$$W_{1e}^i = (1+r_e)\rho\pi_e^i C_0^i.$$

From the proof of (iv), since $C_0^i = W_0(1+\rho)^{-1}$ by closure condition (10),

$$W_{1e}^i = (1+r_e)\rho\pi_e^i C_0,$$

where $C_0 \equiv C_0^M/I$. Summing over all individuals, using closure condition (11) and dividing by I,

$$W_{1e} = (1+r_e)\rho\pi_e C_0, \tag{a'''}$$

where $W_{1e} \equiv W_{1e}^M/I$ and $\pi_e \equiv \sum_i \pi_e^i/I$, all e. Also using closure conditions (10) and (11), from (9)

$$W_0 = C_0 + \sum_e W_{1e}(1+r_e)^{-1}. \tag{b'''}$$

From this point the argument is identical to (v). Q.E.D.

Appendix 2

Optimal sharing rules

Many of the results of this paper are better understood from the perspective of optimal *sharing rules*. With the homogeneity conditions of case (ii), it can be shown that

$$W_{1e}^i = \frac{AW_0^i - A_iW_0}{A\phi + BW_0} + \frac{A_i\phi + BW_0^i}{A\phi + BW_0} W_{1e}, \qquad \text{(all } e \text{ and } i\text{)},$$

where $\phi \equiv 1 + (1+r_F)^{-1}$. The last term in the summation has the natural interpretation of a state-dependent *dividend* and the second term a certain *side payment*. Observe that

$$\sum_i (AW_0^i - A_iW_0)(A\phi + BW_0)^{-1} = 0,$$

$$\sum_i (A_i\phi + BW_0^i)(A\phi + BW_0)^{-1} = I.$$

Consequently, despite the variety of securities available, perhaps a complete market, every individual chooses to hold the market portfolio M in proportion

$(A_i\phi + BW_0^i)(A\phi + BW_0)^{-1}$ and to borrow or lend $(AW_0^i - A_iW_0)(A\phi + BW_0)$ $(1+r_F)^{-1}$ at certain rate of return r_F (universal portfolio separation). When composite parameter $A = 0$, the sharing rule reduces to $W_{1e}^i = (W_0^i/W_0)W_{1e}$, and no side payments are made.

With the homogeneity conditions of cases (iii) or (v) (exponential utility), the sharing rule can be shown to be

$$W_{1e}^i = A_i\ln(\pi_e^i/\pi_e) + A_i[\ln(\rho_i/\rho) - \sum_e (1+r_e)^{-1}\ln(\pi_e^i/\pi_e)]\phi^{-1}$$
$$+ (AW_0^i - A_iW_0)(A\phi)^{-1} + (A_i/A)W_{1e}, \qquad \text{(all } e \text{ and } i\text{)}.$$

The last term is a dividend, the second the third terms a side payment, and the first term may be interpreted as a *side bet*. Observe that

$$\sum_i A_i\ln(\pi_e^i/\pi_e) = 0,$$

$$\sum_i A_i[\ln(\rho_i/\rho) - \sum_e (1+r_e)^{-1}\ln(\pi_e^i/\pi_e)]\phi^{-1} = 0,$$

$$\sum_i (AW_0^i - A_iW_0)(A\phi)^{-1} = 0,$$

$$\sum_i (A_i/A) = I.$$

When all individuals have the same beliefs for all states e, then no side bets are made and there is universal portfolio separation [case (iii)]. Moreover, the use of side bets only arises for those states e for which $\pi_e^i \neq \pi_e$ for some individuals i. Consequently, a complete market is not strictly required in case (v). For those states e for which $\pi_e^i = \pi_e$ for all individuals i, state contingent claims need not exist.

Rubinstein (1973d, p. 25) suggests that speculative volume can be used as an index of the 'information-efficiency' of the securities market (i.e., the extent to which security prices fully reflect information). With exponential utility, as has been shown, speculative volume is measured by

$$\sum_i A_i[\ln(\pi_e^i/\pi_e) - \phi^{-1}\sum_e (1+r_e)^{-1}\ln(\pi_e^i/\pi_e)].$$

If beliefs are the same for all individuals for all states, then there will be no speculative trading, independent of tastes as measured by $\{A_i\}$. However, given the decision to speculate, the volume of speculative trading does depend on $\{A_i\}$.

With the homogeneity conditions of cases (iv) and (vi) (logarithmic utility), Arrow (1972) has shown that the optimal portfolio proportions are determined by the surprisingly simple decision rule $\beta_e^i = \pi_e^i$ for all e and i. Clearly, to the extent individuals agree on the probability of some states, the securities market may be incomplete.

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