Time-Delay Interferometry

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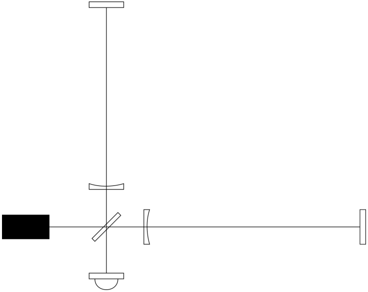
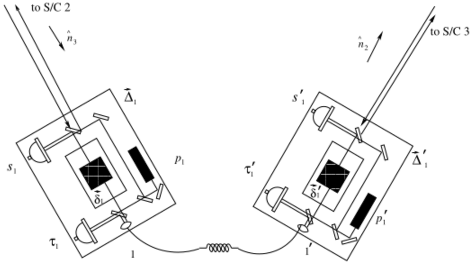
Ground-Based Gravitational-Wave Detectors



- Fixed armlengths
- Fixed geometry
- Single laser
- Static detector

Space-Based Gravitational-Wave Detectors



[M. Tinto, S. V. Dhurandhar, "Time-Delay Interferometry"]

- Variable armlengths
- Variable geometry
- Multiple lasers
- Moving detector

Time-Delay Interferometry

The complicated setup requires the development of a new technique to cancel out the optical noise in the data streams: time-delay interferometry.

By combining laser phase measurements from different spacecrafts at different times, we can construct combinations of those which cancel out the optical noise.

These combinations will respond to a passing gravitational wave in a specific frequency-dependent manner.

Single Arm Response

We can write the instrument response to a passing gravitational wave as the acquired phase during a photon's travel from one spacecraft to the next.

$$\Delta\phi_{ij} = \int_{\text{path}} \omega_\ell dt, \tag{1}$$

or, equivalently, the optical path length is

$$L_{ij} = \frac{\Delta\phi_{ij}}{\omega_\ell} = \int_{\text{path}} dt. \tag{2}$$

Single Arm Response

Assuming a gravitational wave travelling in the z direction, the spacetime through which the photon travels can be described by the metric

$$\begin{aligned} ds^2 &= -dt^2 + dx^2 + dy^2 + dz^2 + h_{ab}(t,z)dx^a dx^b, && (3) \\ &= -dudv + dx^2 + dy^2 + h_{ab}(u)dx^a dx^b, && (4) \end{aligned}$$

with retarded time $u = t - z$, and advanced time $v = t + z$.
The photon travelling along a null geodesic, we have

$$dt^2 = dx^2 + dy^2 + dz^2 + h_{ab}dx^a dx^b. \quad (5)$$

Single Arm Response

Assuming that the photon is emitted from one spacecraft with coordinates $x^\alpha(0)$ and received at another with coordinates $x^\alpha(1)$, we can parametrize the photon path with affine parameter λ as

$$x^\alpha(\lambda) = x^\alpha(0) + \lambda r^\alpha, \tag{6}$$

$$r^\alpha = x^\alpha(1) - x^\alpha(0). \tag{7}$$

We then get along the photon path

$$dt = d\lambda\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2 + h_{ab}(\lambda)\dot{x}^a\dot{x}^b} \tag{8}$$

$$= d\lambda L_0\sqrt{1 + h_{ab}(\lambda)\hat{r}^a\hat{r}^b}. \tag{9}$$

Single Arm Response

The optical path length is

$$L_{ij} = \int_{\text{path}} dt \tag{10}$$

$$= L_0 \int_0^1 d\lambda \left[1 + \frac{1}{2} h_{ab}(\lambda) \hat{r}^a \hat{r}^b \right] . \tag{11}$$

Single Arm Response

Using the retarded time $u = t - z$, we get

$$L_{ij} = L_0 + \frac{L_0}{2} \int_0^1 d\lambda \, h_{ab}(\lambda) \hat{r}^a \hat{r}^b \tag{12}$$

$$= L_0 + \frac{1}{2} \frac{\hat{r}^a \hat{r}^b}{1 - \hat{\boldsymbol{z}} \cdot \hat{\boldsymbol{r}}} \int_{u_r - \Delta u}^{u_r} du \, h_{ab}(u), \tag{13}$$

when the gravitational wave travels along the $\hat{\boldsymbol{z}}$ direction.

Single Arm Response

If we define the Fourier transform of our signal

$$\tilde{h}_{ab}(f) = \int du\, h_{ab}(u) e^{2\pi i f u}, \tag{14}$$

we can rewrite the response as

$$\frac{\Delta L_{ij}}{L_0} = \int df\, \frac{1}{2} \hat{r}^a \hat{r}^b \tilde{h}_{ab}(f) \mathcal{T}(f, \hat{\boldsymbol{z}}) e^{-2\pi i f u_r}, \tag{15}$$

with transfer function

$$\mathcal{T}(f, \hat{\boldsymbol{z}}) = \mathrm{sinc}\left[\pi f L_0 \left(1 - \hat{\boldsymbol{z}} \cdot \hat{\boldsymbol{r}}\right)\right] e^{i\pi f L_0 (1 - \hat{\boldsymbol{z}} \cdot \hat{\boldsymbol{r}})}. \tag{16}$$

Single Arm Response

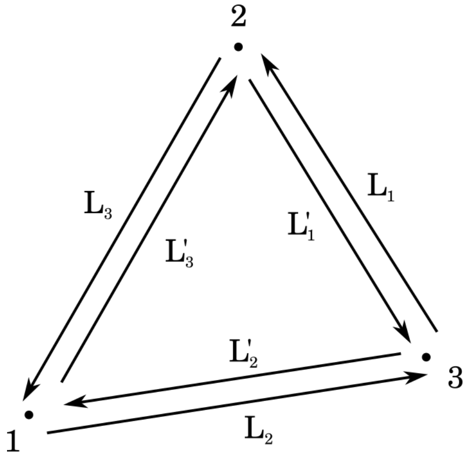
In particular, if we assume that the gravitational wave is monochromatic,

$$h_{ab}(u) = A_+ e^+_{ab} \cos(2\pi f_0 u) + A_\times e^\times_{ab} \sin(2\pi f_0 u), \tag{17}$$

the response becomes

$$\frac{\Delta L_{ij}(u_r)}{L_0} = \frac{1}{2} \Re \left[\mathcal{T}(f_0, \boldsymbol{z}, u_r) \hat{r}^a(u_r) \hat{r}^b(u_r) (A_+ e^+_{ab} - i A_\times e^\times_{ab}) e^{-2i\pi f_0 u_r} \right]. \tag{18}$$

Triangle Configuration



Phase Measurements Combinations

We can combine different phase measurements in order to create a set of six effective measurements, all of which respond to a passing gravitational wave along one of the arms in one direction according to the single arm response computed earlier, and each containing a particular combination of optical noises.

We write the combination that contains a signal travelling from spacraft i to spacecraft k along arm j by y_{ijk}.

First Generation TDI Variables

We can define a time-delay operator as

$$D_i x \equiv x_{,i} = x(t - L_i). \tag{19}$$

First generation TDI variables are constructed by assuming that the constellation is static on timescales comparable to a light travel around it. This results in the approximations that the armlengths are equal in both directions ($L'_i = L_i$), and that time-delay operators commute ($x_{,ij} = x_{,ji}$).

The laser noise contained in the y_{ijk}'s depend on three noise functions ϕ_i, as

$$N[y_{ijk}] = \phi_{i,j} - \phi_k. \tag{20}$$

The first generation TDI variables are linear combinations of time-delayed signals such that the laser noise in them cancels.

First Generation Sagnac-like TDI Variables

One can build a TDI combination so that the noise cancels, in such a way that it corresponds to the difference between the signal travelling once around the constellation clockwise, and once anticlockwise.

$$\alpha = y_{231} + y_{312,3} + y_{123,13} - y_{32'1} - y_{21'3,2'} - y_{13'2,1'2'}. \tag{21}$$

We can check that the noise in α cancels with the first generation TDI assumptions.

We can build two similar TDI combinations β and γ by permutation of the indices.

First Generation Michelson-like TDI Variables

One can build another set of TDI combinations that also cancel the noise, but correspond to the difference between two Michelson-like interferometers.

$$
\begin{aligned}
X = & \, y_{231} + y_{13'2,3} + y_{32'1,3'3} + y_{123,2'3'3} \\
& - y_{32'1} - y_{123,2'} - y_{231,22'} - y_{13'2,322'} .
\end{aligned}
\tag{22}
$$

We can check again that the noise in X cancels with the first generation TDI assumptions.

We can then build two similar TDI combinations Y and Z by permutation of the indices.

Second Generation TDI Variables

The first generation TDI variables are very useful to simulate data from a detector. However, in a real setting those will not cancel the optical noise enough. In order to improve upon them, we need to relax our static constellation assumption.

We define a non-commutative time-delay operator as

$$\mathcal{D}_i x \equiv x_{;i} = x[t - L_i(t)], \tag{23}$$

$$\mathcal{D}_j(\mathcal{D}_i x) = x_{;ij} = x\{t - L_j(t) - L_i[t - L_j(t)]\}. \tag{24}$$

Second Generation Sagnac-like TDI Variables

Relaxing our static detector assumption, one can no longer build a TDI combination so that the noise exactly cancels. However, by constructing a combination as

$$\begin{aligned}\alpha_1 &= y_{231} + y_{312;3} + y_{123;13} \\ &\quad + y_{32'1;213} + y_{21'3;2'213} + y_{13'2;1'2'213} \\ &\quad - y_{32'1} - y_{21'3;2'} - y_{13'2;1'2'} \\ &\quad - y_{231;3'1'2'} - y_{312;33'1'2'} - y_{123;133'1'2'},\end{aligned} \tag{25}$$

one can check that the noise in α_1 reduces to the commutator

$$N[\alpha_1] = [\mathcal{D}_3\mathcal{D}_1\mathcal{D}_2, \mathcal{D}_{2'}\mathcal{D}_{1'}\mathcal{D}_{3'}]\phi_1. \tag{26}$$

We can build two similar TDI combinations α_2 and α_3 by permutation of the indices.

Second Generation Michelson-like TDI Variables

Similarly, we can build Michelson-like TDI combinations,

$$
\begin{aligned}
X_1 = {} & y_{231} + y_{13'2;3} + y_{32'1;3'3} + y_{123;2'3'3} \\
& + y_{32'1;22'3'3} + y_{123;2'22'3'3} + y_{231;22'22'3'3} + y_{13'2;322'22'3'3} \\
& - y_{32'1} - y_{123;2'} - y_{231;22'} - y_{13'2;322'} \\
& - y_{231;3'322'} - y_{13'2;33'322'} - y_{32'1;3'33'322'} - y_{123;2'3'33'322'} .
\end{aligned}
\tag{27}
$$

One can check that the noise in X_1 again reduces to a commutator

$$
N[X_1] = [\mathcal{D}_3\mathcal{D}_{3'}\mathcal{D}_{2'}\mathcal{D}_2, \mathcal{D}_{2'}\mathcal{D}_2\mathcal{D}_3\mathcal{D}_{3'}]\phi_1 . \tag{28}
$$

We can build two similar TDI combinations X_2 and X_3 by permutation of the indices.

Optimal Signal-to-Noise Ratio

It can be shown that looking for a combination of TDI variables with maximum SNR can be reduced to finding the optimum set of linear combinations of α, β and γ with independent noise.

Such a linear combination can be written in the Fourier domain as

$$\eta = a_1\tilde{\alpha} + a_2\tilde{\beta} + a_3\tilde{\gamma}. \tag{29}$$

Optimal Signal-to-Noise Ratio

The optimal SNR can then be shown to be

$$\rho_{\mathrm{opt}}^2 = \int df\, \boldsymbol{x}_i^{(s)\,*} C_{ij}^{-1} \boldsymbol{x}_j^{(s)}, \tag{30}$$

with

$$\boldsymbol{x} = (\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}), \tag{31}$$

a vector containing the signal part of the TDI variables, the noise correlation matrix

$$C_{ij} = \left\langle \boldsymbol{x}_i^{(n)} \boldsymbol{x}_j^{(n)\,*} \right\rangle, \tag{32}$$

and where superscipt (s) denotes the signal part of the variable, and superscript (n) denotes the noise part of it.

Optimal Signal-to-Noise Ratio

We therefore find that a system of TDI variables from spacecraft in a triangle configuration exchanging laser signals is equivalent to three detectors with uncorrelated noise.

If we make the assumption that the noises in the three variables are equal, we get the following form for the noise correlation matrix

$$C = \begin{pmatrix} S_{\alpha\alpha} & S_{\alpha\beta} & S_{\alpha\beta} \\ S_{\alpha\beta} & S_{\alpha\alpha} & S_{\alpha\beta} \\ S_{\alpha\beta} & S_{\alpha\beta} & S_{\alpha\alpha} \end{pmatrix}. \tag{33}$$

Optimal Signal-to-Noise Ratio

This matrix has one two-dimensional and one one-dimensional eigenspaces. An orthonormal basis of eigenvectors can be chosen as

$$\boldsymbol{v}_A = \frac{1}{\sqrt{2}}(-1, 0, 1), \quad \boldsymbol{v}_E = \frac{1}{\sqrt{6}}(1, -2, 1), \quad \boldsymbol{v}_T = \frac{1}{\sqrt{3}}(1, 1, 1), \tag{34}$$

corresponding to the noise-uncorrelated TDI combinations

$$A = \frac{1}{\sqrt{2}} (\tilde{\gamma} - \tilde{\alpha}), \tag{35}$$

$$E = \frac{1}{\sqrt{6}} \left(\tilde{\alpha} - 2\tilde{\beta} + \tilde{\gamma}\right), \tag{36}$$

$$T = \frac{1}{\sqrt{3}} \left(\tilde{\alpha} + \tilde{\beta} + \tilde{\gamma}\right). \tag{37}$$

As $\boldsymbol{v}_A$ and $\boldsymbol{v}_E$ are part of the same eigenspace, the noise PSD in A and E are equal.

References

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