On compressible Navier–Stokes equations with density dependent viscosities in bounded domains

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Abstract

The present note extends to bounded domains recent results about barotropic compressible Navier-Stokes systems with density dependent viscosity coefficients. We show how to get the existence of global weak solutions for both classical Dirichlet and Navier boundary conditions on the velocity, under appropriate constraints on the initial density profile and domain curvature. An additional turbulent drag term in the momentum equation is used to handle the construction of approximate solutions.

Keywords: Barotropic compressible Navier-Stokes equations, weak solutions, bounded domains.
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1 Introduction

This paper is devoted to the problem of global existence of weak solutions for compressible Navier–Stokes equations, in bounded domains, when viscosity coefficients vanish on vacuum. It mainly deals with barotropic flows, which satisfy:

$$\partial_t \rho + \text{div}\,(\rho u) = 0, \tag{1}$$
$$\partial_t(\rho u) + \text{div}\,(\rho u \otimes u)$$
$$= \text{div}\,(2\mu(\rho)D(u)) + \nabla(\lambda(\rho)\text{div}\,u) - \nabla p(\rho) - r_0\rho|u|u, \tag{2}$$

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where ρ and u are as usual the density and velocity of the fluid. The r.h.s. of (2) splits into the viscous term, with Lamé coefficients μ and λ, and the pressure term $p = p(\rho)$. For the sake of simplicity, we shall consider classical power laws as pressure functions:

$$p(\rho) = a\,\rho^\gamma, \quad a > 0, \quad \gamma > 1, \tag{3}$$

and restrict to the three dimensional space case. The last term in the momentum equation corresponds to a turbulent drag force, where r_0 is a positive constant coefficient.

This system, with $r_0 = 0$, has been widely studied, starting from the case of constant coefficients μ, λ and pressure laws of type (3) (see notably [12], [13], [10], [11], [16]). More recently, many studies have focused on density dependent viscosity coefficients $\mu = \mu(\rho)$, $\lambda = \lambda(\rho)$ in space dimensions 2 or 3. These studies were originally developed on Korteweg and shallow water models, corresponding to $\gamma = 2$, $\mu(\rho) = \rho$, $\lambda(\rho) = 0$ (see [7], [3]). They all rely on a new mathematical entropy (*the BD entropy*), that has been discovered in its general form in [4]. It requires that the following algebraic relation holds

$$\lambda(\rho) = 2(\rho\mu'(\rho) - \mu(\rho)). \tag{4}$$

This entropy provides bounds on density gradients, that yield weak compactness of sequences of approximate solutions of (1)-(2) (see recent work [14] on the barotropic equations, and [6] for extension to the full compressible equations). This compactness property on solutions allows, together with a careful regularization scheme, to prove existence of global in time weak solutions, see [6]. All these works concern the whole space case $\mathbb{R}^3$, or the periodic case $\mathbb{T}^3$. Let us observe that the construction procedure of approximate solutions derived in [6] strongly assumes additional drag terms for the barotropic Navier–Stokes equations or adequate cold pressure laws for the compressible Navier–Stokes equations with heat conduction.

The aim of the present paper is to extend such results to a bounded domain Ω. We will consider either Dirichlet boundary condition

$$\rho u = 0, \quad \text{on} \quad \mathbb{R}^+ \times \partial\Omega, \tag{5}$$

on the momentum, or Navier's condition of the type

$$\rho u \cdot n = 0, \quad \mu(\rho)(D(u)n)_\tau = -\alpha\mu(\rho)u_\tau, \quad \text{on} \quad \mathbb{R}^+ \times \partial\Omega, \tag{6}$$

where α is a positive constant. We remind that (6) is a friction condition, expressing that the shear stress at the boundary is proportional to the tangential velocity. It is widely used in the simulation of geophysical flows, and can be justified in the framework of rough boundaries (see [15], [1]).

In order to preserve the BD entropy, an additional boundary condition on the density will be involved, namely that

$$\mu(\rho)\nabla\varphi(\rho) \times n = 0, \tag{7}$$

where $\varphi'(\rho) = \mu'(\rho)/\rho$. This condition expresses that the density should be constant on each connected component of $\partial\Omega$. System (1)-(2) is also supplemented with initial conditions

$$\rho|_{t=0} = \rho_0, \qquad \rho u|_{t=0} = m_0. \tag{8}$$

The precise meaning of (5) to (7), as well as precise assumptions and statements, will be given in the next section. In brief, we will show two results:

i) System (1)-(2), with boundary conditions (5)-(7) and initial condition (8), has a global in time weak solution.

ii) System (1)-(2), with boundary conditions (6)-(7) and initial condition (8), has a global in time weak solution, *under an additional geometrical condition on $\partial\Omega$*.

The existence theorems of weak solutions are stated in Section 2. Their proof is given in Section 3. Let us remark that extension of these theorems to the compressible Navier–Stokes equations with heat conduction may be obtained assuming standard boundary conditions on the temperature. We may just have to replace the drag term by a cold pressure close to vacuum as in [5], where the periodic and whole space cases are treated.

2   Main results

2.1   Assumptions, weak solutions

Viscosity coefficients. As mentioned above, the key assumption is the algebraic relation (4) between $\mu(\rho)$ and $\lambda(\rho)$. Following [14], we add the

following hypothesis: there exists a positive constant $\nu \in (0, 1)$ such that

$$\mu'(\rho) \geq \nu, \qquad \mu(0) \geq 0$$
$$|\lambda'(\rho)| \leq \frac{1}{\nu}\mu'(\rho) \qquad\qquad\qquad (9)$$
$$\nu\mu(\rho) \leq 2\mu(\rho) + 3\lambda(\rho) \leq \frac{1}{\nu}\lambda(\rho).$$

Moreover, if $\gamma \geq 3$, we require that

$$\liminf_{\rho \to +\infty} \frac{\mu(\rho)}{\rho^{\gamma/3+\varepsilon}} > 0$$

for some small ε. As stressed in [14], hypothesis (4) and (9c) imply that

$$\begin{cases} C\,\rho^{2/3+\nu/3} \leq \mu(\rho) \leq C\,\rho^{2/3+1/(3\nu)}, & \rho \geq 1, \\ C\,\rho^{2/3+1/(3\nu)} \leq \mu(\rho) \leq C\,\rho^{2/3+\nu/3}, & \rho \leq 1. \end{cases} \qquad (10)$$

Initial data. We assume as usual that the initial data (8) satisfy

$$\rho_0 \geq 0, \qquad \frac{|m_0|^2}{\rho_0} = 0 \ \text{ on } \ \rho_0^{-1}(\{0\}).$$

and impose the following integrability properties:

$$\rho_0 \in L^\gamma(\Omega), \qquad \frac{|m_0|^2}{\rho_0} \in L^2(\Omega). \qquad (11)$$

Moreover, we shall need

$$\sqrt{\rho_0}\nabla\varphi(\rho_0) \in L^2(\Omega), \qquad\qquad\qquad (12)$$

where φ satisfies $\varphi'(\rho) = \mu'(\rho)/\rho$. From these properties, we shall deduce following [5, 14] that

$$\mu(\rho_0)\nabla\varphi(\rho_0) \in L^1(\Omega).$$

Indeed, writing

$$\mu(\rho_0)\nabla\varphi(\rho_0) = \frac{\mu(\rho_0)}{\sqrt{\rho_0}}\,\sqrt{\rho_0}\nabla\varphi(\rho_0),$$

it is enough to show that $\mu(\rho_0)/\sqrt{\rho_0} \in L^2(\Omega)$. Let us show that one has even

$$\mu(\rho_0)/\sqrt{\rho_0} \in H^1(\Omega).$$

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First,
$$\nabla\left(\frac{\mu(\rho_0)}{\sqrt{\rho_0}}\right) = 2\mu'(\rho_0)\nabla\sqrt{\rho_0} - \frac{\mu(\rho_0)}{2\rho_0^{3/2}}\nabla\rho_0.$$

With
$$2\mu'(\rho)\rho = 2\mu(\rho) + \lambda(\rho) \geq \frac{2\mu(\rho)}{3} + \lambda(\rho) \geq \frac{\nu}{3}\mu(\rho).$$

We deduce that
$$\left|\nabla\left(\frac{\mu(\rho_0)}{\sqrt{\rho_0}}\right)\right| \leq C\,\mu'(\rho_0)|\nabla\sqrt{\rho_0}| \;=\; \frac{C}{2}\sqrt{\rho_0}\nabla\varphi(\rho_0) \in L^2(\Omega).$$

Besides, from the various bounds on μ, we have easily that
$$(\mu(\rho_0)/\sqrt{\rho_0})^s \in L^2(\Omega), \quad \text{for some} \quad 0 < s < 1.$$

A classical bootstrap argument provides $\mu(\rho_0)/\sqrt{\rho_0} \in L^2(\Omega)$ which ends the proof (see [14]) for all details).

Thus, we have $\mu(\rho_0)\nabla\varphi(\rho_0) \in L^1(\Omega)$, and
$$\mathrm{curl}\,(\mu(\rho_0)\nabla\varphi(\rho_0)) = \mathrm{curl}(\nabla\psi(\rho_0)) = 0, \quad \text{where} \quad \psi'(\rho) = \mu(\rho)\,\varphi'(\rho).$$

We are then able to define a tangential trace
$$\mu(\rho_0)\nabla\varphi(\rho_0) \times n \in L^\infty(\partial\Omega).$$

We impose that
$$\mu(\rho_0)\nabla\varphi(\rho_0) \times n = 0, \quad \text{on } \partial\Omega. \tag{13}$$

We point out that, similarly to $\mu(\rho_0)\nabla\varphi(\rho_0)$, we have
$$\left|\nabla\sqrt{\mu(\rho_0)}\right| = \frac{1}{2}\left|\frac{\mu'(\rho_0)}{\sqrt{\mu(\rho_0)}}\nabla\rho_0\right| \leq \frac{1}{2\sqrt{\nu}}|\sqrt{\rho_0}\,\nabla\varphi(\rho_0)| \in L^2(\Omega) \tag{14}$$

and $\sqrt{\mu(\rho_0)}^s \in L^2(\Omega)$ for some $0 < s < 1$, which leads to
$$\sqrt{\mu(\rho_0)} \in H^1(\Omega).$$

Weak solutions. We now precise our notion of weak solution in the bounded domain Ω. It depends whether we deal with Dirichlet condition (5) or Navier

condition (6) on the velocity. In both cases, we ask for the following regularities (that echoes those of the initial data): for all $T > 0$,

$$\rho \in L^\infty(0, T; L^\gamma(\Omega)), \quad \sqrt{\rho}u \in L^\infty(0, T; L^2(\Omega)), \quad \rho^{1/3}u \in L^3((0, T) \times \Omega)$$

$$\sqrt{\mu(\rho)}\nabla u \in L^2((0, T) \times \Omega), \quad \sqrt{\rho}\nabla\varphi(\rho) \in L^\infty(0, T; L^2(\Omega)). \tag{15}$$

Let us show that these regularities imply:

$$\mu(\rho)/\rho \in L^\infty(0, T; H^1(\Omega)), \quad \sqrt{\mu(\rho)} \in L^\infty(0, T; H^1(\Omega)),$$
$$\mu(\rho)\,\nabla\varphi(\rho) \in L^\infty(0, T; L^1(\Omega)), \quad \rho u \in L^2(0, T; W^{1,1}(\Omega)), \tag{16}$$
$$\mu(\rho)u \in L^2(0, T; W^{1,1}(\Omega)).$$

The first three regularity properties are obtained with the same reasoning as for the initial data. Then, to obtain the regularity of ρu, we write

$$\rho u \,=\, \sqrt{\rho}\,(\sqrt{\rho}u) \in L^\infty(0, T; L^{2\gamma/(1+\gamma)}(\Omega)) \subset L^\infty(0, T; L^1(\Omega)).$$

We also compute the gradient, which yields

$$|\nabla(\rho u)| \leq |\rho\,\nabla u| \,+\, |\nabla\rho \otimes u|$$

$$\leq \sqrt{\rho}\,(\sqrt{\rho}|\nabla u|) \,+\, \frac{1}{\nu}\frac{\mu'(\rho)}{\sqrt{\rho}}\,(\sqrt{\rho}u)$$

$$\leq \sqrt{\nu}\sqrt{\rho}\,\left(\sqrt{\mu(\rho)}\nabla u\right) \,+\, \frac{1}{\nu}\,(\sqrt{\rho}\nabla\varphi(\rho))\,(\sqrt{\rho}u),$$

with r.h.s. clearly in $L^2(0, T; L^1(\Omega))$. Finally, to control $\mu(\rho)u$, we notice that

$$\mu(\rho)u = \frac{\mu(\rho)}{\sqrt{\rho}}\,(\sqrt{\rho}u) \in L^\infty(0, T; L^1(\Omega)),$$

and

$$\nabla(\mu(\rho)u) = \sqrt{\mu(\rho)}\,\left(\sqrt{\mu(\rho)}\nabla u\right) \,+\, \frac{\nabla\mu(\rho)}{\sqrt{\rho}} \otimes (\sqrt{\rho}u) \in L^2(0, T; L^1(\Omega)).$$

We are ready for the following definitions

Definition 1 (weak solutions)

1. *We shall say that (ρ, u) is a (global) weak solution of (1)-(2) with boundary conditions (5)-(7) if it satisfies regularity properties (15), as well as (5) in $L^2(0, T; L^1(\partial\Omega))$, (7) in $L^2(0, T; L^\infty(\partial\Omega))$, and (1)-(2) in $\mathcal{D}'(0, T \times \Omega)$ for all $T > 0$.*

2. *We shall say that (ρ, u) is a (global) weak solution of (1)-(2) with boundary conditions (6)-(7) if it satisfies regularity properties (15), as well as $\rho u \cdot n = 0$ in $L^2(0, T; L^1(\partial\Omega))$, (1) in $\mathcal{D}'(0, T \times \Omega)$, and for all $v \in C_c^\infty(0, T; \overline{\Omega})^3$:*

$$
- \int_\Omega (\rho u \cdot \partial_t v + \rho u \otimes u : \nabla v + p(\rho) \operatorname{div} v) \; + \; \int_\Omega 2\mu(\rho) D(u) : D(v)
$$
$$
+ \int_\Omega \lambda(\rho) \operatorname{div} u \operatorname{div} v + \int_\Omega r_0 \rho |u| u \cdot v = - \int_{\partial\Omega} \alpha \mu(\rho) u_\tau \cdot v_\tau,
$$
(17)

for almost every $t \in (0, T)$, for all $T > 0$.

We point out that in virtue of (15), (16), all boundary terms are well-defined, notably the surface integral in (17) (using the Sobolev regularity of $\mu(\rho)u$). As usual, we deduce from (15) and the Navier–Stokes system itself that ρ and ρu are continuous in time with values in $W^{-1,1}(\Omega)$, which allows to define their initial values.

We now state our existence results:

Theorem 1 (Weak solutions for Dirichlet condition)
Under above hypothesis on the viscosity coefficients and initial data (ρ_0, m_0), there exists a global weak solution (ρ, u) of (1)-(2) with boundary conditions (5)-(7), such that $\rho|_{t=0} = \rho_0$, $\rho u|_{t=0} = m_0$.

Theorem 2 (Weak solutions for Navier condition)
Under above hypothesis on the viscosity coefficients and initial data (ρ_0, m_0), and under the assumption

$$
\alpha Id - \kappa(x) \geq 0, \quad \forall x \in \partial\Omega
$$
(18)

where κ is the Weingarten endomorphism of $\partial\Omega$, there exists a global weak solution (ρ, u) of (1)-(2) with boundary conditions (6)-(7), such that $\rho|_{t=0} = \rho_0$, $\rho u|_{t=0} = m_0$.

We remind that the Weingarten endomorphism is defined by

$$
\kappa(x) : T_x(\partial\Omega) \mapsto T_x(\partial\Omega), \quad v_\tau \mapsto D_{v_\tau} n(x).
$$

In two dimensions, it reduces to the scalar curvature. Condition (18) expresses that friction must be large enough near parts of the boundaries that are far from being flat.

The rest of the paper is devoted to the proof of these theorems and possible extensions.

3 Proof

This section contains the proof of theorems 1 and 2. It relies of course on the analysis carried in domains without boundaries ($\mathbb{R}^3$ or $\mathbb{T}^3$). The main idea is to obtain good energy estimates, notably using the BD entropy. This will provide enough compactness on a sequence of approximate solutions to pass to the limit and obtain a global weak solution.

3.1 Energy estimates and BD entropy

The keypoint of the proof is to derive two estimates:

i) the classical energy estimate on ρ^γ and $\rho|u|^2$.

ii) the BD entropy estimate on $\rho|u + \nabla\varphi(\rho)|^2$.

We show in this paragraph how these estimates are preserved by our choice of boundary conditions. The classical estimate i) comes from the multiplication of (2) by u, of (1) by $|u|^2/2$, and addition of the resulting equations. It reads (say for smooth solutions (ρ, u) with ρ bounded from below by a positive constant):

$$\frac{d}{dt}\int_\Omega \left(\rho\frac{|u|^2}{2} + \frac{1}{\gamma-1}\rho^\gamma\right) + \int_\Omega \left(2\mu(\rho)D(u) : D(u) + \lambda(\rho)|\operatorname{div} u|^2\right)$$
$$+r_0\int_\Omega \rho|u|^3 = \int_{\partial\Omega}\left(2\mu(\rho)D(u)n \cdot u - (p + \rho\frac{|u|^2}{2} - \lambda(\rho)\operatorname{div} u)(u \cdot n)\right)ds.$$

With boundary condition (5), the surface integral at the r.h.s. vanishes. In the case of Navier condition (6), it reduces to $-\alpha\int_{\partial\Omega}\mu(\rho)|u_\tau|^2 \leq 0$. In both cases,

$$\frac{d}{dt}\int_\Omega \left(\rho\frac{|u|^2}{2} + \frac{1}{\gamma-1}\rho^\gamma\right) + \int_\Omega \left(2\mu(\rho)D(u) : D(u) + \lambda(\rho)|\operatorname{div} u|^2\right)$$
$$+r_0\int_\Omega \rho|u|^3 \leq 0. \tag{19}$$

Then, the key estimate ii) involves the algebraic relation (4). From the mass equation, renormalization techniques provide (with the usual summation convention over repeated indices)

$$2\partial_t\partial_i\mu(\rho) + \partial_j(2\mu(\rho)\partial_i u_j) + \partial_j(u_j\partial_i 2\mu(\rho)) + \partial_i(\lambda(\rho)\partial_j u_j) = 0, \tag{20}$$

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which added to the momentum equation

$$\partial_t(\rho u_i) + \partial_j(\rho u_i u_j) + \partial_i p - \partial_j(\mu(\rho)(\partial_i u_j + \partial_j u_i)) - \partial_i(\lambda(\rho)\partial_j u_j) = 0, \quad (21)$$

implies the following identity

$$\partial_t(\rho w_i) + \partial_j(\rho u_j w_i) + \partial_i p - \partial_j(\mu(\rho)(\partial_j u_i - \partial_i u_j)) = 0, \qquad (22)$$

where $w = u + 2\nabla\varphi(\rho)$ and $\rho\varphi'(\rho) = \mu'(\rho)$.

It suffices now to multiply this equation by w_i and sum up with respect to i and to use the mass equation multiplied by $|w_i|^2/2$ to get the following extra information on w

$$\frac{d}{dt}\int_\Omega \rho\left(\frac{|w|^2}{2} + e(\rho)\right) + \int_\Omega 2\mu(\rho)A(u) : A(u) + 2\int_\Omega \nabla p \cdot \nabla\varphi(\rho)$$

$$+ r_0\int_\Omega \rho|u|^3 = \int_{\partial\Omega}\left(2\mu(\rho)w \cdot (A(u) \cdot n) - \frac{|w|^2}{2}\rho u \cdot n\right) ds$$

$$- r_0\int_\Omega \rho|u|u \cdot \nabla\varphi(\rho). \qquad (23)$$

Dirichlet condition. We first remark that

$$\int_{\partial\Omega} 2\mu(\rho)\nabla\varphi(\rho) \cdot (A(u) \cdot n)\, ds = \int_{\partial\Omega} 2\mu(\rho)\nabla\varphi(\rho) \cdot (\operatorname{curl} u \times n)\, ds$$

$$= -2\int_{\partial\Omega}(\mu(\rho)\nabla\varphi(\rho) \times n) \cdot \operatorname{curl} u\, ds$$

which vanishes because of condition (7). Thus, when (5) holds, the first term in the r.h.s. vanishes. Concerning the last term, integrating by parts,

$$-\int_\Omega \rho|u|u \cdot \nabla\varphi(\rho) = \int_\Omega \mu(\rho)\left(|u|\operatorname{div} u + \frac{u}{|u|} \cdot (u \cdot \nabla)u\right). \qquad (24)$$

This term may be controlled since

$$\int_\Omega \mu(\rho)|u||\nabla u| \leq \int_\Omega \frac{\sqrt{\mu(\rho)}}{\rho^{1/3}}\rho^{1/3}|u|\sqrt{\mu(\rho)}|\nabla u|1_{\rho\leq 1}$$

$$+ \int_\Omega \sqrt{\mu(\rho)}|u|\sqrt{\mu(\rho)}|\nabla u|1_{\rho>1}.$$

We now use (10) and the fact that $\rho^{1/3}u \in L^3((0,T) \times \Omega)$ from (19). This gives

$$\frac{\sqrt{\mu(\rho)}}{\rho^{1/3}} 1_{\rho \leq 1} \in L^\infty((0,T) \times \Omega), \quad u 1_{\rho > 1} \in L^3((0,T) \times \Omega)$$

and therefore

$$\int_\Omega \mu(\rho)|u||\nabla u| \leq c\|\sqrt{\mu(\rho)}\nabla u\|_{L^2((0,T) \times \Omega)}\left(1 + \|\sqrt{\mu(\rho)}\|_{L^6(\Omega)}\right). \tag{25}$$

Using now (23), (24), (25) and estimates similar as (14), we get

$$\frac{d}{dt}\int_\Omega \rho\left(\frac{|w|^2}{2} + e(\rho)\right) + \int_\Omega 2\mu(\rho)A(u) : A(u) + 2\int_\Omega \nabla p \cdot \nabla\varphi(\rho)$$

$$+r_0\int_\Omega \rho|u|^3 \leq c\|\sqrt{\mu(\rho)}\nabla u\|_{L^2((0,T) \times \Omega)}\left(1 + \|\sqrt{\rho}\nabla\varphi(\rho)\|_{L^2(\Omega)}\right) \tag{26}$$

with $c \in L^3(0,T)$. We now conclude, using the identity $2ab \leq \varepsilon a^2 + \frac{1}{\varepsilon}b^2$ and Gronwall Lemma, summing (19), (26) and using that $\sqrt{\rho}u \in L^2(0,T; L^2(\Omega))$.

Navier condition. To handle the Navier condition, we use the following lemma:

Lemma 1 *Let $u \in \mathcal{C}^\infty\left(\overline{\Omega}\right)^d$, $d = 2, 3$, satisfying $u \cdot n = 0$. We have*

$$(D(u)n)_\tau = \frac{1}{2}\left(\frac{\partial u}{\partial n}\right)_\tau - \frac{1}{2}\kappa(x)u_\tau, \tag{27}$$

where κ is the scalar curvature for $d = 2$, the Weingarten endomorphism of $\partial\Omega$ for $d = 3$.

Proof. Assume first that $d = 2$. Let (τ, s) the direct Fresnet frame. We write for any vector field v

$$v = v_\tau + v_n n = v_t\tau + v_n n.$$

We compute

$$(D(u) \cdot n)_t = \frac{\partial_i u_j}{2}n_j\tau_i + \frac{\partial_j u_i}{2}n_j\tau_i.$$

On the one hand,

$$n_j\tau_i\partial_j u_i = \frac{\partial u_i}{\partial n}\tau_i = \left(\frac{\partial u}{\partial n}\right)_t.$$

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On the other hand,

$$
\begin{aligned}
n_j \tau_i \partial_i u_j &= n_j \tau_i \partial_i (u_t \tau_j) \\
&= n_j \tau_i \tau_j \partial_i u_t + u_t n_j \tau_i \partial_i \tau_j \\
&= u_t ((\tau \cdot \nabla) \tau) \cdot n = -u_t \kappa.
\end{aligned}
$$

Equality (27) follows. Assume now that $d = 3$. We denote by (τ^a, τ^b, n) a direct orthonormal basis, depending smoothly on $x \in \partial\Omega$. We write for any vector field v

$$
v = v_\tau + v_n n = v_a \tau^a + v_b \tau^b + v_n n.
$$

Computing as previously, we get

$$
(D(u) \cdot n)_{a,b} = \left(\frac{\partial u}{\partial n} \right)_{a,b} + u_a \left((\tau^{a,b} \cdot \nabla) \tau^a \right) \cdot n + u_b \left((\tau^{a,b} \cdot \nabla) \tau^b \right) \cdot n.
$$

This ends the proof of the lemma. □

We are now able to control the boundary term. The direct combination of (6), (27) yields

$$
-\frac{1}{2} \left(\frac{\partial u}{\partial n} \right)_\tau + \frac{1}{2} \kappa(x) u_\tau = \alpha u_\tau, \tag{28}
$$

Note that

$$
A(u) \cdot n = \operatorname{curl} u \times n, \quad x \in \partial\Omega,
$$

so that $(A(u) \cdot n) \cdot n = 0$. We deduce, using (6) and (28)

$$
\begin{aligned}
A(u) n = (A(u) \cdot n)_\tau &= \left(\frac{\partial u}{\partial n} \right)_\tau - (D(u) \cdot n)_\tau \\
&= \left(\frac{\partial u}{\partial n} \right)_\tau + \alpha u_\tau = (-\alpha + \kappa) u_\tau.
\end{aligned}
$$

As a consequence, using (7),

$$
\begin{aligned}
\int_{\partial\Omega} 2\mu(\rho) w \cdot (A(u) \cdot n) \, ds &= \int_{\partial\Omega} 2\mu(\rho) u \cdot (A(u) \cdot n) \, ds \\
&= \int_{\partial\Omega} 2\mu(\rho) \left((-\alpha \operatorname{Id} + \kappa) \cdot u_\tau \right) \cdot u_\tau \, ds.
\end{aligned}
$$

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we can conclude that this term is negative if (18) holds, leading to

$$\frac{d}{dt}\int_\Omega \rho\left(\frac{|w|^2}{2}+e(\rho)\right)+\int_\Omega 2\mu(\rho)A(u):A(u)+r_0\int_\Omega \rho|u|^3$$
$$+2\int_\Omega \nabla p\cdot\nabla\varphi(\rho)\le c\|\sqrt{\mu(\rho)}\nabla u\|_{L^2(\Omega)}\big(1+\|\sqrt{\rho}\|_{L^6(\Omega)}\big). \qquad (29)$$

We now conclude as in the Dirichlet condition part.

4 Approximate solutions and compactness

The a priori estimates (19) and (29) are the key ingredient of the existence results. Broadly, as soon as "approximate solutions" $(\rho^\varepsilon, u^\varepsilon)$ of (1)-(2) satisfy such bounds, compactness properties are sufficient to extract a subsequence that converges to a weak solution (thanks notably to the strong convergence of $\sqrt{\rho^\varepsilon}u^\varepsilon$ in $L^2((0,T)\times\Omega)$). For the sake of brevity, we do not detail the compactness arguments, which are exactly the same as for the whole space and periodic case given in [14].

Note that we consider an additional drag term $r_0\rho|u|u$ in the momentum equation compared to the standard barotropic Navier–Stokes equations. This term provides the additional integrability $\rho^{1/3}u\in L^3((0,T)\times\Omega)$ which replaces the integrability on $\sqrt{\rho}u$ obtained in [14] by the use of the multiplier $(1+\ln(1+|u|^2))u$ in the momentum equations. Such integrability is sufficient to obtain enough compactness on $\sqrt{\rho^\varepsilon}u^\varepsilon$. The main reason of this drag term addition relies on the construction of approximate solutions that has been performed in [6]. The first smoothing operator is introduced as an additional force at the right hand side of the momentum equation, inspired from capillary forces $\varepsilon\rho\nabla\Delta\rho$. Such capillary forces already yield $L^\infty(0,T;L^2)$ bounds on $\varepsilon^{1/2}\nabla\rho$ and $L^2(0,T;L^2)$ bounds on $\varepsilon^{1/2}\Delta\rho$ by using the new BD entropy. One way to obtain higher smoothing effects, still preserving the mathematical entropy, is to consider modified capillary forces such as $\varepsilon\rho\nabla\Delta^s\rho$ with large enough parameter s in order to have high Sobolev bounds in space on the density. Once the density has been suitably regularized, it remains to add some hyperdiffusive terms in the right hand side of the momentum and temperature equation to end up with a globally well posed system.

Such procedure does not seem to apply to standard Navier–Stokes equations without additional drag terms or cold pressure for the compressible Navier-Stokes equations with heat conduction [5] (the so called cold pressure

component is just the zero Kelvin isothermal relating pressure and density variables) since it seems incompatible with the multiplier introduced in [14]. Extension to such physical cases is an interesting question which is still in progress in [9].

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