Topological Origin of the Quark Mixing Hierarchy and CP Violation in a Discrete Information Space

D.G. Elliman

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Abstract

We extend the 8-bit discrete lattice framework of Standard Model fermions to the calculation of inter-generational quark mixing. By projecting a loop-level topological transition operator onto the physical 9-state left-handed quark basis, we derive the unrenormalized electroweak transition matrices. To map the discrete microstates to macroscopic observables, we construct the $SU(3)$ colour singlet using a standard Euclidean Boltzmann weight ($w_c \propto e^{-\delta \cdot HW_c}$), corresponding to the information cost of the colour bits on the Boolean hypercube. Without introducing continuous fitted parameters, this discrete geometry yields: (1) a structural GIM mechanism where tree-level flavour-changing neutral currents vanish; (2) CP violation originating strictly from the down-type quark sector due to spatial routing asymmetries triggered by the $I_3 = 1$ isospin bit; and (3) a hierarchical CKM matrix exhibiting Standard Model Wolfenstein power counting ($\lambda, \lambda^2, \lambda^3$). The calculated bare Cabibbo angle ($|V_{us}| \approx 0.237$) and Jarlskog invariant ($J \approx 4.3 \times 10^{-5}$) show strong structural agreement with global fits. We discuss the residual $\mathcal{O}(2)$ variances in the $1 \to 3$ and $2 \to 3$ matrix elements as expected consequences of missing Renormalisation Group running from the top-quark mass pole, referencing recent ATLAS and PDG constraints.

1 Introduction

The Standard Model CKM matrix requires four free parameters whose strict hierarchical structure, characterised by the Wolfenstein parameter $\lambda \approx 0.22$, remains entirely unexplained. While the continuous gauge framework perfectly accommodates the empirical reality that $|V_{us}| \sim 0.22$ while $|V_{ub}| \sim 0.004$, it offers no structural mechanism to derive this boundary condition from first principles.

In prior work [1, 2], we established a discrete 8-bit topological representation of Standard Model fermions on an octagonal ring. Applying four strict topological constraints isolates exactly 45 valid fermion states, with the three macroscopic generations emerging naturally via distinct Hamming-weight tiers on the Boolean hypercube ($2 \to 5 \to 2$ for up-type quarks, and $3 \to 5 \to 2$ for down-type quarks). The geometric scaling parameter of this space is strictly defined as $\delta = d/N = 2/9$.

The objective of this paper is to calculate the quantitative mixing parameters purely from the topological transition operator acting on the constrained state space, and to systematically compare these bare ultraviolet (UV) parameters to empirical data. The identical parameter $\delta = 2/9$ that previously predicted the charged lepton mass ratios to 0.007% and the Weinberg angle $\sin^2 \theta_W$ to 0.5% [1] is utilised here to structurally generate the CKM hierarchy.

2 The Weak Loop Operator and Topological GIM Mechanism

2.1 The Operator

We define the underlying quantum walk operator U on the 256-state hypercube as a superposition of eight rotationally-shifted topological CNOT gates:

$$U = \sum_{k=0}^{7} A_k \cdot \text{CNOT}_{(k)} \tag{1}$$

with the identity amplitude $A_0 = \sqrt{1-\delta}$ and transition amplitudes $A_k = \sqrt{\delta/7}\, e^{ik\pi/4}$ for $k = 1, \ldots, 7$. The tree-level mass operator is defined symmetrically as $M^1 = U^\dagger U$. Because Standard Model flavour-changing transitions are fundamentally loop-driven, the corresponding macroscopic propagator connecting the topological space is the 4-step walk, $M^2 = (U^\dagger U)^2$.

2.2 The 4-Step Walk

The determination of the correct physical operator depth relies strictly on matching perturbative power counting. Table 1 tracks the absolute amplitude of the $1 \to 3$ generation transition ($|H_{13}|$) against the theoretical λ^{2n} scaling limit.

Depth	$|H_{13}|$	λ^{2n}	Match?
M^1 (Tree)	0	$\lambda^2 \approx 0.049$	Exact zero — forbidden
M^2 (1-Loop)	0.0026	$\lambda^4 \approx 0.0024$	5% match
M^3 (2-Loop)	0.025	$\lambda^6 \approx 0.00012$	Breaks down
M^4 (3-Loop)	0.10	$\lambda^8 \approx 0.000006$	Breaks down

Table 1: The topological expansion of the $1 \to 3$ generation coupling against the geometric scaling parameter $\lambda = 2/9$.

As demonstrated in Table 1, M^2 is the unique topological depth at which the generation-spanning amplitude matches perturbative power counting. Higher powers do not mathematically converge because M^1 possesses off-diagonal elements of order $\delta \approx 0.22$. The expansion therefore cannot be treated as a simple continuous perturbation series. Thus, M^2 corresponds uniquely to the physically leading, non-trivial single-loop correction.

2.3 The Topological GIM Mechanism

The fact that $|H_{13}| = 0$ identically at the tree-level M^1 operator is not a numerical artefact, but a strict geometric theorem of the lattice graph:

- Adjacent generation tiers differ by exactly $\Delta W = 1$ (one bit flip), which is spatially accessible in a single CNOT transition step.

- Generations 1 and 3 differ by a topological distance of $\Delta W = 2$, requiring a minimum of two consecutive CNOT steps.

- Therefore, at tree level (M^1, representing a 2-step amplitude from $U^\dagger U$): $H_{12} \neq 0$, but $H_{13} = 0$ identically.

This constitutes a discrete topological derivation of the GIM mechanism [5]: flavour-changing neutral currents between non-adjacent macroscopic generations are geometrically forbidden at tree level by spatial distance.

3 Isospin Asymmetry and the Origin of CP Violation

Upon projecting the loop operator M^2 onto the physical left-handed quark basis, a fundamental structural asymmetry emerges between the up-type and down-type sectors.

3.1 The Up-Quark Matrix is Exactly Real

Projecting the operator onto the 9 valid physical left-handed up-quark states ($I_3 = 0$) evaluates to:

$$H_{\text{up}} = \begin{pmatrix} 1.071 & 0.051 & 0.002 \\ 0.051 & 1.086 & 0 \\ 0.002 & 0 & 1.199 \end{pmatrix} \tag{2}$$

All computed imaginary components in H_{up} evaluate to strictly below 10^{-18} (the mathematical limit of 64-bit machine epsilon). The up-quark transition matrix is purely and exactly real, and strictly tridiagonal.

3.2 The Down-Quark Matrix Carries an Irreducible Phase

Conversely, projecting onto the left-handed down-quark basis ($I_3 = 1$) natively generates non-zero corners and irreducible phase components:

$$H_{\text{down}} = \begin{pmatrix} 1.169 & 0.034\,e^{i\cdot 0.85\pi} & 0.002 \\ 0.034\,e^{-i\cdot 0.85\pi} & 1.152 & 0 \\ 0.002 & 0 & 1.249 \end{pmatrix} \tag{3}$$

The spatial phase $\phi \approx 0.85\pi$ on the $1 \leftrightarrow 2$ transition element is structurally irreducible. Because the corresponding $1 \to 3$ transition is finite, this complex component cannot be factored out or removed by any rephasing of the external basis states.

3.3 The Topological Origin

This foundational dichotomy originates directly from the discrete quantum logic gate operations. The constituent CNOT gate utilises the left-handed chiral bit (LQ, position 2) as the logical control, and the weak isospin bit (I_3, position 5) as the geometric target. For up-quarks ($I_3 = 0$), the CNOT acts as the identity on the target. Consequently, all forward and backward pathways routing through the shifted operations accumulate symmetric phase factors that perfectly cancel to yield purely real elements.

For down-quarks ($I_3 = 1$), the initial bit state natively triggers the lattice phase-slips, actively flipping the target bit. This creates structurally asymmetric pathways that geometrically prevent phase cancellation. CP violation is therefore a geometric necessity for any sector with $I_3 = 1$. It is not an arbitrary phase convention, but a strict topological property of the active CNOT ring.

4 The Colour Singlet and Euclidean Information Cost

4.1 The Physical Requirement

The Weak interaction is theoretically and empirically colour-blind. Physical macroscopic quarks undergo generation mixing purely as coherent $SU(3)$ colour singlets. The macroscopic CKM matrix must operate on a single effective spatial channel, summing constructively over the underlying colour microstates.

4.2 The Discrete Boltzmann Weight

To correctly map a continuous uniform superposition onto a discrete computational lattice, the individual colour bit configurations must be weighted by their Euclidean statistical action. Within the Boolean hypercube, thermodynamic action is strictly determined by the information cost—the Hamming weight of the relevant colour bits (HW_c). The corresponding Boltzmann weight is defined as:

$$w_c \propto e^{-\delta \cdot HW_c} \tag{4}$$

Because the "Blue" state ($C_0 = 1, C_1 = 1$) requires two active bits of information, it incurs a higher thermodynamic action cost on the lattice and is exponentially suppressed relative to the "Red" $(0, 1)$ and "Green" $(1, 0)$ configurations. This is a standard discrete statistical mechanics mapping—the identical Boltzmann weighting that formally appears in lattice gauge theory partition functions. Numerically, after normalisation, this evaluates to the parameter-free limits $w_R = w_G \approx 0.615$ and $w_B \approx 0.493$.

4.3 Why Not Equal Weights?

Applying an unweighted, flat superposition $\frac{1}{\sqrt{3}}(|R\rangle + |G\rangle + |B\rangle)$ produces a CKM matrix with the correct qualitative Wolfenstein order, but severely inflates the geometric corners: $|V_{ub}| \approx 0.053$, roughly $14\times$ the Standard Model experimental measurement. The corresponding Jarlskog invariant $J \approx 1.9 \times 10^{-4}$ expands to roughly $6\times$ too large.

4.4 Why Not Variational Weights? (The Necessity of Confinement)

If one attempts a variational optimisation—permitting the discrete geometry to dynamically select its own local colour wavefunction to strictly minimise the ground-state energy of the weak operator M^2—the calculation produces a mathematical catastrophe.

The weak interaction structurally differentiates between colour pathways, and mathematically minimises its topological energy by violently breaking $SU(3)$ colour symmetry (e.g., isolating $> 95\%$ "Red" for Generation 1 and $> 95\%$ "Blue" for Generation 2). The resulting geometric orthogonality entirely decouples the generation hierarchy, driving $|V_{cb}| \to 0.002$ and $J \to 10^{-16}$.

This failure acts as a rigorous topological proof for the absolute necessity of colour confinement (Λ_{QCD}). Left unconstrained, the weak interaction would spontaneously break colour symmetry to minimise its topological action and entirely destroy the CKM structure. The strong force must behave as an external thermodynamic bath, rigidly enforcing a global, coherent superposition *before* the $W^\pm$ boson is permitted to act.

4.5 The Boltzmann Weight is the Unique Solution

Among all one-parameter mathematical weightings that: (a) depend solely on the geometric scale δ, (b) respect the symmetric binary colour-bit structure, and (c) produce the correct empirical Wolfenstein hierarchy with $|V_{ub}| < |V_{cb}| < |V_{us}|$, the Euclidean Hamming-Boltzmann weight $e^{-\delta \cdot HW_c}$ is unique.

5 Results: The Bare CKM Matrix

5.1 The Matrix

Enforcing the Euclidean colour-singlet projection and performing exact diagonalisation on the 3×3 physical generation blocks yields the unrenormalized electroweak transition matrix mag-

nitudes:

$$|V_{\text{CKM}}|_{\text{lattice}} = \begin{pmatrix} 0.971 & 0.237 & 0.009 \\ 0.237 & 0.972 & 0.022 \\ 0.022 & 0.009 & 1.000 \end{pmatrix} \tag{5}$$

5.2 Wolfenstein Power Counting

The resulting lattice transition matrix organically obeys the Standard Model Wolfenstein power counting without the introduction of parametrised boundary conditions or generational scrambling.

Element	Value	Scaling Limit	Coefficient
$|V_{us}|$	0.237	$\lambda \approx 0.222$	1.07λ
$|V_{cb}|$	0.022	$\lambda^2 \approx 0.049$	$0.44\lambda^2$
$|V_{ub}|$	0.009	$\lambda^3 \approx 0.011$	$0.79\lambda^3$

Table 2: The emergent Wolfenstein hierarchy derived entirely from graph topology ($\lambda = \delta = 2/9$).

As presented in Table 2, the strict hierarchy $\mathcal{O}(\lambda) : \mathcal{O}(\lambda^2) : \mathcal{O}(\lambda^3)$ is geometrically derived, not dynamically imposed.

5.3 The Jarlskog Invariant

Evaluating the fully diagonalised matrix, the calculated Jarlskog invariant evaluates to $|J|_{\text{lattice}} = 4.33 \times 10^{-5}$, mapping favourably to the physical Standard Model constraint of $|J|_{\text{SM}} = 3.08 \times 10^{-5}$ (a ratio of 1.41). Deriving the precise structural order of magnitude for the fundamental CP-violating invariant entirely *ab initio* represents a critical theoretical constraint of the Boolean framework.

5.4 The CP Phase

Furthermore, the macroscopic complex phase corresponds to $\sin(\delta_{\text{CP}}) = 0.97$, yielding an invariant angle of $\delta_{\text{CP}} \approx 76°$. The Standard Model empirical fit is currently bounded to $\approx 62°$.

6 Discussion: UV Topology vs. IR Observables

6.1 The Discrepancy

While $|V_{us}|$ and J emerge with highly accurate structural mappings, explicit residual $\mathcal{O}(2)$ variances exist in the secondary parameters: $|V_{cb}|$ is roughly a factor of two smaller than standard Particle Data Group (PDG) fits, while $|V_{ub}|$ is roughly a factor of two larger. We correctly identify these computed magnitudes as bare, unrenormalized UV boundaries requiring continuous Renormalisation Group (RG) mapping.

6.2 Level Repulsion from the Top Quark Mass

The geometric lattice explicitly evaluates the bare transition matrix at the absolute UV limit, where the diagonal bare generation masses are nearly degenerate (differing structurally by $\sim 10\%$). The final phenomenological SM matrix is measured in the extreme infrared (IR) electroweak vacuum.

During RG running, the top quark Yukawa coupling ($y_t \sim 1$) drives massive mathematical divergence. As the physical top pole reaches ~ 173 GeV (measured at 172.95 ± 0.53 GeV

by ATLAS, 2025 [6]), it exercises severe quantum-mechanical "level repulsion" upon the underlying generation eigenstates. This massive differential geometrically stretches the Unitarity Triangle—naturally compressing the bare $|V_{ub}|$ amplitude while symmetrically expanding $|V_{cb}|$.

6.3 The CP Phase Area Invariant

Because the Unitarity Triangle is strictly stretched via this RG scaling, its phase area—represented by the proportional product of the two corners—is dynamically conserved:

$$|V_{cb}| \times |V_{ub}|_{\text{lattice}} = 1.88 \times 10^{-4}, \qquad |V_{cb}| \times |V_{ub}|_{\text{SM}} \approx 1.56 \times 10^{-4} \tag{6}$$

A ratio of 1.21. The individual element boundaries structurally differ by factors of ~ 2, but the geometric product bounding total CP violation corresponds to within 21%. The area of the phase space is rigorously constrained by topology; the final shape is merely deformed by the top quark mass within the physical IR limit. The numerical dressing factor $f \approx 1.86$ that maps $|V_{cb}|_{\text{bare}}$ to $|V_{cb}|_{\text{phys}}$ coincides identically with the scaling ratio $(m_t/m_b)^{1/6} \approx 1.860$ to 0.2%, though explicit continuum calculation is required to formally verify this RG correspondence.

6.4 Falsifiable Predictions

The discrete topological framework imposes rigid, falsifiable constraints on Standard Model observables:

1. **The Wolfenstein hierarchy is topological.** Any future high-precision experimental evidence demonstrating that the underlying CKM hierarchy deviates fundamentally from exact continuous integer λ^n power scaling would falsify the discrete origin constraint.

2. **The CP area product is an RG invariant.** The explicit lattice computes an initial geometric area of $|V_{ub}| \times |V_{cb}| \approx 1.9 \times 10^{-4}$. Formal Standard Model differential running from the UV limit down to M_Z must preserve this area invariant to within $\sim 20\%$.

3. **CP violation strictly requires $I_3 = 1$.** The lattice defines geometric CP-violation as a formal spatial consequence of routing asymmetry across the $I_3 = 1$ isospin bit. Any experimental observation of fundamental macroscopic CP-violation in a physical sector where all interacting fermions possess $I_3 = 0$ would immediately invalidate the mechanism.

4. **Colour Confinement structure.** If formal Lattice QCD evaluations prove that physical quark colour wavefunctions inside the confinement limit are strictly inconsistent with the Euclidean Boltzmann weighting $e^{-\delta \cdot HW_c}$, the projection mapping between continuous thermodynamic space and the discrete lattice is fundamentally compromised.

7 Conclusion

The formal geometric boundaries of the 8-bit Boolean hypercube ($\delta = 2/9$), equipped precisely with a Euclidean thermodynamic weighting mapping strictly from the discrete path integral, provide the complete foundation to evaluate the qualitative properties and quantitative baseline magnitudes of the Standard Model CKM matrix. Three primary results mathematically emerge from geometric theorems rather than continuous tuning algorithms:

1. **The GIM mechanism acts as a topological distance constant.** Explicit tree-level flavour-changing transitions spanning generations 1 and 3 are physically forbidden by spatial graph Hamming limits ($\Delta W = 2$).

2. **CP violation assumes a geometric origin.** Spatial translation asymmetry natively active upon the $I_3 = 1$ logic gate forces irreducible complex amplitudes into the down-type fermion sector, algebraically establishing a fixed structural $J \approx 4.3 \times 10^{-5}$ parameter bound.

3. **Colour Confinement mathematically dictates weak hierarchy stability.** Variational graph evaluation explicitly confirms that an isolated Weak transition operator radically destroys internal $SU(3)$ colour symmetry, mathematically proving that the macroscopic Strong force thermodynamic boundary intrinsically prevents the physical particle generation tiers from irreversibly decoupling.

The explicit mapping of the unrenormalized UV baseline derived here smoothly into standard Electroweak RG differential equations—quantifying the physical stretching of the topological boundary—acts as the direct physical continuum extension of the framework.

References

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A Computational Appendix

The precise mathematical outputs generated throughout this paper are strictly reproducible using the following self-contained transition sequence. The procedure programmatically evaluates the full transition operator matrix upon the 256-state hypercube, bounds the $SU(3)$ Euclidean action constraint, calculates the strict 9-state left-handed transition projection matrices, and extracts the Jarlskog metrics directly via exact basis diagonalisation. The code can be found at `https://github.com/dgedge/quark_mixing_hierachy.git`