Waveform Modelling and Numerical Relativity: What do We Need?

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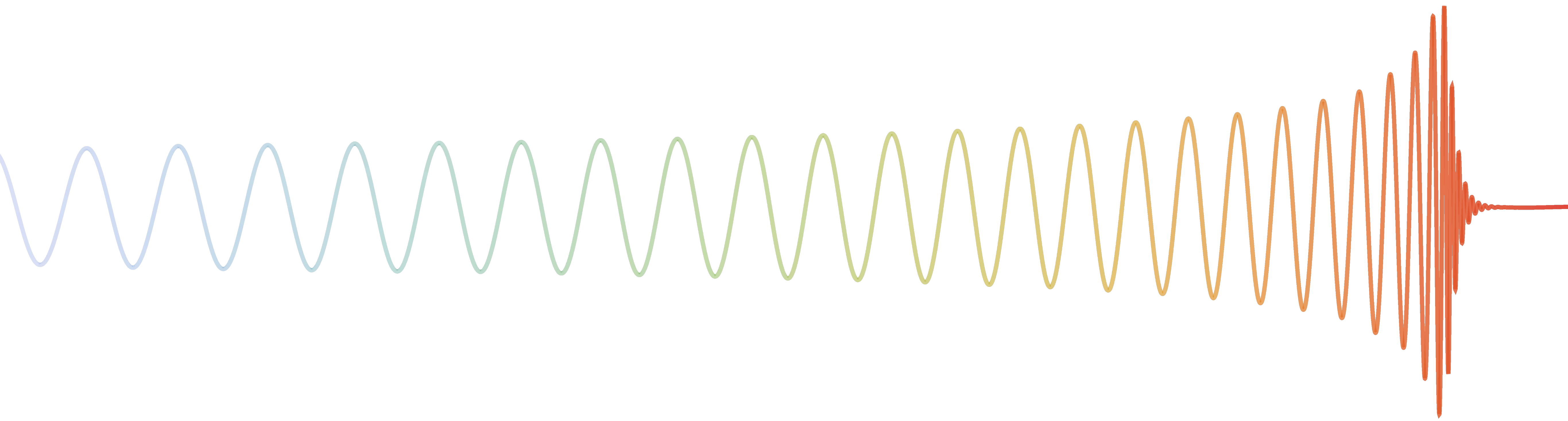
GrEAT, University of Birmingham, 2019

Phenomenology of Binary Black Holes



Inspiral:
Post-Newtonian Theory
Effective One Body

Analytical approximations begin to break down

Merger:
No analytical model

Ringdown:
Black Hole Perturbation Theory

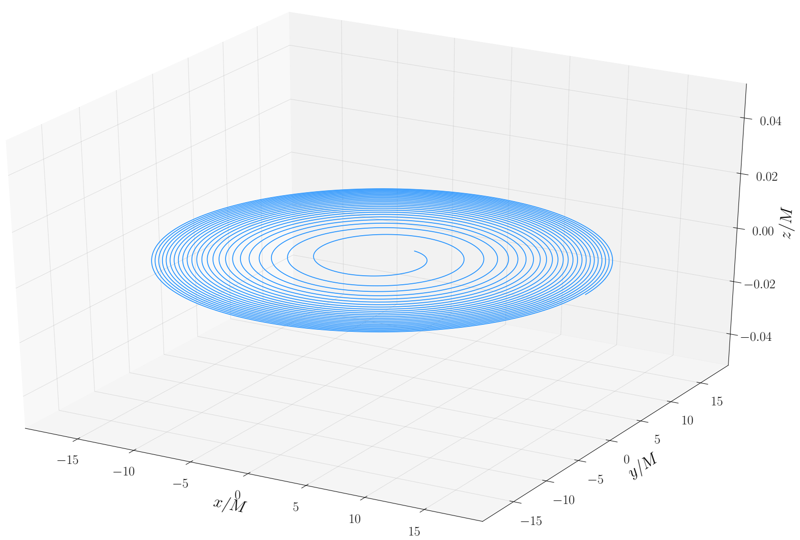
Flux Balance:

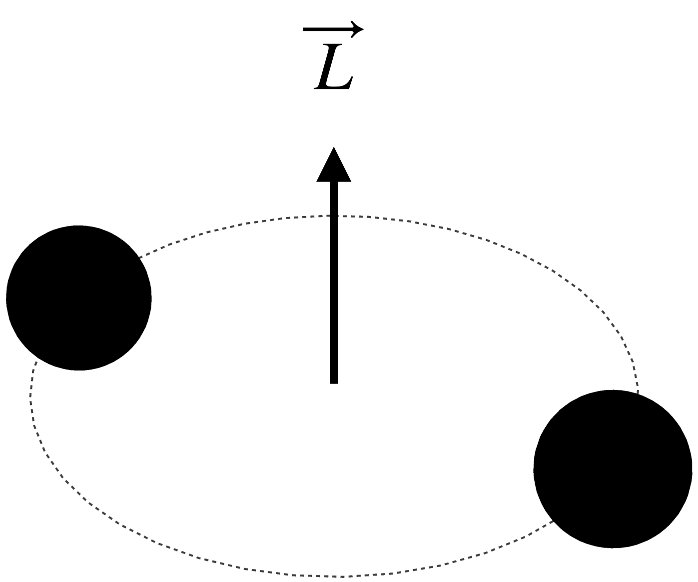
$$\frac{dE_{\rm orbital}}{dt} = \mathcal{L}_{\rm GW} \approx \frac{32}{5}\frac{c^5}{G}\frac{(m_1 m_2)^2}{M^4}\left(\frac{v}{c}\right)^5$$

Phenomenology of Binary Black Holes

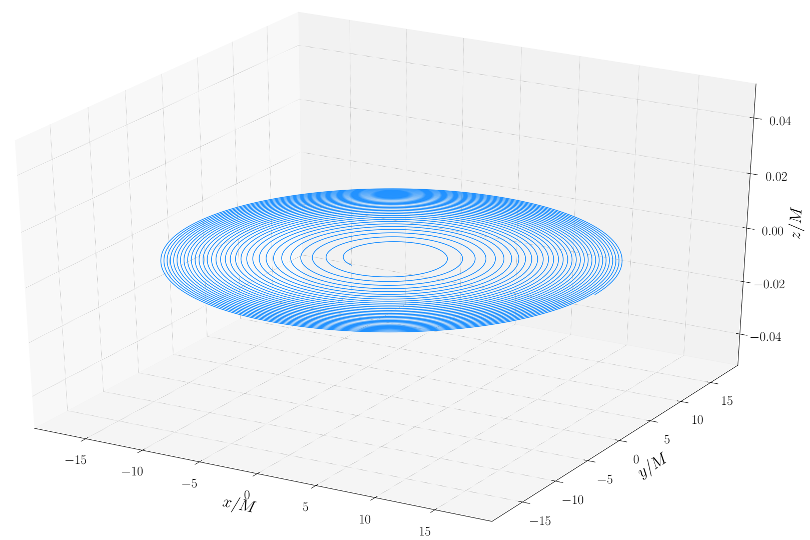
- Spin-orbit and spin-spin couplings can change the phenomenology of the binary dramatically

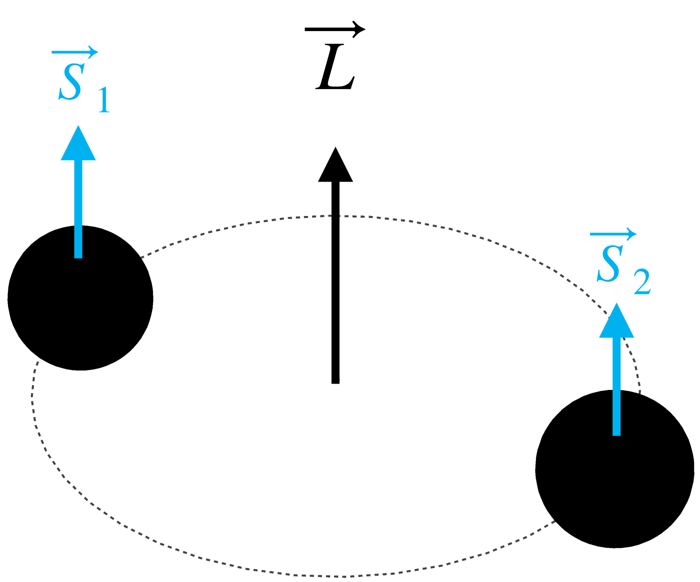
Non-Spinning



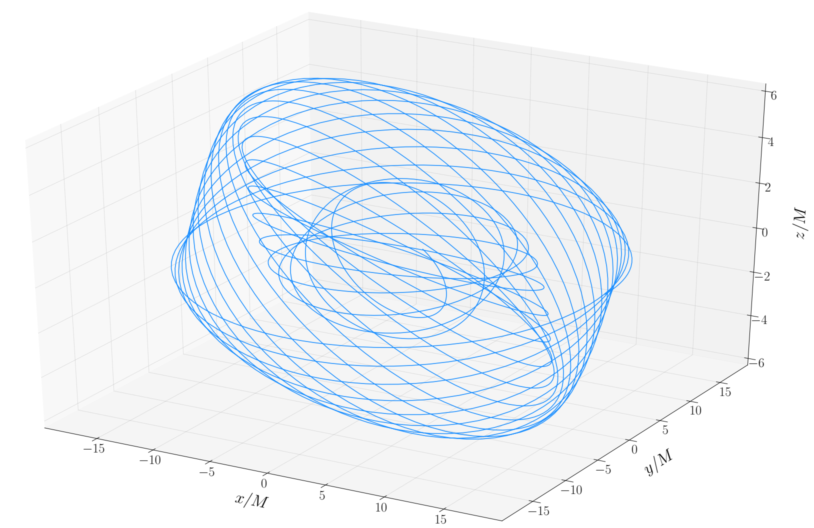


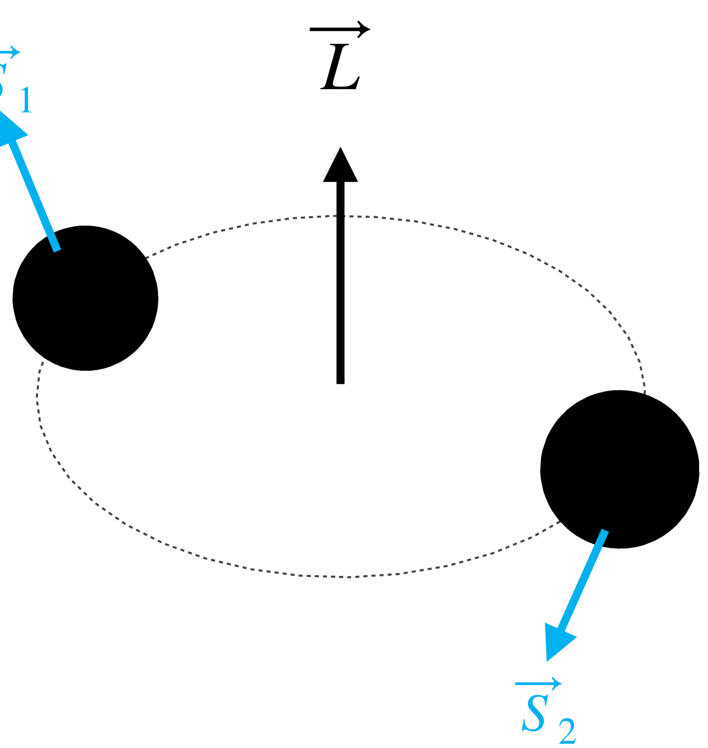
Aligned Spins





Precessing Spins





Waveform Modelling: Key Approaches

Numerical Relativity

- Solve full Einstein Field Equations
- Coupled system of PDEs
- Time integration

Effective One Body

- Hamiltonian framework
- System of ODEs
- Time integration
- Evolve dynamics, obtain GW emission

Phenomenological

- Direct fit to GW signal
- Closed-form expressions
- Frequency domain model

Computational Efficiency →

← Model Complexity

Numerical Relativity

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Numerical Relativity: 50 years of history and growing!

1962 ADM 3+1 Formulation

1964 Hahn-Lindquist, 2 wormholes

1984, Unruh, Excisions

1992, Choptuik; Abrahams & Evans, Critical Phenomena

1997, Brandt-Brügmann, Puncture Data

99-00, AEI/PSU, Grazing Collisions

00, Choptuik; Schnitter; Brügmann, Mesh Refinement

2005, Pretorius, IMR w/ Harmonic Gauge

2005-06,UTB; NASA; IMR w/ BSSN + punctures

2006-08, Caltech/ Cornell, IMR w/ Spectral

2007, Ajith, AEI, Jena, Phenom GW Models

2009, UMD, SXS, EOB GW Models

2011, Schmidt ea, Radiation aligned frame & precession

2011, Lousto ea, q100

2014, Schmidt ea, Precessing waveform models

2015, Szilagyi ea, 175 orbits

1964 ~1999 ~2005 2019+

1975-77, Smarr-Eppley, Head on Collisions

1979, York, Kinematics & Dynamics of GR

1989-95, Bona-Masso, Modified ADM (hyperbolicity!)

1994, Cook, Bowen-York Initial Data

1994, NCSA-Wash., Improving head on collisions

99-05, JW York, Cornell, Caltech, LSU, Hyperbolic Formulations

1999, BSSN

1999,York, Conformal Think Sandwich ID

2000-02, Alcubierre, Gauge Conditions

2000 Ashtekar, Isolated Horizons

2004, Brügmann et al, One Orbit

2006-07, Baker ea, Gonzalez ea, Non-spinning BBH Kicks

2007, SXS, PN-NR

2007-11, RIT; Jena; AEI, BBH Superkicks

2010, Bernuzzi et al, CCZ4/Z4c

2007-11, RIT; Jena; AEI, BBH Superkicks

2008, All of NR, NINJA

2011, Lovelace ea, Spins ~ 0.97

2009-11, Bishop, CCE

2011-, Le Tiec ea, self-force

2015-, All NR, GW150914 NR Follow-up

Adapted from Lousto; Pfeiffer

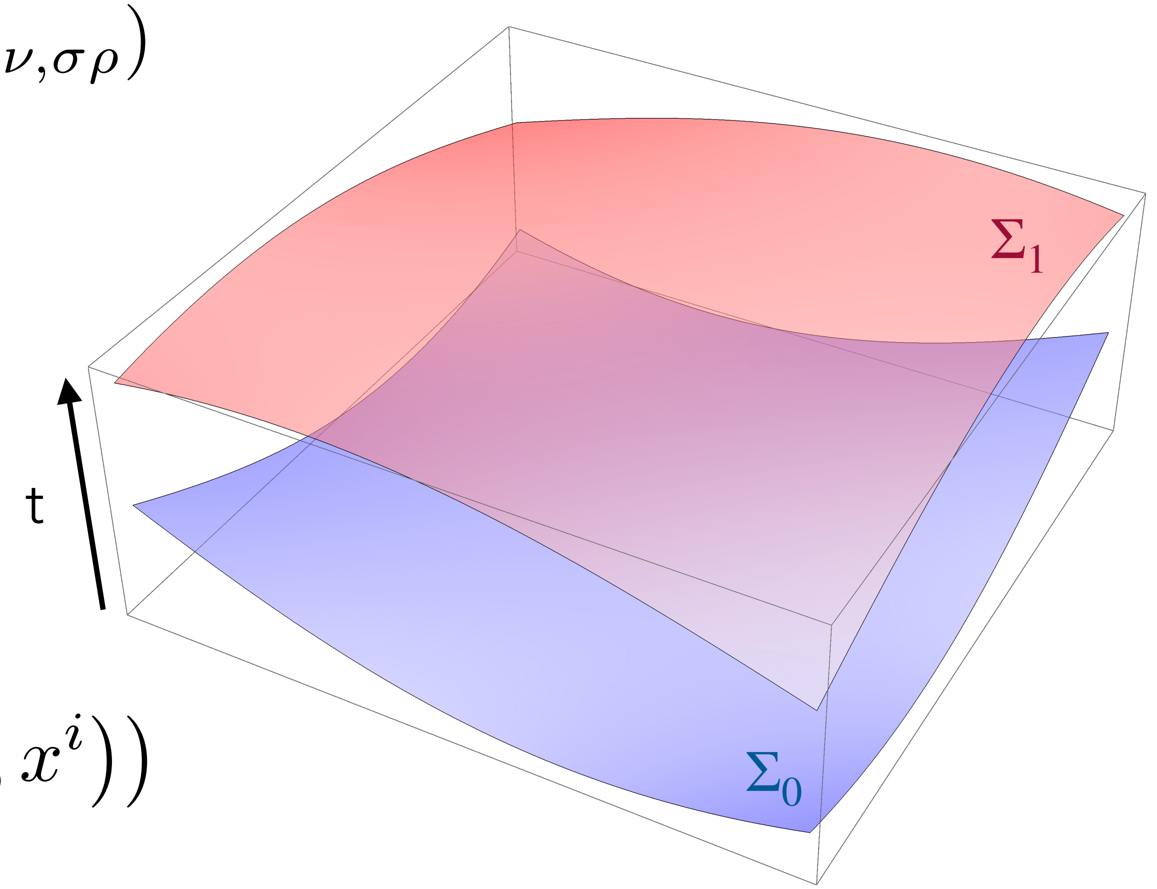
Numerical Relativity: Broad Overview

- Einstein Field Equations: 10 coupled non-linear 2nd order partial differential equations

$$R_{\mu\nu} \equiv \frac{1}{2} g^{\sigma\rho} \left(g_{\sigma\nu,\mu\rho} + g_{\mu\rho,\sigma\nu} - g_{\sigma\rho,\mu\nu} - g_{\mu\nu,\sigma\rho} \right) + g^{\sigma\rho} \left(\Gamma^{m}_{\mu\rho} \Gamma_{m\sigma\nu} - \Gamma^{m}_{\mu\nu} \Gamma_{m\sigma\rho} \right)$$

$$\Gamma^{\mu}_{\nu\sigma} \equiv \frac{1}{2} g^{\mu\rho} \left(g_{\rho\nu,\sigma} + g_{\rho\sigma,\nu} - g_{j\sigma,\rho} \right)$$

- Formulate as an initial boundary value problem



$$\frac{\partial}{\partial t} u\left(t, x^i\right) = F\left(u\left(t, x^i\right), \partial u\left(t, x^i\right), \partial^2 u\left(t, x^i\right)\right)$$

- Initial data $u(0,\mathbf{x}) = f(\mathbf{x})$ evolved forward in time with evolution equations $\partial_t u(t,\mathbf{x}) = F(u, \partial u, \partial^2 u)$

- Constraint equations can be defined on surfaces of equal time Σ_t

Numerical Relativity: The Two Main Approaches

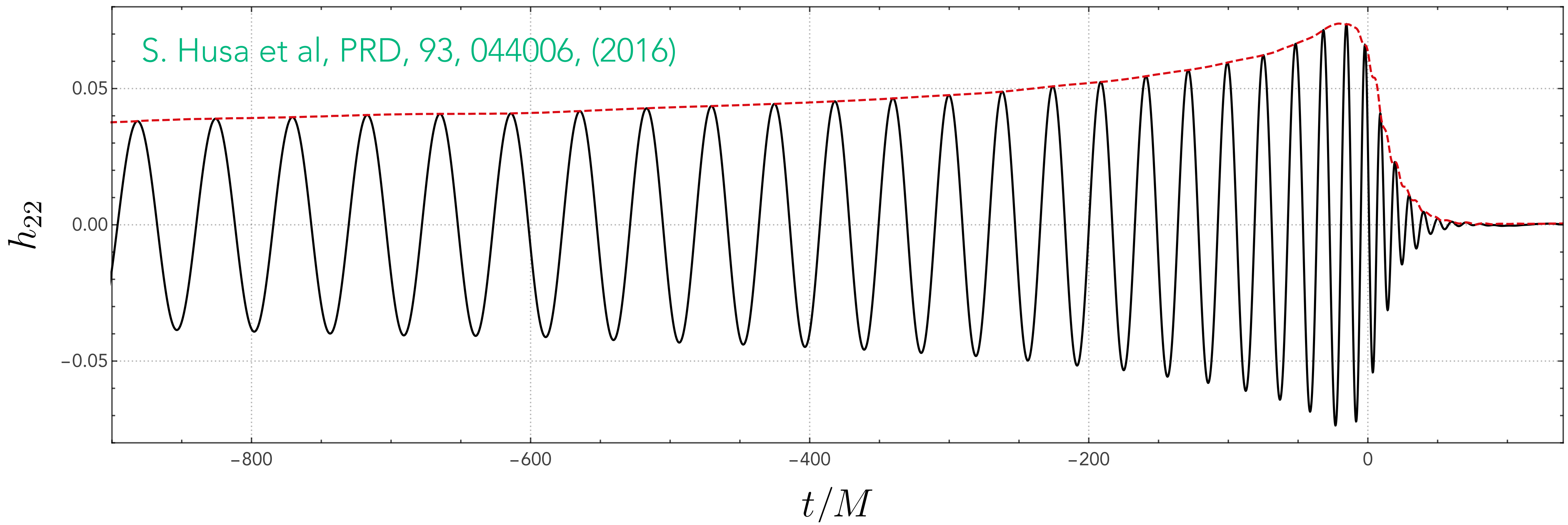
- Puncture initial data (Brandt & Brügmann 97)
- BSSN or CCZ4 formulation with moving punctures (Campanelli ea 06, Baker ea 06, Bernuzzi et 10, Alec ea 12)
- Gauge conditions: 1+log, Gamma-driver, Sommerfeld outer BC
- Finite differences with AMR
- Historic groups: RIT, GATech, Goddard, Jena, UIB Palma, Cardiff, Perimeter
- Codes: BAM, Einstein Toolkit, LEAN, GRChombo, SACRA, …

- Quasi-equilibrium excision initial-data (Cook&Pfeiffer 04, Lovelace ea 08)
- Generalized harmonic with constraint damping (Gundlach ea 05, Pretorius 05)
- Gauge conditions: damped harmonic gauge, constraint preserving, minimally reflective outer BCs
- Multi-domain spectral methods
- Historic groups: The SXS collaboration, Cornell, Caltech, CITA, WSU, Fullerton, AEI
- Codes: SpEC, BAMPS, …

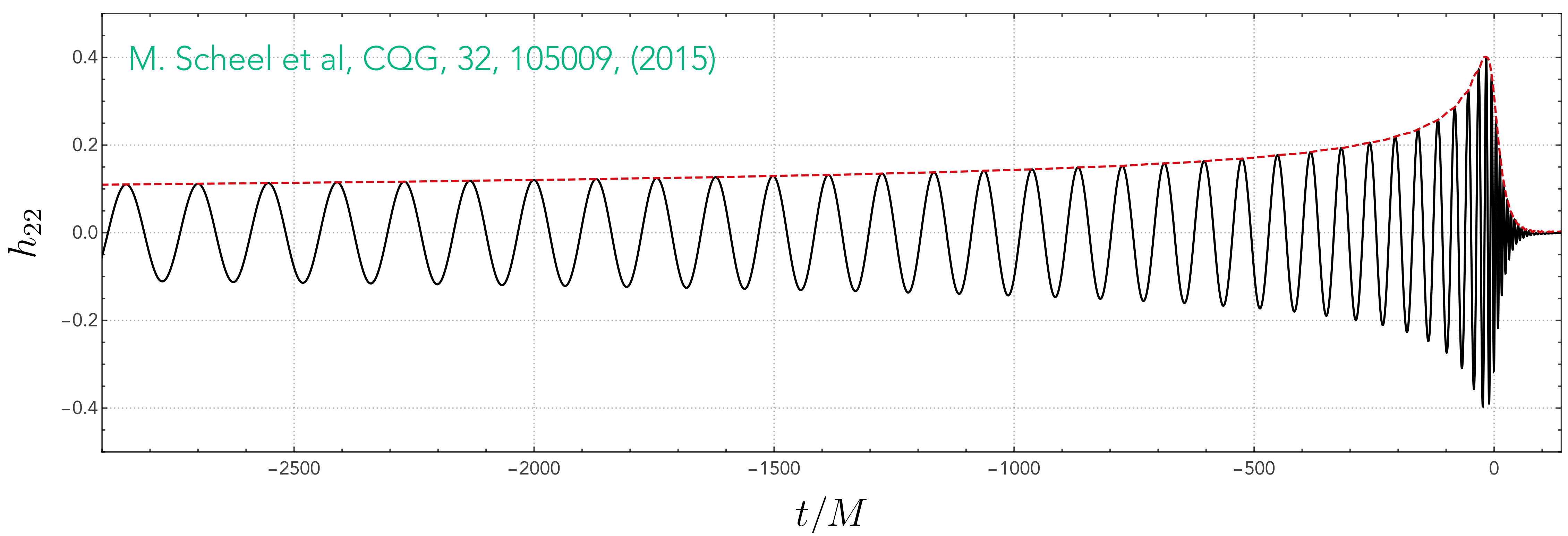
- Some overlapping approaches, e.g. Pretorius used GH + FD + AMR

Numerical Relativity: Extreme Cases

- High mass ratio q18 series produced with BAM code: $\chi_1 \in \{-0.8, -0.4, 0.0, +0.4, +0.8\}$



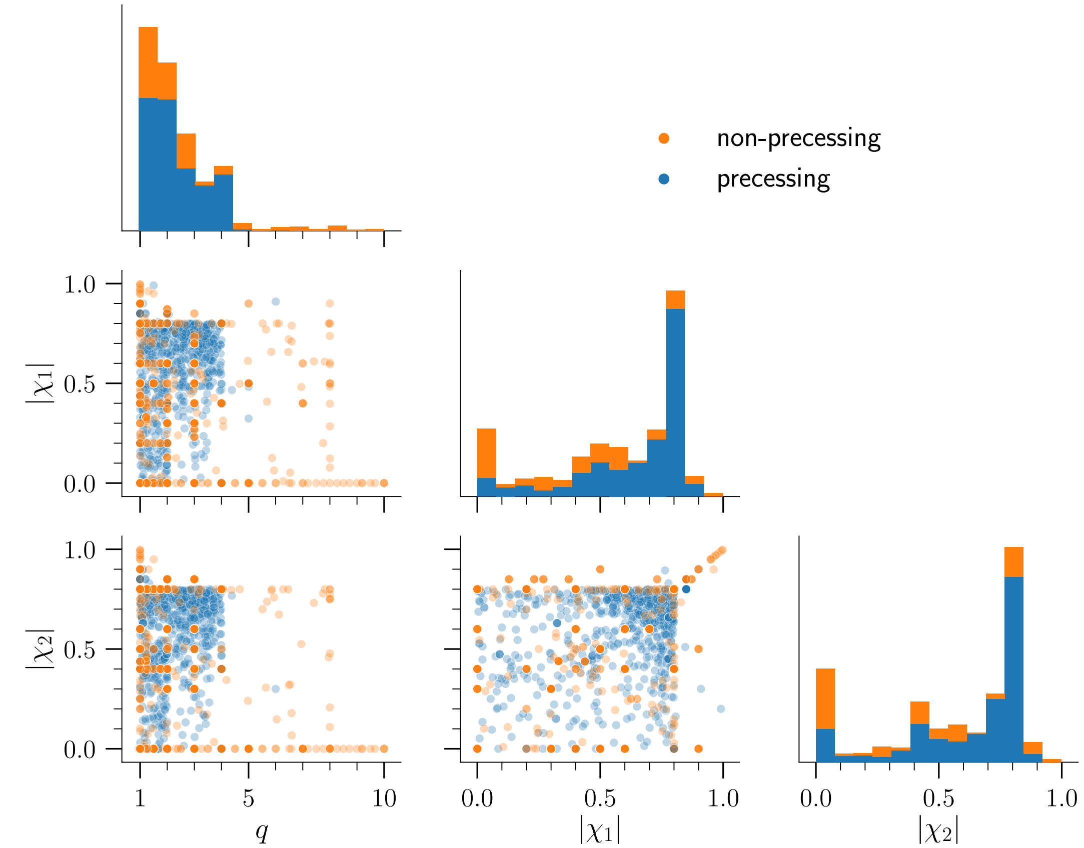
- High spin, equal-mass series produced with SpEC code, e.g. $\chi_1 \in \{0.995\}$

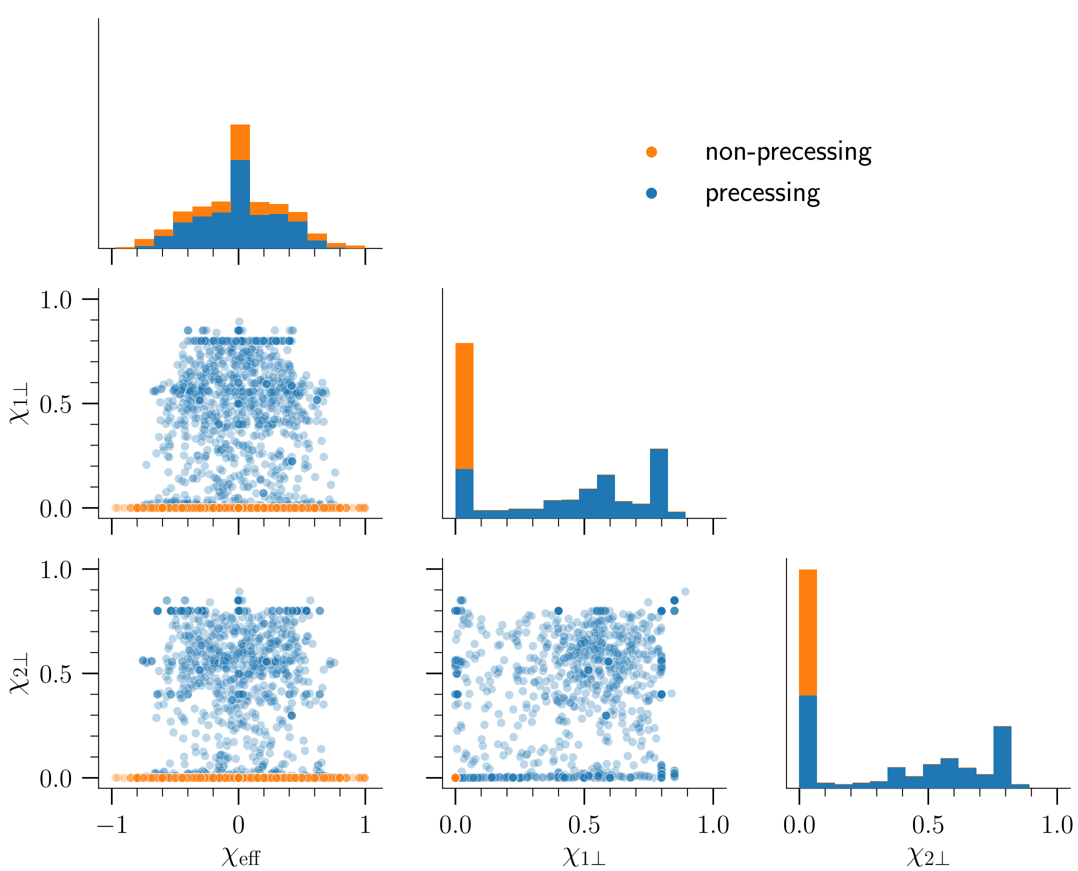


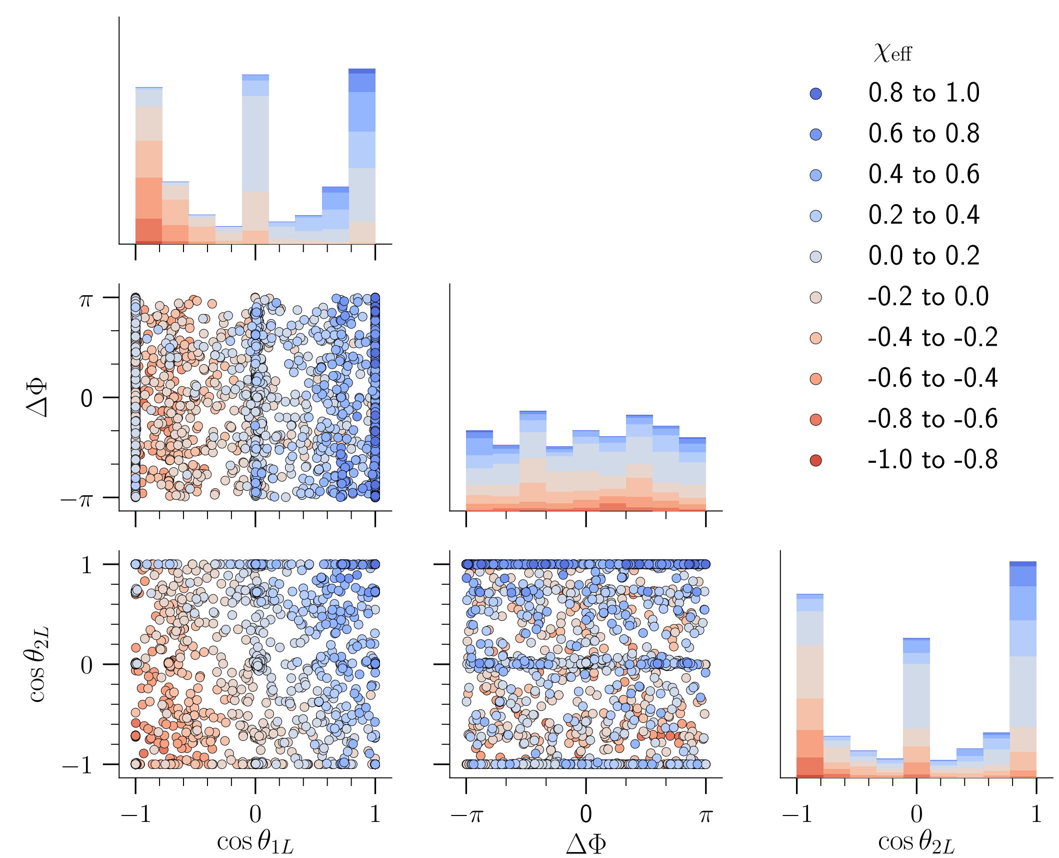
Numerical Relativity: Coverage of Parameter Space

- SXS Catalog, Boyle et al, CQG, 36, 195006, (2019): 2018 distinct configurations
- 1426 precessing configurations (covers mass ratios up to q ~ 4)
- Median length ~ 39 cycles

$$q = m_1/m_2 \qquad \chi_i = \frac{c}{G}\frac{S_i}{m_i^2}$$
$$\eta = (m_1 m_2)/M^2$$

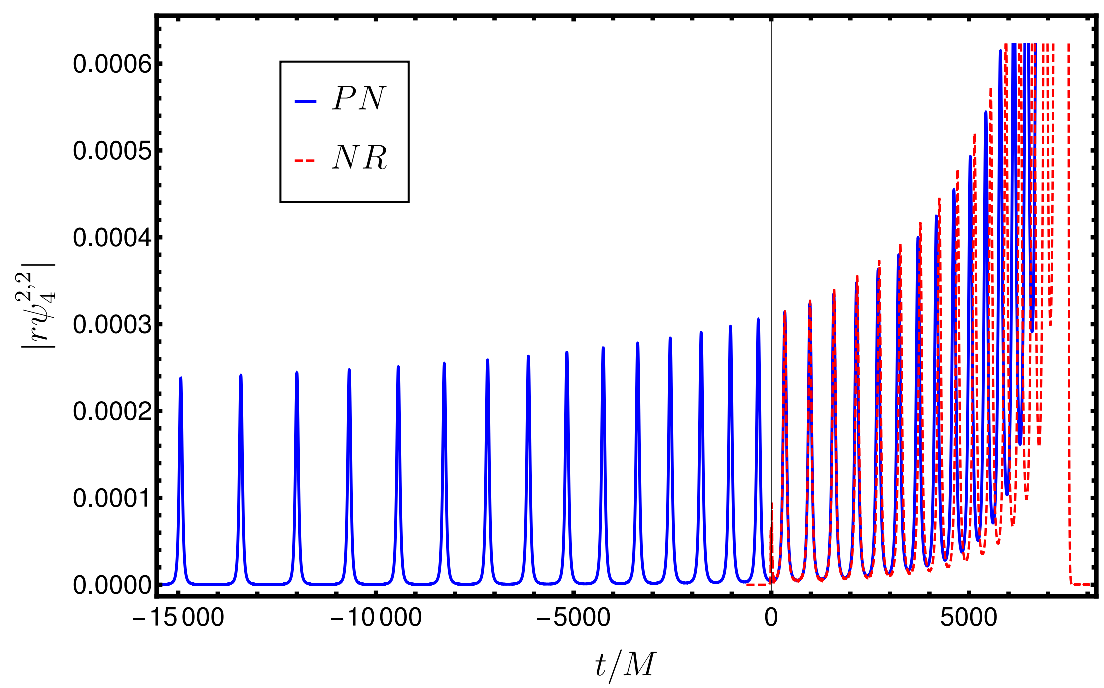


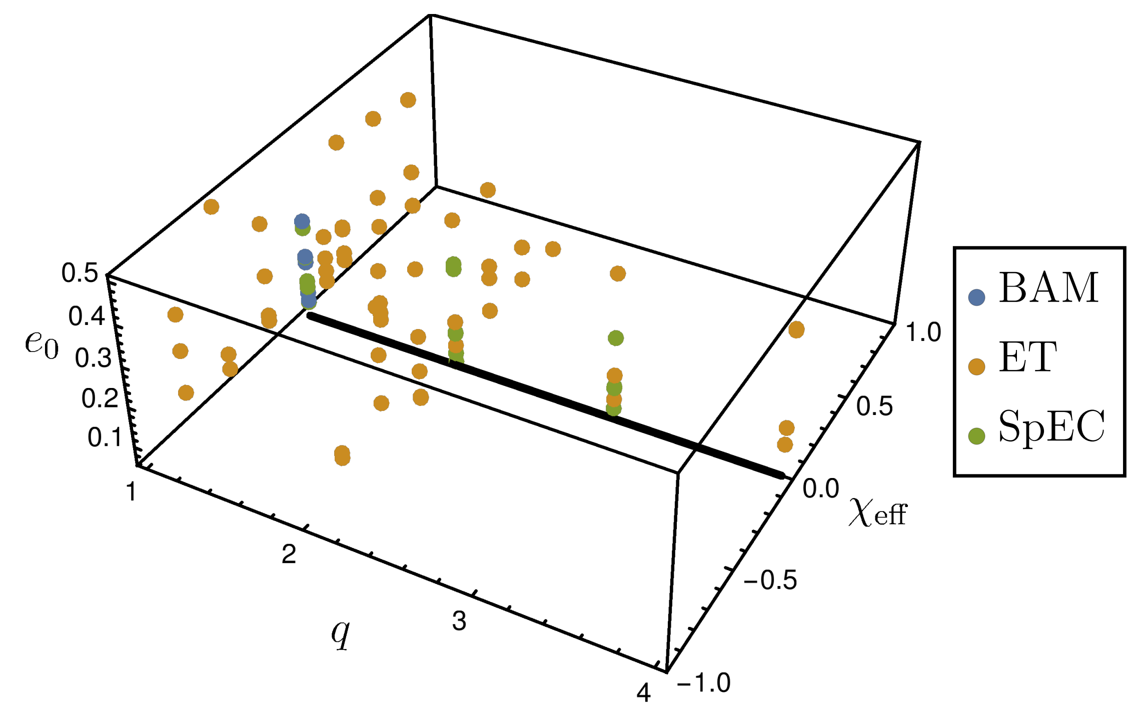


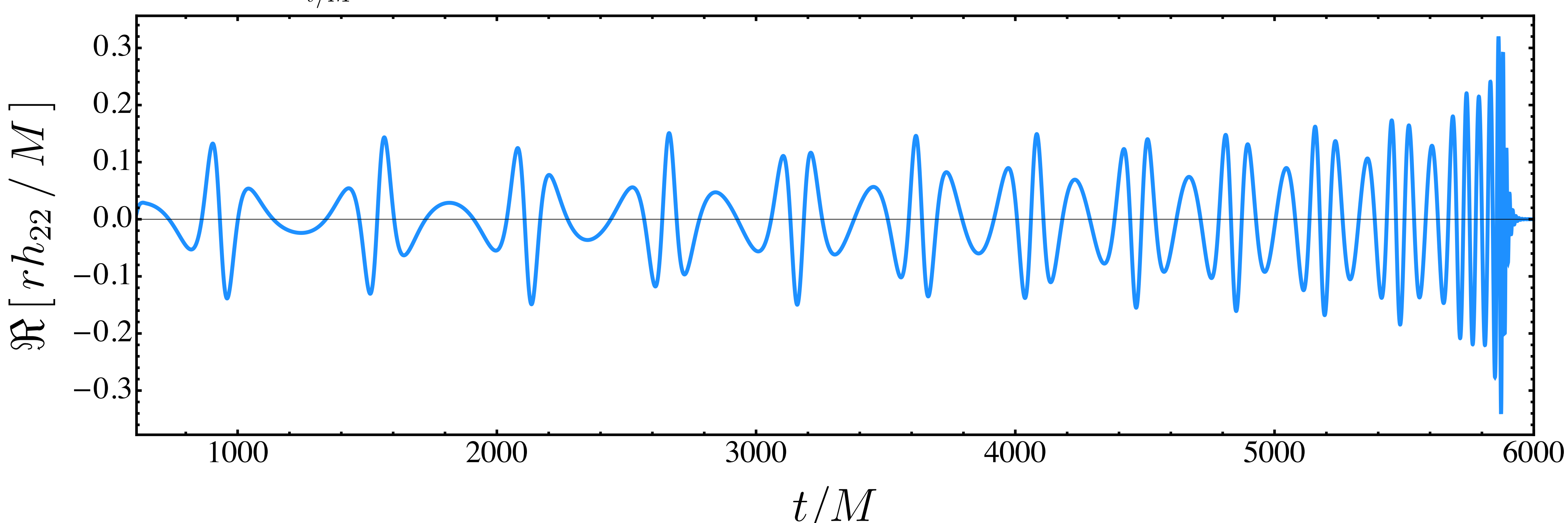


Numerical Relativity: Coverage of Parameter Space

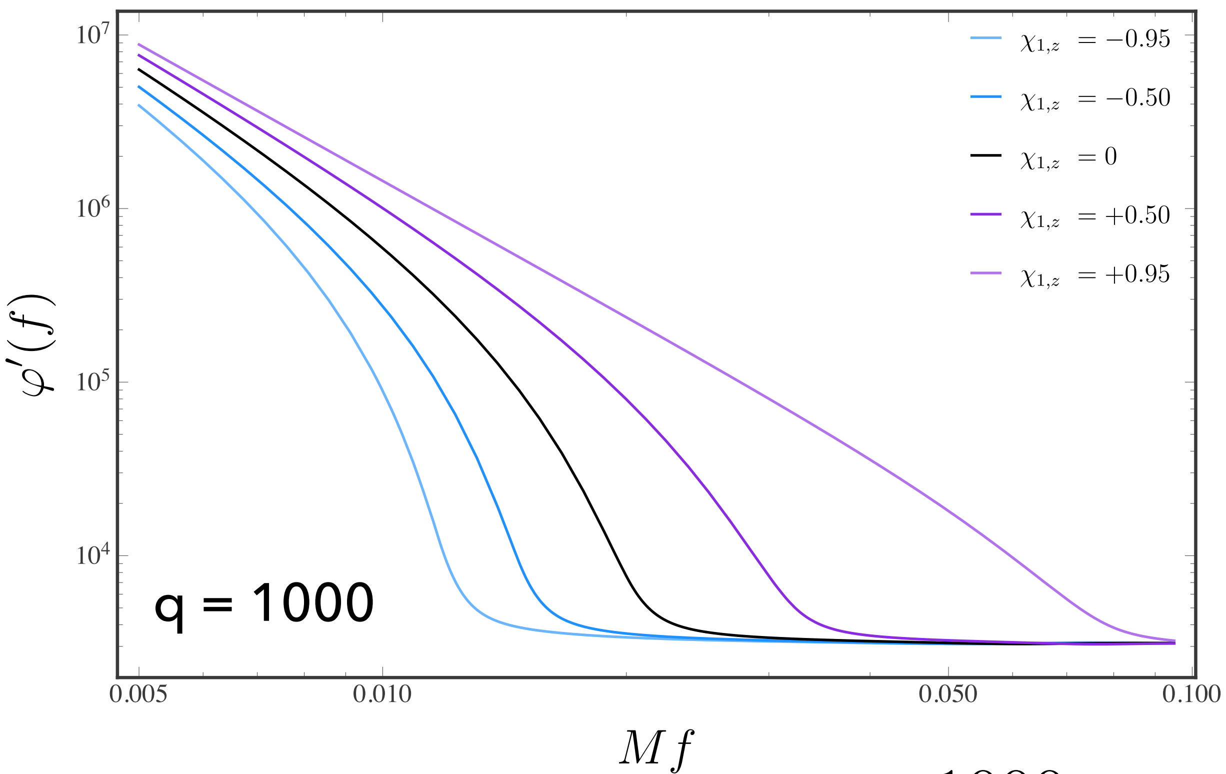
- Eccentric ET/BAM/SXS Catalog: A. Ramos-Buades et al + GP, arXiv:1909.11011, (2019)
- Explore parameter space systematically: non-spinning, aligned-spin, precessing, etc
- NR waveforms hybridised against eccentric PN, good for low-mass studies



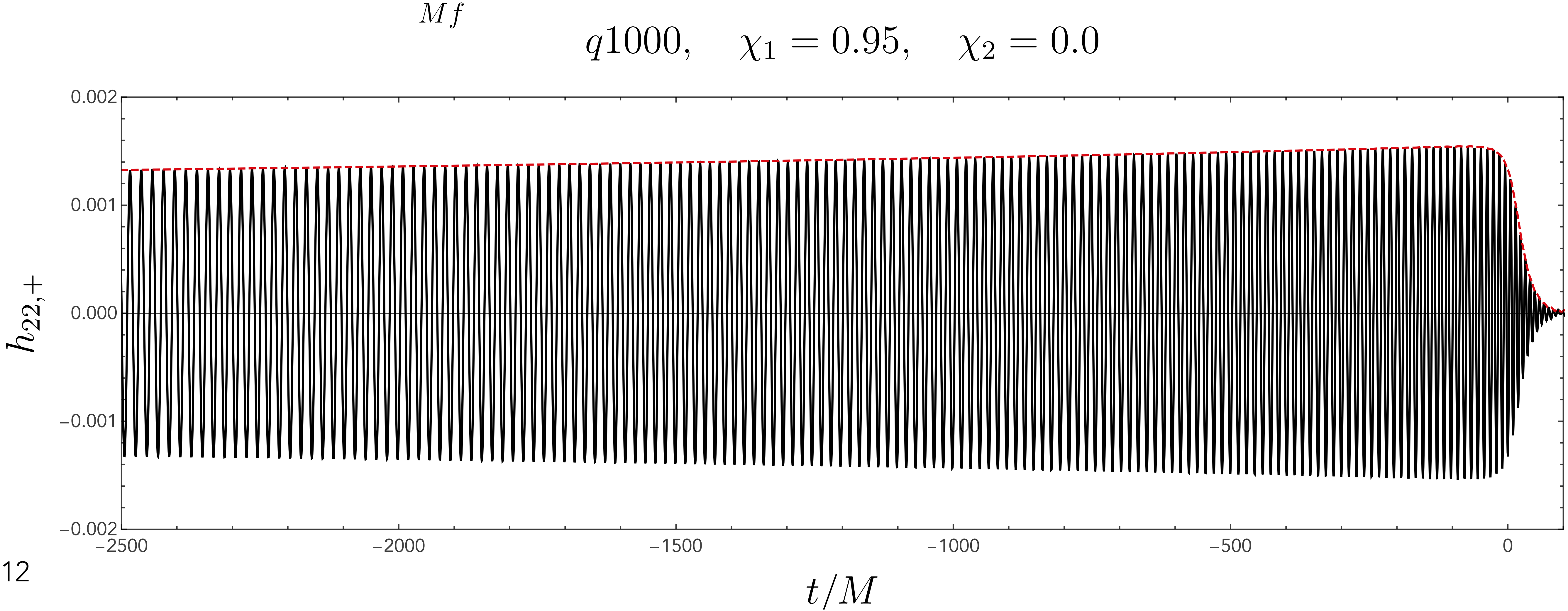




Numerical Relativity: Bridging the Test Particle Limit?



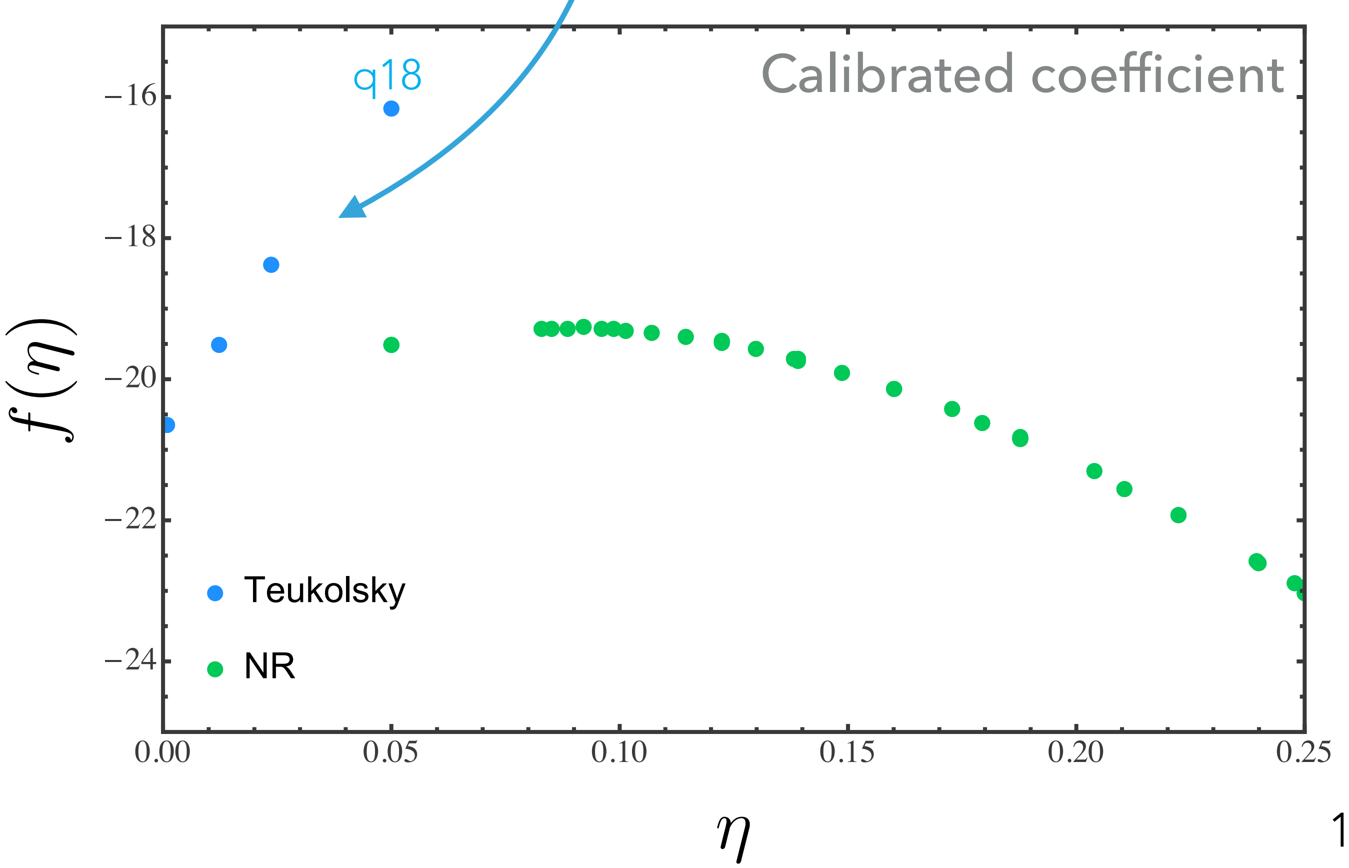
- NR simulations diverge in time as $\eta \to 0$
- Numerically solve black hole perturbation equations in extreme mass ratio limit
- Positive aligned spins ~ highly adiabatic ~ many quasi-circular orbits before merger
- Negative aligned spins ~ un-adiabatic ~ rapid plunge

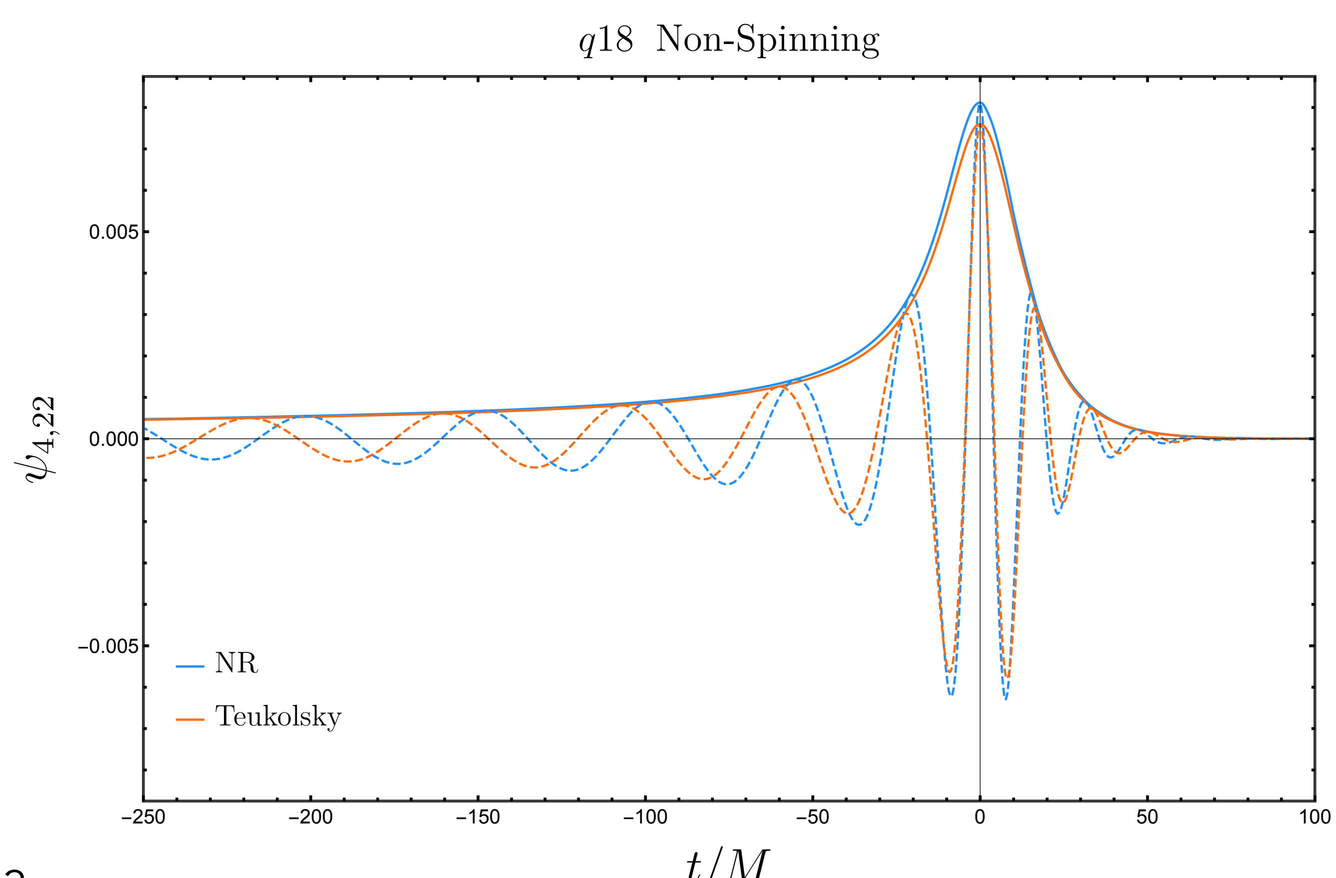


Numerical Relativity: Bridging the Test Particle Limit?

- Can we extrapolate these results to the comparable mass regime?
- Not yet! Work to do on both numerical and analytical side to bridge the gap!
- Comparisons between full NR and Teukolsky code show disagreements
- Tackle problem from both directions: push NR to higher mass ratios and invest in significantly improving analytical knowledge (e.g. second order self-force, see talk by Niels Warburton!)

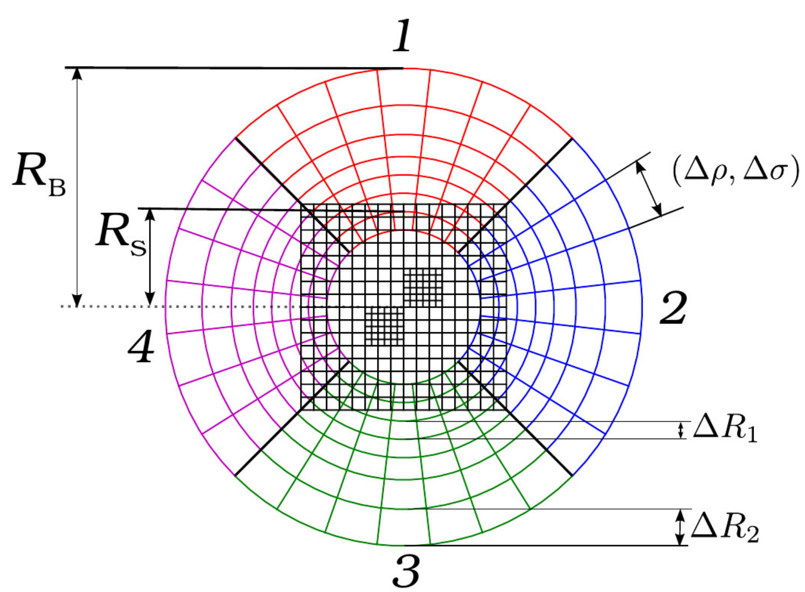
Test particle waveforms follow discrepant branches to NR when calibrating waveform models





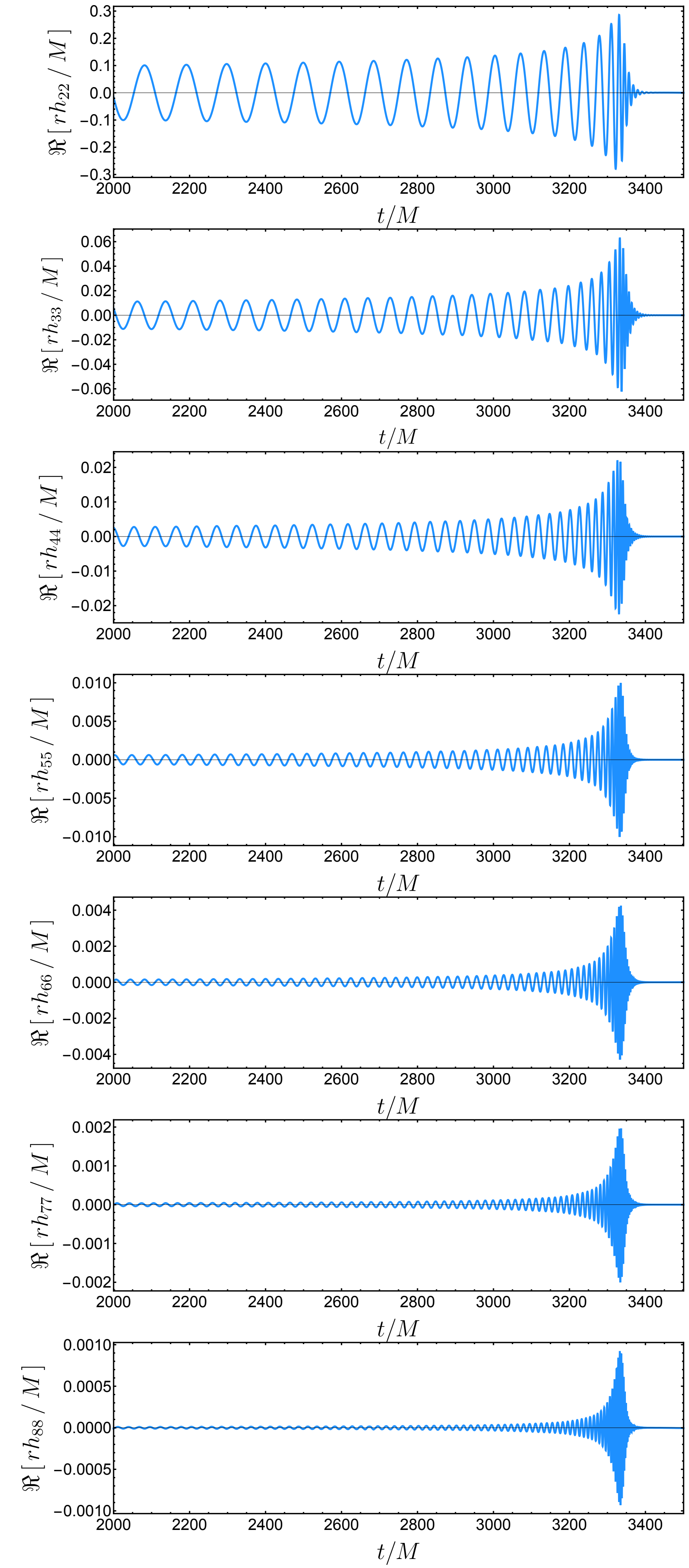
Numerical Relativity: Directions?

- Need improved understanding of systematics in NR codes
    - Comparable mass regime is well understood - need to explore more challenging regions of the parameter space
    - High mass ratios and high spins or particular importance
- Ever improving initial data vital for exploring extremal spin configurations - also needs more detailed understanding of gauge conditions!
- Cartesian AMR schemes hugely wasteful - use spherical topology of wave extraction zones (spectral methods, multi-patch grid)



Pollney et al, PRD, 83, 044045, (2011)

- Paradigm shift? Consider novel AMR schemes based on alternative methods? Wavelets? Much ongoing work, too early to say!



Phenomenological Waveform Models

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Phenomenological Waveform Models: The Basic Framework

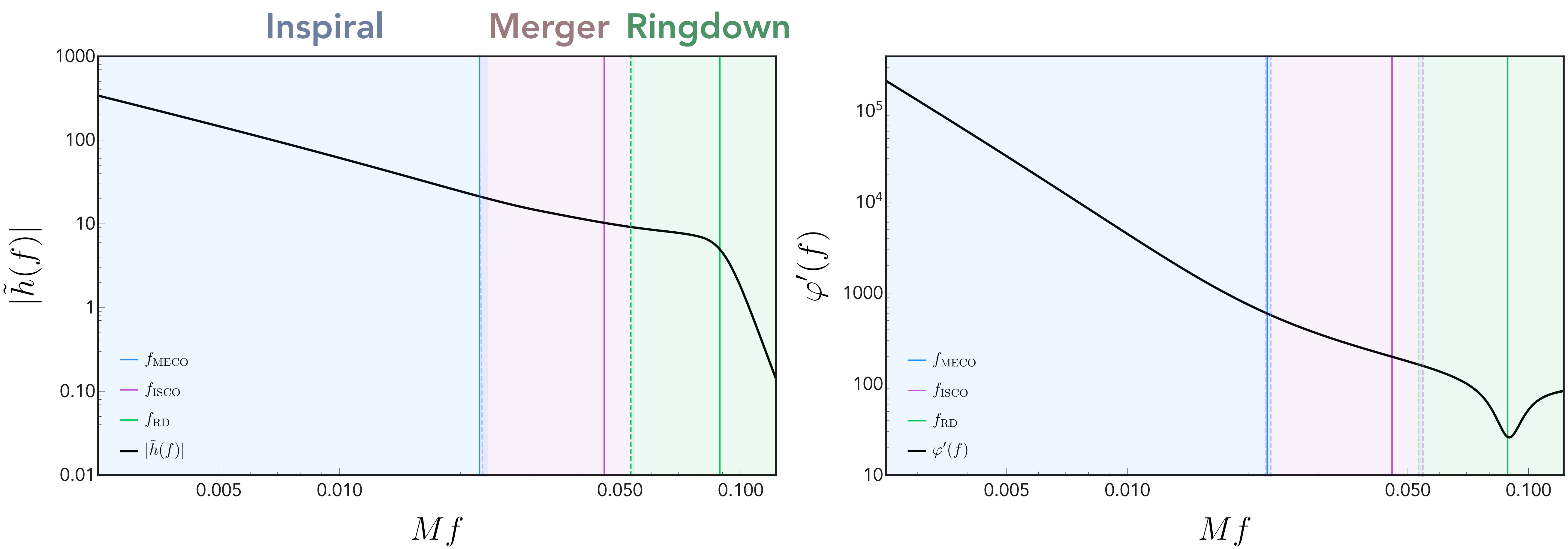
- Data-driven approach: want accurate but computationally efficient waveforms
    - Modern parameter estimation can require $\sim \mathcal{O}(10^6)$ or more waveform evaluations
- Basic ideology: model the amplitude and phase of the GW signal directly

$$h(f) = \sum_{\ell,m} {}_{-2}Y_{\ell m}\, h_{\ell m} \approx \underbrace{{}_{-2}Y_{22}\, h_{22}}_{\text{Initially focus on modelling the dominant harmonic}} = A(f) e^{i\varphi(f)}$$

- Initial focus on the dominant harmonic in non-precessing limit
    - Forms the basis for nearly all extensions in the Phenom framework: higher multipoles, precession, etc
    - Non-precessing limit is well understood: the perfect playground for understanding subdominant effects in a clean way
    - Will discuss the extensions in a few slides!

Phenomenological Waveform Models: The Basic Framework

- Modular approach that splits waveform into 3 key regions: inspiral, merger and ringdown
- Gauge ambiguities from freedom in time and phase shifts (these are not physical)
- Working with phase derivative helps with some of these ambiguities



Phenomenological Waveform Models: The Basic Framework

- Inspiral built from post-Newtonian theory + stationary phase approximation (TaylorF2)
  - Capture as-of-yet unknown PN terms via calibration to hybridised NR waveforms
  - Hybrid baseline is SEOBNRv4 (leads to better low-frequency behaviour)

$$\phi_{\rm Ins} = \phi_{\rm TF2}(Mf;\boldsymbol{\theta}) + \frac{1}{\eta}\left(\sigma_0 + \sigma_1 f + \frac{3}{4}\sigma_2 f^{4/3} + \frac{3}{5}\sigma_3 f^{3/5} + \frac{1}{2}\sigma_4 f^2\right)$$

- Intermediate region has no known analytical form
  - Use a simple polynomial ansatz to smoothly connect inspiral to rigndown

$$\eta\phi'_{\rm lnt} = c_{\rm lnt} + a_4 f^{-4} + a_2 f^{-2} + a_1 f^{-1} - \frac{4 a_0 a_\phi}{\left(f - f_{\rm RD}\right)^2 + \left(2 f_{\rm damp}\right)^2}$$

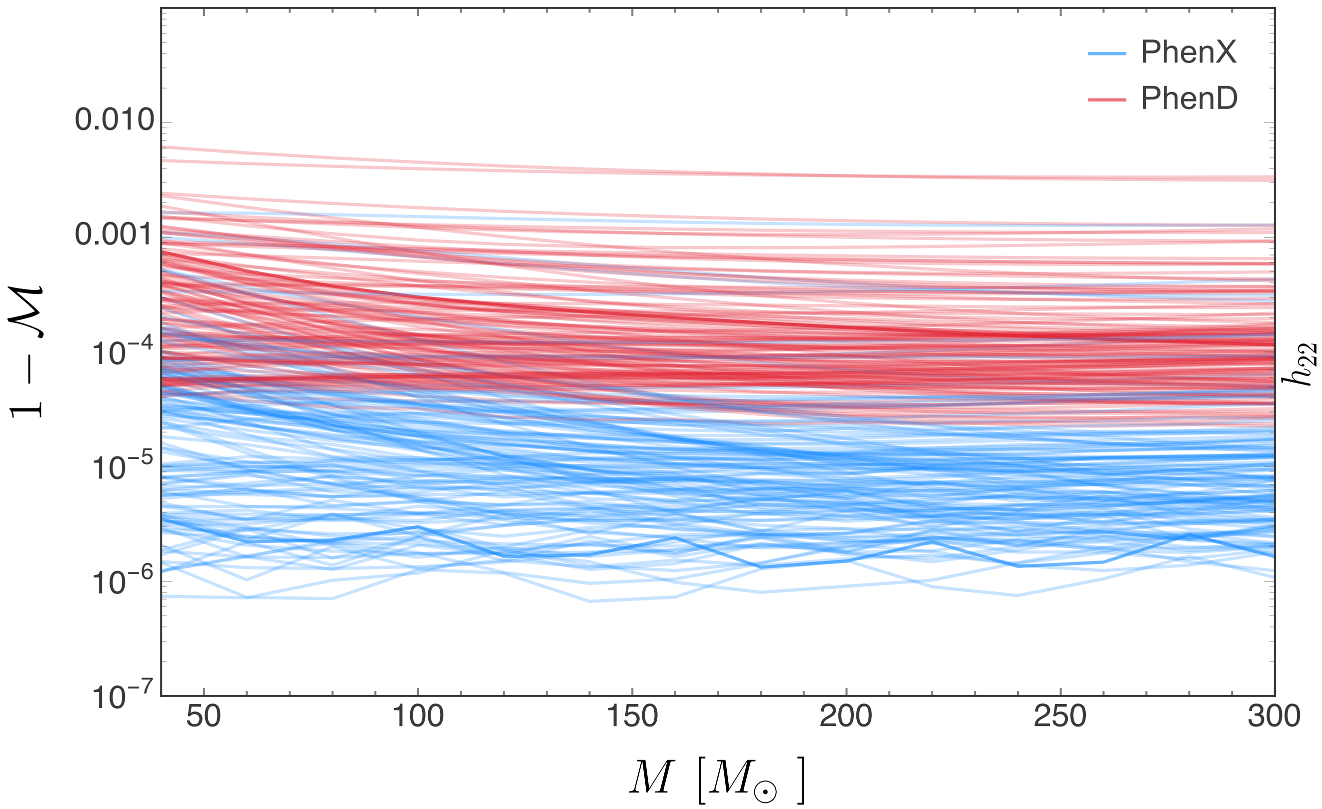
- Ringdown from black hole perturbation theory
  - Well captured by deformed Lorentzian in non-linear background from merger

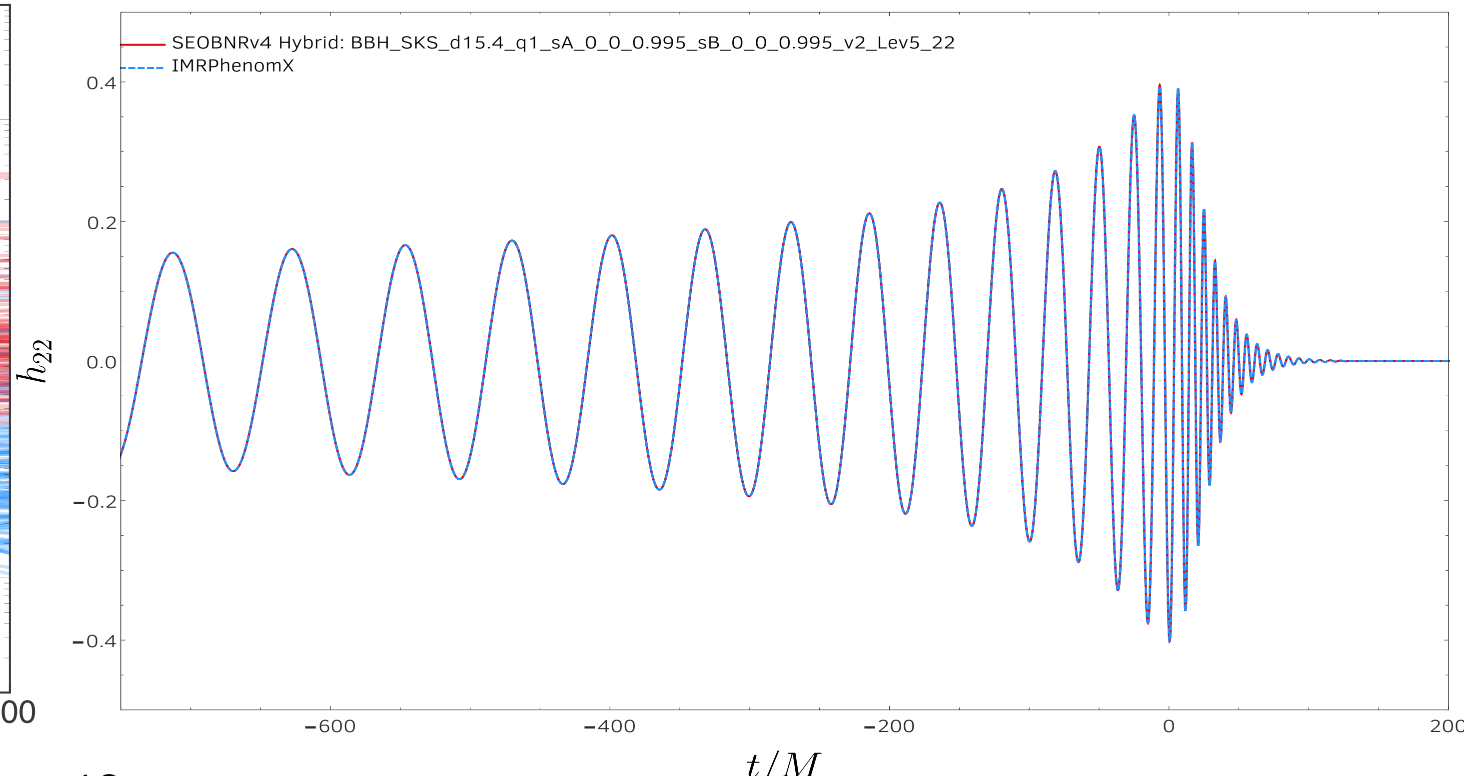
$$\eta\phi'_{\rm RD} = c_{\rm RD} + \sum_{i}^{n} c_i f^{-p_i} + \frac{c_0 a_\varphi}{f_{\rm damp}^2 + \left(f - f_{\rm RD}\right)^2}$$

Phenomenological Models: Next Generation of Models?

- State of the art Phenomenological waveform model IMRPhenomX [G. Pratten et al, in prep. (2019)] + extension to higher modes [C. García Quirós et al + *GP*, in prep., (2019)]
- Calibration against large suite of NR hybrids (~ 376 waveforms)
- Order(s) of magnitude improvement over previous generation of Phenom models

$$\langle h_1, h_2 \rangle = 4\,\mathrm{Re} \int_{f_{\min}}^{f_{\max}} \frac{\tilde{h}_1(f)\tilde{h}_2^*(f)}{S_n(f)} df$$





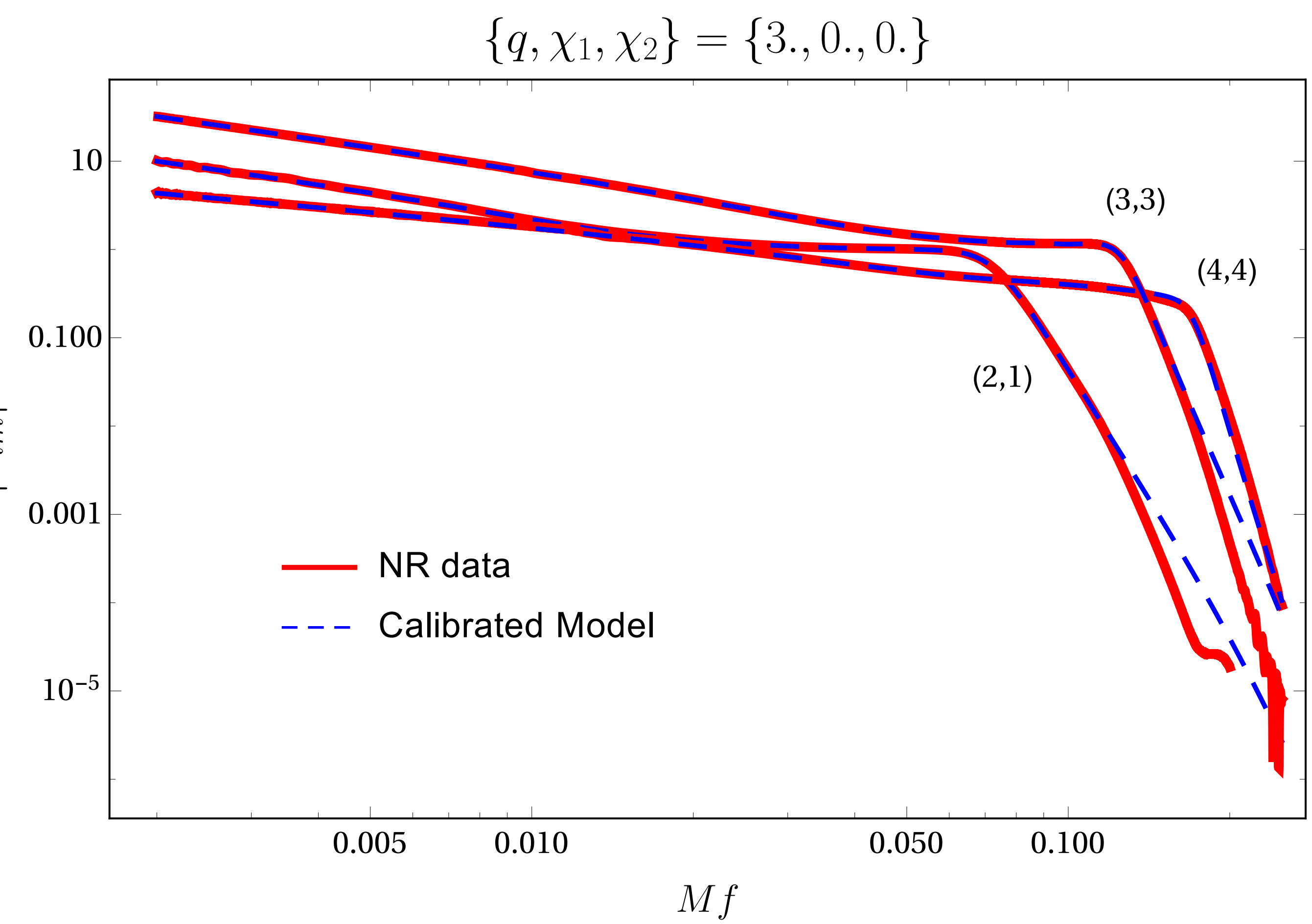
Phenomenological Models: Higher Multipoles

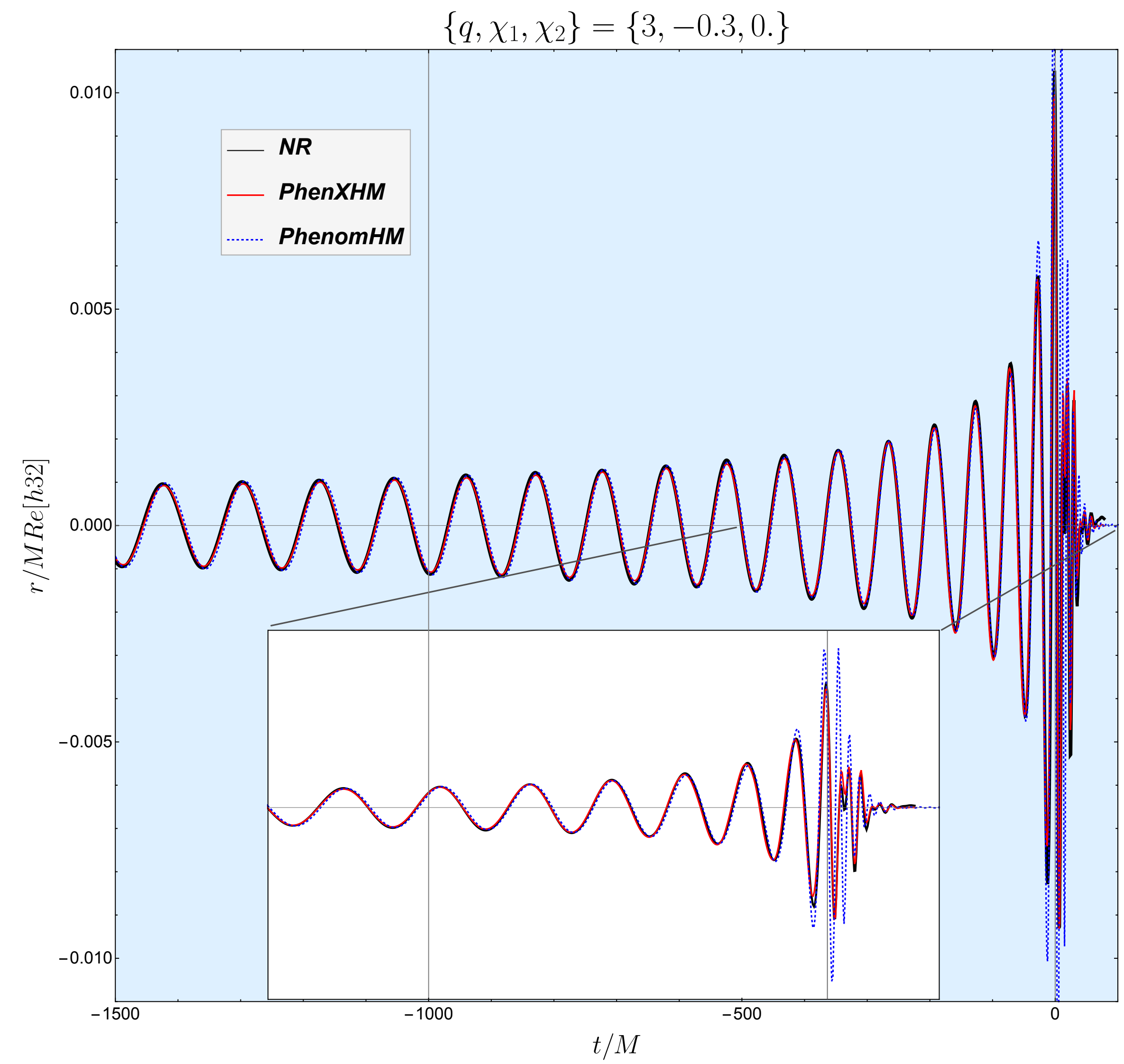
- Model subdominant spherical harmonics

$$h(t,\theta,\varphi) = \sum_{\ell=2}^{\ell_{\max}} \sum_{m=-\ell}^{+\ell} h_{\ell m}(t) \; {}_{-2}Y_{\ell m}(\theta,\varphi)$$

- Two key approaches: use simple PN-inspired rescaling (IMRPhenomHM) or calibrate the waveform model against NR (IMRPhenomXHM)

- Mode-mixing requires special treatment!





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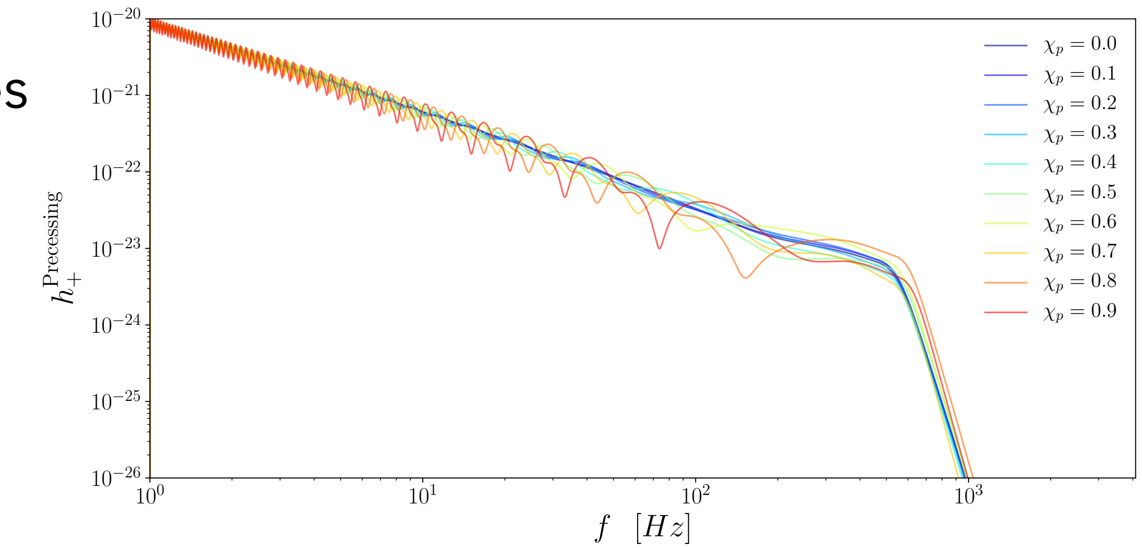
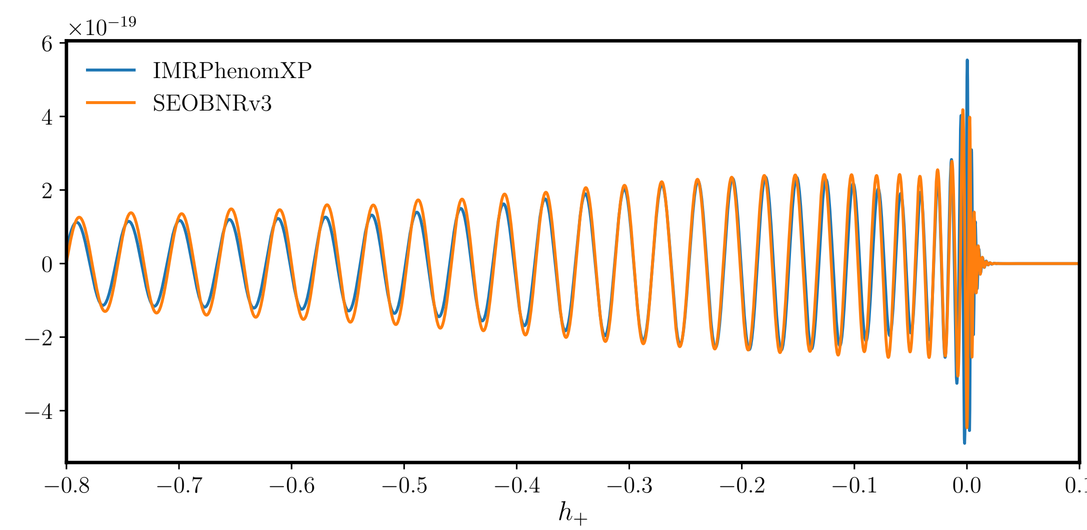
Phenomenological Models: Precession

- Spins mis-aligned with orbital angular momenta induces precession of orbital plane and spins = complex amplitude and phase modulations!
- **Key insight**: approximate decoupling between inspiral dynamics and precession dynamics (Schmidt+11)
- **Key insight**: waveforms in a preferred radiation frame approximately resemble aligned-spin waveforms (Schmidt+11, Schmidt+12)
- Use aligned-spin waveforms as basis for precessing waveform model





$$h^{\rm I}_{\ell m} = \sum_{m=-\ell}^{\ell} \mathcal{D}^{\ell *}_{m m'}(\alpha,\beta,\gamma) h^{\rm P}_{\ell m'}$$

Encodes precession of orbital plane: [P. Schmidt et al, PRD, 84, 024046, (2011); P. Schmidt et al, PRD, 86, 104063, (2012); M. Hannam & P. Schmidt et al + *GP*, PRL, 113, 151101, (2014)]

- Effective spin parameterisation ~ dominant effect

$$\chi_p = \frac{S_p}{A_2 m_2^2} \quad S_p = \mathrm{Max}\left(A_1 S_{1\perp}, A_2 S_{2\perp}\right) \quad A_1 = 2 + \frac{3q}{2}; A_2 = 2 + \frac{3}{2q}$$

Phenomenological Models: Waveform Generation Cost?

- Phenom models defined in frequency domain - closed form expressions make these exceptionally quick to evaluate!

- Two main paths to improving performance: multibanding and reduced order quadrature

- Multibanding ~ sampling across non-uniform grid to adaptively resolve features in waveform [S. Vinciguerra et al, CQG, 34, 11, (2017); C. Garciá-Quiros et al, in prep, (2019]

$$\hat{A}_k = A_j + \frac{\hat{f}_k - f_j}{f_{j+1} - f_j} \left(A_{j+1} - A_j \right)$$

$$\hat{\phi}_k = \phi_j + \frac{\hat{f}_k - f_j}{f_{j+1} - f_j} \left(\phi_{j+1} - \phi_j \right)$$

Accelerate waveform generation cost!

- Reduced order quadrature (ROQ) ~ approximation to overlap integral using empirical interpolant [P. Canizares et al, PRL, 114, 071104, (2015)]

$$\left(h_{\rm A}(\vec{\lambda}), h_{\rm B}(\vec{\lambda}) \right)_{\rm ROQ} \approx \sum_{k=1}^{N_{\rm O}} \psi_k \tilde{h}_{\rm A} \left(\mathcal{F}_k; \vec{\lambda} \right) \tilde{h}^*_{\rm B} \left(\mathcal{F}_k; \vec{\lambda} \right)$$

- Enables rapid approximation to likelihood significantly improving parameter estimation costs

Phenomenological Models: Key Models on the Market

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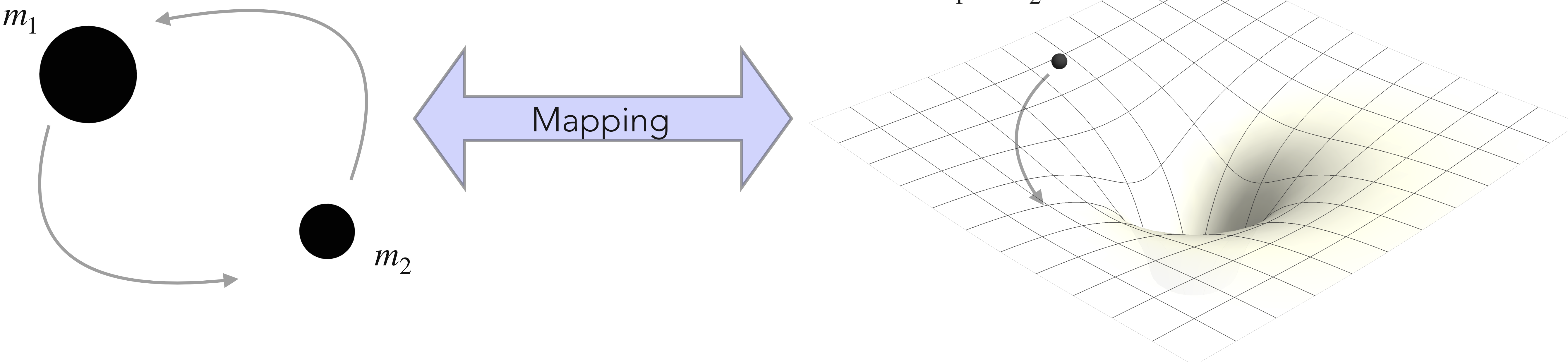
- **IMRPhenomD** [Husa et al, PRD, 93, 044006, (2016); Khan et al, PRD, 93, 044007, (2016)]:
  - Aligned-spins, calibration up to q ~ 18, spins ~ 0.85
- **IMRPhenomPv1/v2** [M. Hannam & P. Schmidt et al + *GP*, PRL, 113, 151101, (2014)]:
  - First precessing IMR waveform model and currently used in LVC analysis of GW data
- **IMRPhenomHM** [London et al, PRL, 120, 161102, (2018)]:
  - Calibrated 22-only higher-multipole model
- **IMRPhenomPv3** [Khan et al, PRD, 100, 024059, (2019)]:
  - Improved treatment of PN inspiral to help capture double spin effects
- **IMRPhenomX** [G. Pratten et al, In. Prep., (2019); C. García Quirós et al + *GP*, In. Prep., (2019]
  - New aligned spin model, calibrated up to q ~ 1000, calibrated higher modes
  - Extensions to precession (IMRPhenomXP), higher modes (IMRPhenomXHM) and precessing higher modes (IMRPhenomXPHM)
  - Improvements (both significant and incremental) in all aspects of model construction

Effective One Body Formalism

Effective One Body Formalism: Introduction

- Replace two-body dynamics (m_1, m_2) with dynamics of particle of mass μ in an effective metric $g^{\text{eff}}_{\mu\nu}(u)$

$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$



- Effective mapping first given by Buonanno & Damour [PRD, 59, 084006, (1999)]

- Many significant breakthroughs in EOB modelling since

Effective One Body Formalism: The General Framework

PN Dynamics → Resummed → EOB Hamiltonian

PN Rad. Reac. → Resummed → EOB Rad. Reac. Force

PN Waveform → Resummed → Factorized Waveform

Black Hole Perturbation Theory → QNM Spectra → Matching at merger time

$$\frac{dr}{dt} = \left(\frac{A}{B}\right)^{1/2} \frac{\partial \hat{H}_{\mathrm{EOB}}}{\partial p_{r_*}}$$

$$\frac{dp_{r_*}}{dt} = -\left(\frac{A}{B}\right)^{1/2} \frac{\partial \hat{H}_{\mathrm{EOB}}}{\partial r}$$

$$\Omega \equiv \frac{d\varphi}{dt} = \frac{\partial \hat{H}_{\mathrm{EOB}}}{\partial p_{\varphi}}$$

$$\frac{dp_{\varphi}}{dt} = \hat{\mathcal{F}}_{\varphi}$$

$$h_{\ell m} = h_{\ell m}^{(N,\epsilon)} \hat{h}_{\ell m}^{(\epsilon)}$$
$$\hat{h}_{\ell m}^{(\epsilon)} = \hat{S}_{eff}^{(\epsilon)} T_{\ell m} e^{i\delta_{\ell m}} \rho_{\ell m}^{\ell}$$

$$h_{\ell m}^{\mathrm{ringdown}}(t) = \sum_N C_N^{+} e^{-\sigma_N^{+}(t-t_m)}$$

EOB Waveform

$$h_{\ell m}^{\mathrm{EOB}} = \theta(t_m - t)\, h_{\ell m}^{\mathrm{insplunge}}(t) + \theta(t - t_m)\, h_{\ell m}^{\mathrm{ringdown}}(t)$$

Effective One Body Formalism: The Hamiltonian

- EOB Hamiltonian

$$H = M\sqrt{1 + 2\nu\left(\frac{H_{\text{eff}}}{\mu} - 1\right)}$$

- Decompose the effective Hamiltonian into orbital (includes all terms even in the spins) and two spin-orbit pieces

$$H^{\text{eff}}_{\text{orb}} = \sqrt{p^2_{r_*} + A\left(r, \nu, S_1, S_2\right)\left(\mu^2 + \frac{L^2}{r_c^2} + Q_4\right)}$$

Centrifugal radius with NLO spin-spin terms:

$$r_c^2 = r^2 + \hat{a}_0^2\left(1 + \frac{2}{r}\right) + \delta\hat{a}^2$$

~ quadrupole deformation of each BH induced by rotation

- PN expansion of EOB radial potential where $u = 1/r$

$$A^{\text{PN}}_{\text{orb}}\left(u_c\right) = 1 - 2u_c + 2\nu u_c^3 + \nu a_4 u_c^4 + \nu\left(a_5^c + a_5^{\log}\ln u_c\right)u_c^5 + \nu\left(a_6^c + a_6^{\log}\ln u_c\right)u_c^6$$

Resummation!

$$\longrightarrow \quad A_{\text{orb}}\left(u_c, \nu\right) = P_5^1\left[A^{\text{PN}}_{\text{orb}}\right]\left(u_c\right)$$

- Spin-orbit interactions enter the Hamiltonian as

$$\hat{H}^{S,S^*}_{\text{eff}} = \frac{g_S^{\text{eff}}}{r^3}\mathbf{L}\cdot\mathbf{S} + \frac{g_{S^*}^{\text{eff}}}{r^3}\mathbf{L}\cdot\mathbf{S}^*$$

Hamiltonian and constituent components is non-unique! Many different formulations possible!

Effective One Body Formalism: Post-Merger and Ringdown?

- New paradigm: multiplicative decomposition, factorize fundamental QNM [Damour & Nagar, PRD, 90, 024054, (2014)]

$$h(\tau) = e^{\sigma_1 \tau - \mathrm{i}\phi_0} \bar{h}(\tau)$$

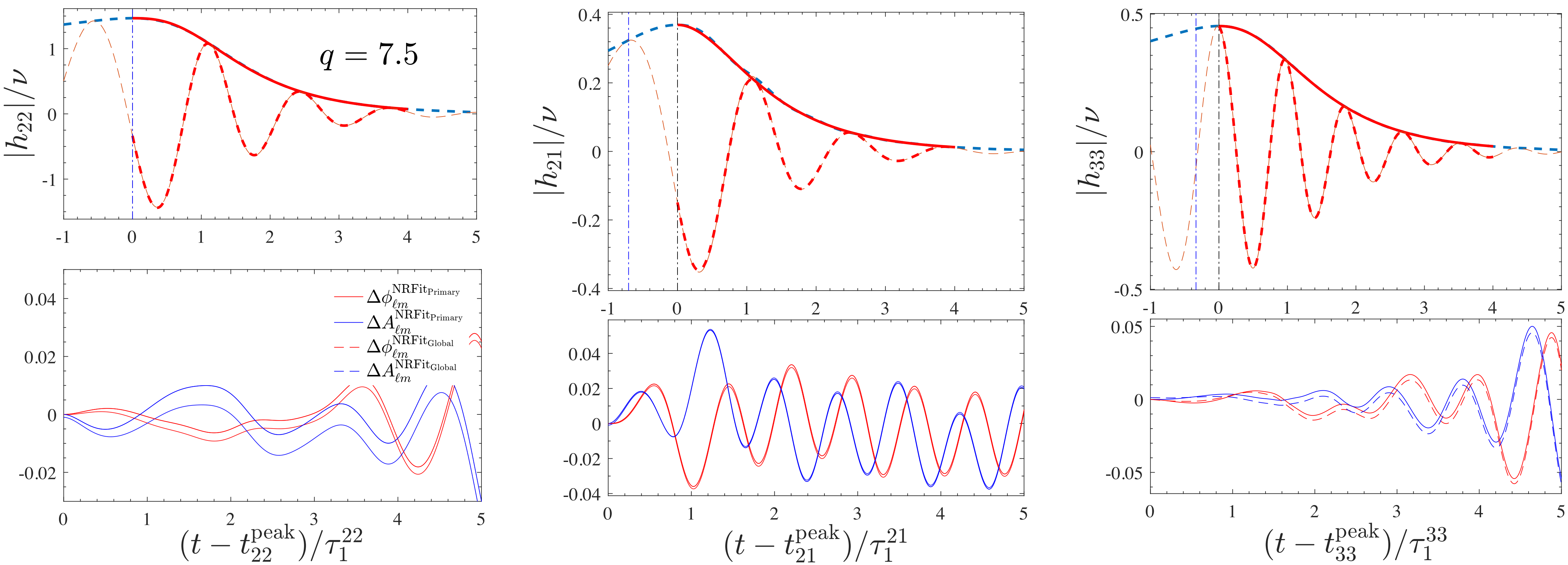
$$\bar{h}(\tau) \equiv A_{\bar{h}} e^{\mathrm{i}\phi_{\bar{h}}(\tau)}$$

- Phenomenological ansatz for remainder:

$$A_{\bar{h}}(\tau) = c_1^A \tanh\left(c_2^A \tau + c_3^A\right) + c_4^A$$

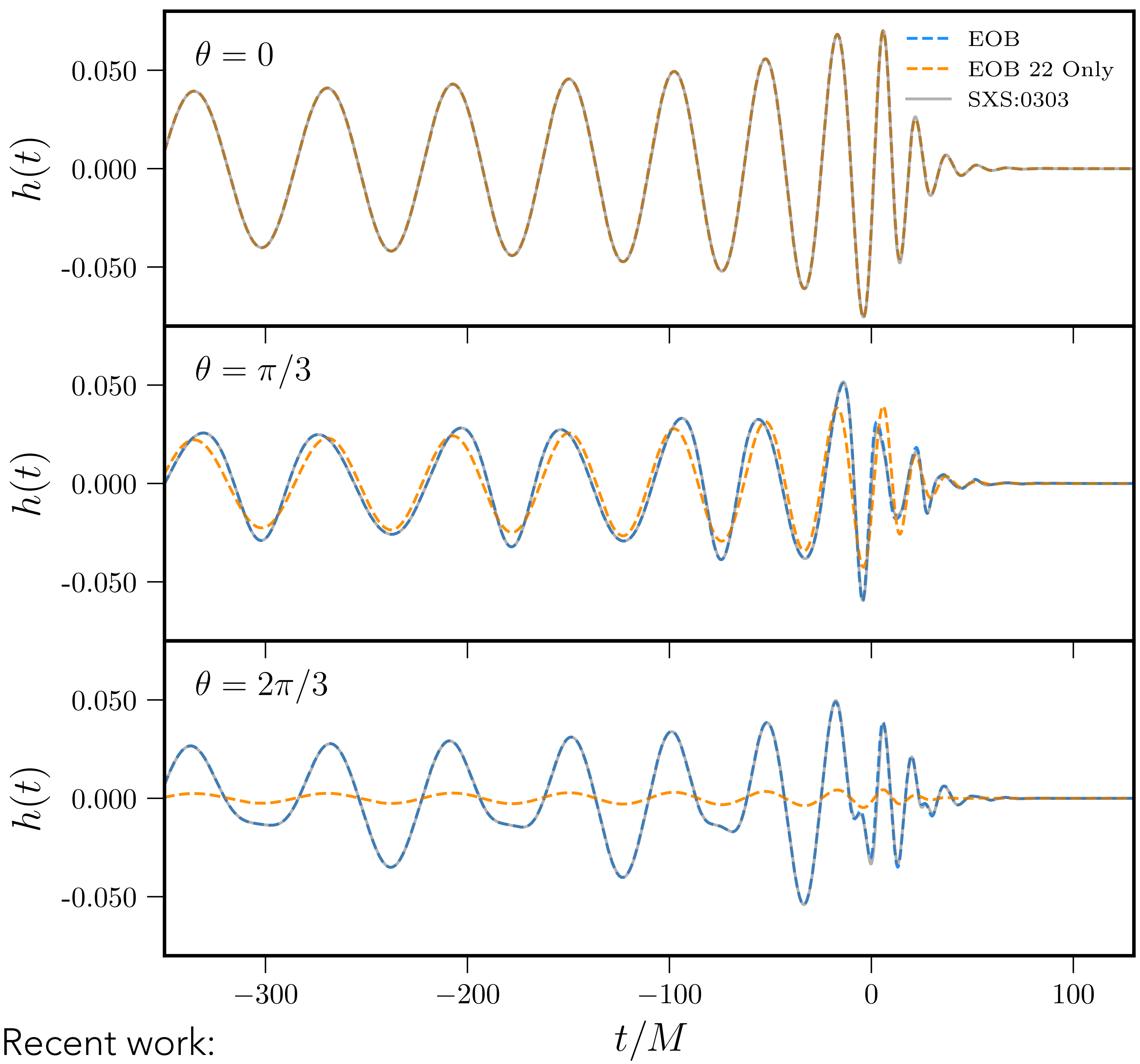
$$\phi_{\bar{h}}(\tau) = -c_1^{\phi} \ln\left(\frac{1 + c_3^{\phi} e^{-c_2^{\phi}\tau} + c_4^{\phi} e^{-2c_2^{\phi}\tau}}{1 + c_3^{\phi} + c_4^{\phi}}\right)$$

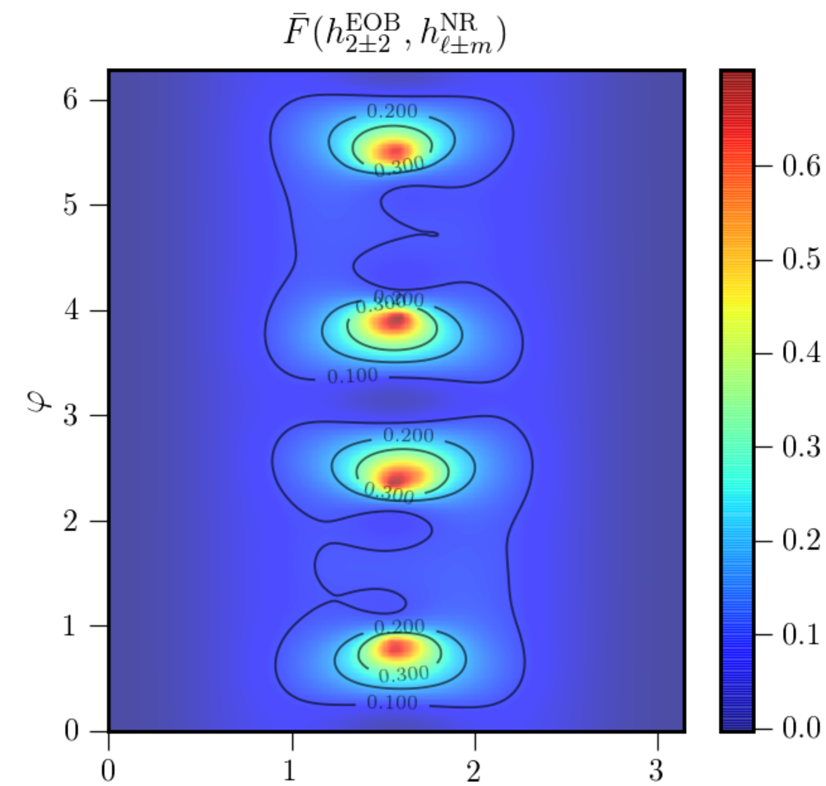
E.g Nagar et al + GP, arXiv:1904.09550, (2019)

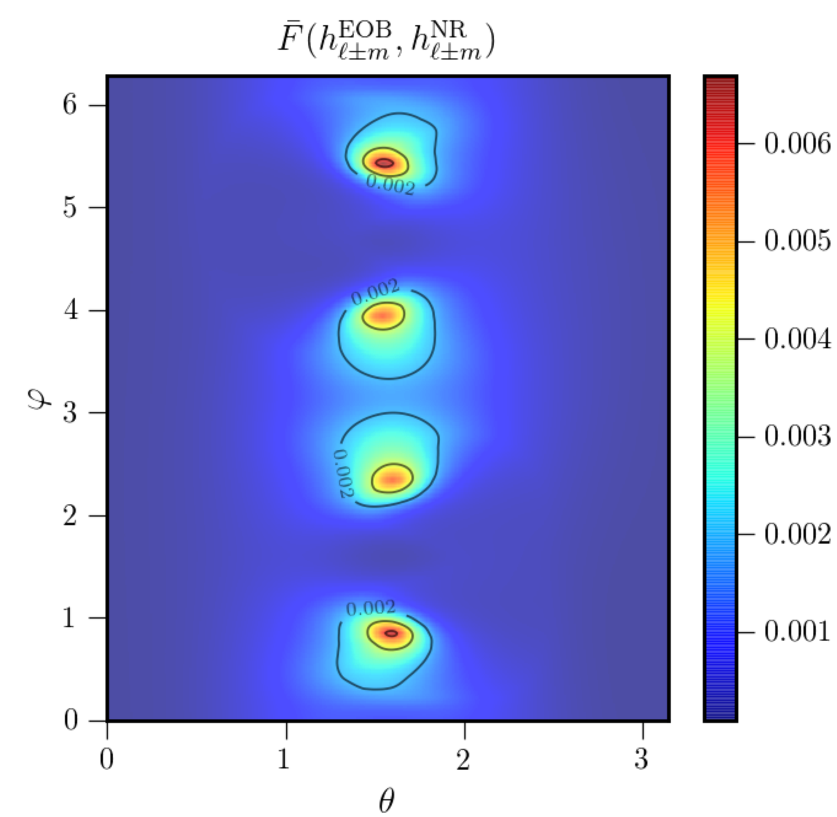


Effective One Body Formalism: Higher Multipoles?

- Framework can be extended to modelling higher multipoles in analogous way to 22 mode







Recent work:

- Cotesta et al, PRD, 98, 084028, (2018)
- Nagar et al + *GP*, arXiv:1904.09550 (2019)
- Riemenschneider et al + *GP*, In. Prep. (2019)

Effective One Body Formalism: Waveform Generation Costs?

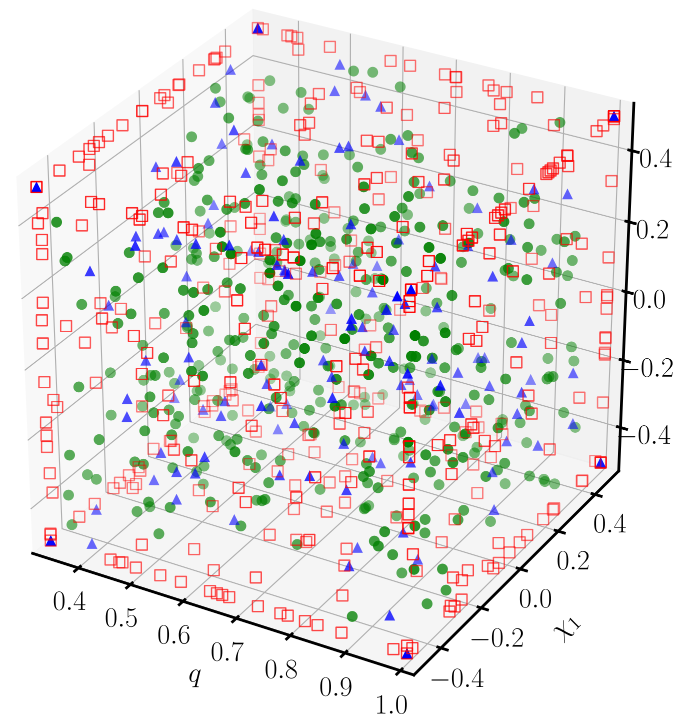
- EOB models intrinsically defined in time domain - computationally expensive to solve system of ODEs and perform FFTs

- Significant effort developing novel tools to improve waveform generation costs

- Reduced order models (ROM):

  - Sample waveform over parameter space & construct orthonormal bases (SVD, greedy, etc)

$$h(f;\vec{\lambda}) \approx \sum_{i=1}^{m} c_i(\vec{\lambda}) e_i(f) \quad c_i(\vec{\lambda}) = \left\langle h(\cdot;\vec{\lambda}), e_i(\cdot) \right\rangle$$



  - Interpolation of coefficients (e.g. tensor-product interpolation)

$$\mathcal{I}_{\otimes}[g](\vec{\lambda}) = \sum_{i=1}^{n_\eta} \sum_{j=1}^{n_{\chi_1}} \sum_{k=1}^{n_{\chi_2}} g_{ijk} \left(\Psi_\eta \otimes \Psi_{\chi_1} \otimes \Psi_{\chi_2} \right)_{ijk} (\vec{\lambda})$$

  - ROM for amplitude and phase separately

$$\tilde{h}_{\mathrm{m}}(\vec{\lambda}; M; f) := A_0(\vec{\lambda}, M) \mathcal{I}_f \left[\mathcal{B}_{\mathcal{A}} \cdot \mathcal{I}_{\otimes} \left[\mathcal{M}_{\mathcal{A}} \right] (\vec{\lambda}) \right] \exp \left\{ i \mathcal{I}_f \left[\mathcal{B}_{\Phi} \cdot \mathcal{I}_{\otimes} \left[\mathcal{M}_{\Phi} \right] (\vec{\lambda}) \right] \right\}$$

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Effective One Body Formalism: Waveform Generation Costs?

- Can we use adiabatic nature of waveform directly? Yes! [Nagar & Rettegno, PRD, 99, 021501, (2019)]

- Adiabatic limit ~ no radiation reaction ~ circular orbits

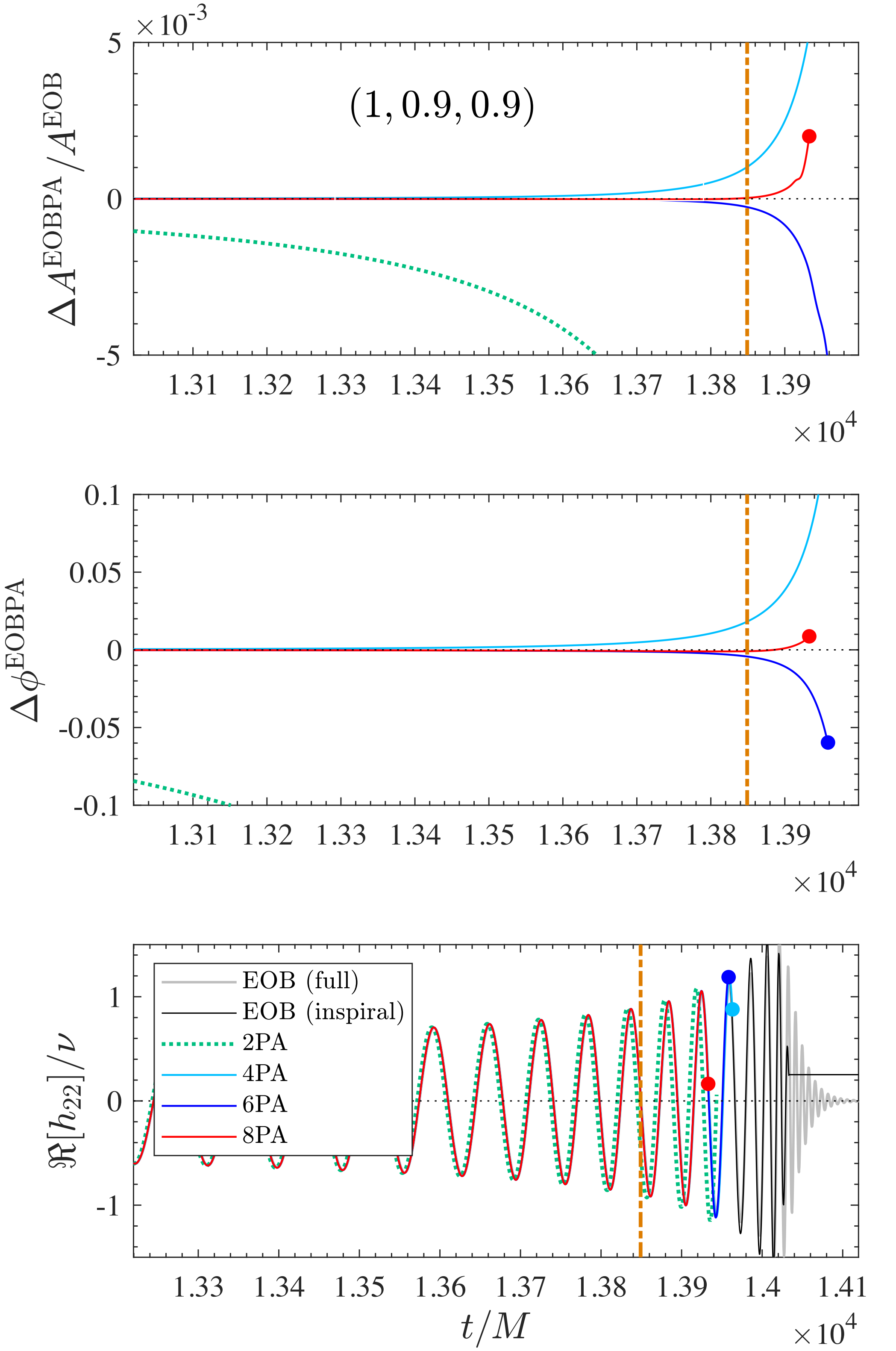
- Perturbatively solve about circular orbit (post-adiabatic correction)

$$\frac{dp_\varphi}{dr} = \hat{\mathcal{F}}_\varphi \left(\frac{dr}{dt} \right)^{-1}$$

- Write solution of EOB equations as formal expansion and iteratively solve = semi-analytic expressions! Time and phase obtained via quadratures

$$p_\varphi^2(r) = j_0^2(r) \left(1 + \sum_{n=1}^{\infty} k_{2n}(r) \varepsilon^{2n} \right)$$

$$p_{r_*}(r) = \sum_{n=0}^{\infty} \pi_{2n+1}(r) \varepsilon^{2n+1}$$



Effective One Body Formalism: Key Waveform Models on the Market

- Two major families of EOB: SEOBNR and TEOBResumS

- SEOBNRv*:
  - Main development by AEI / Maryland

- **SEOBNRv3** [Pan et al, PRD, 89, 084006, (2014)] :
  - Precessing-spins, based on (older) SEOBNRv2 Hamiltonian [Taracchini et al, PRD, 86, 024011, (2012)]
  - Also see [Buonanno et al, PRD, 74, 104005, (2006)

- **SEOBNRv4** [Bohé et al, PRD, 95, 044028, (2017)] :
  - Aligned-spins

- **SEOBNRv4HM** [Cotesta et al, PRD, 98, 084028, (2018)]:
  - Higher multipole extension of SEOBNRv4

- **SEOBNRv4P/v4PHM** [Marsat et al, In. Prep. (2019)]:
  - Precessing variant of SEOBNRv4, extension to precession + higher multipoles

- **TEOBResumS** [Nagar et al + *GP* + *P. Schmidt*, PRD, 98, 104052, (2018]
  - Main development by IHES, Turin, Jena, Birmingham
  - Aligned-spins, higher multipoles [Nagar et al + *GP*, arXiv:1904.09550 (2019): Riemenschneider et al + *GP*, In. Prep. (2019)]
  - Precessing development ongoing