A VERY BRIEF INTRODUCTION TO NUMERICAL RELATIVITY

MPAGS 2019/20

LECTURER: DR PATRICIA SCHMIDT

LOGISTICS

- 3 lectures: **14.11. (PW 106), 21.11. (PW 106), 28.11. (PW 103)**

- Recommended literature:
  - Miguel Alcubierre, "Introduction to 3+1 Numerical Relativity"
  - Thomas Baumgarte, Stuart Shapiro, "Numerical Relativity"
  - Eric Gourgoulhon, "3+1 Formalism and the Bases of Numerical Relativity" (https://arxiv.org/pdf/gr-qc/0703035.pdf)

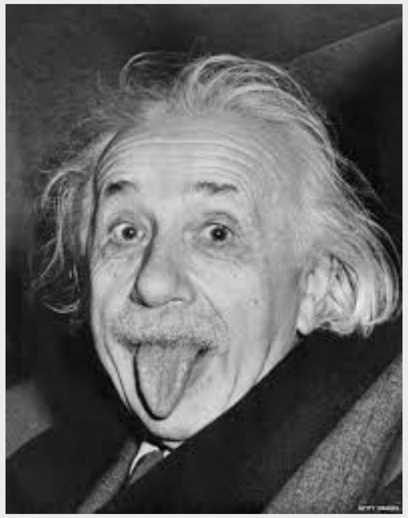
- Assessed problems sheet: TBA

- Residential workshop: Friday, Dec 6th @ 1pm-5pm

3+1 NUMERICAL RELATIVITY

LECTURE 1

WHY NUMERICAL RELATIVITY (NR)?



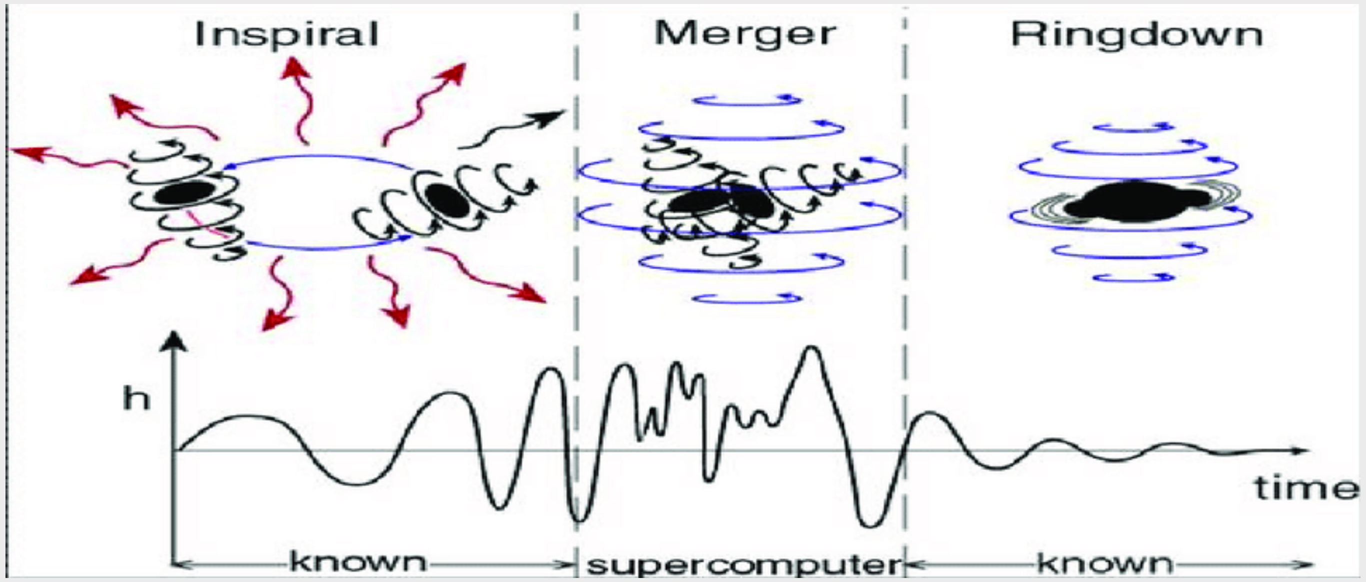
- Note: We use geometric units, i.e. G = c = 1!
- Covariant formulation of General Relativity
  - Einstein field equations:

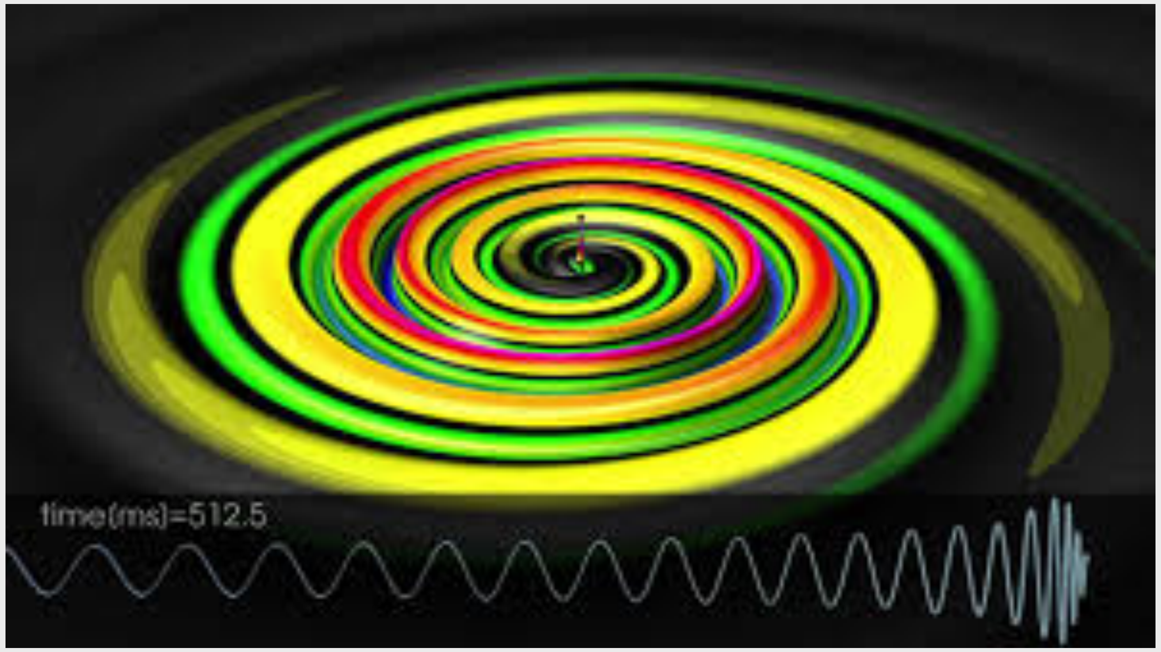
$$\boxed{G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi T_{\mu\nu}}$$

  - 10 non-linear second-order partial differential equations for the metric tensor $g_{\mu\nu}$
  - Analytic solutions exist only for special cases
    - Examples: Schwarzschild, Kerr, Tolman-Oppenheimer-Volkhov (TOV)
  - No known analytic solutions for more general spacetimes

WHY NUMERICAL RELATIVITY (NR)?

- Examples include:
  - Relativistic two-body problem & gravitational waves ("holy grail")





Credit: K. Thorne & UIB

  - Supernovae explosions
  - Perturbations of isolated stars/BHs
  - Cosmological simulations beyond Newton gravity
  - Critical collapse phenomena (e.g. primordial BH formation)
- … and of course most things beyond GR (not discussed here)

NR INGREDIENTS LIST (FOR BINARY BLACK HOLES)

Reformulate the Einstein field equations (EFE) as initial value problem (IVP)

Prove existence of a well-posed initial value problem

Numerically suitable reformulation of the EFE

“good” coordinates (gauge choices)

Initial data

Deal with singularities

“find” the black hole horizons

Extract gravitational waves

1952 Choquet-Bruhat
IVP

1962 ADM
formulation

1964 Hahn-Linquist
2 wormholes

1984 Unruh
excision

1992,3 Choptiuk
Abrahams, Evans
critical phenomena

1997
Brandt&Brügmann
puncture data

1994-98
BBH grand challenge

1999-00
AEI/PSU
Grazing Collisions

2000-04
AEI/UTB/NASA
Lazarus

~2000 Choptuik,
Brügmann, Schnetter
mesh refinement

2005 Gundlach+
constraint damping

2005 Pretorius
IMR w harmonic

2005-06 Campanelli+,
Baker+; IMR w BSSN &
moving punctures

2006-08 SXS
IMR w spectral

2011 Lousto+
q=100

2011 Lovelace+
a=0.97

2015 Szilagyi+
175 orbits

2014-17 Ruchlin+
high spin puncture ID

1950s — 2005 — 2015+

1975-77 Smarr-Eppley
head on collisions

1979 York
Kinematics &
dynamics of GR

1989-95
Bona-Masso
Modified ADM

1994 Cook
Bowen-York ID

94-95 NCSA/WSU
improved head-on

1999-2005
JW York,
Cornell, Caltech, LSU
hyperbolic formulations

1999
BSSN

1999 York
CTS ID

2000-02
Alcubierre
gauge conditions

2004
Brügmann+
one orbit

2003-08
Cook, Pfeiffer ea
improved ID

2000 Ashtekar
isolated horizons

2006,07 Baker+;
Gonzalez+
non-spinning BBH
kicks

2007-11
RIT; Jena; AEI;...
BBH superkicks

2008
NINJA

2009-11
Bishop+
CCE

2010
Bernuzzi+
C4z

Adapted from slides by C. Lousto & H. Pfeiffer

3+1 FORMALISM

- Goal: Solve for the dynamical evolution of the gravitational field in time
  - Initial value problem (Cauchy problem) formulation of GR

    Given a set of adequate initial (and boundary) conditions, the fundamental equations must predict the future (past) evolution of the system.

  - To rewrite the EFE as an IVP, we first separate spacetime into space *and* time (3+1 formalism)

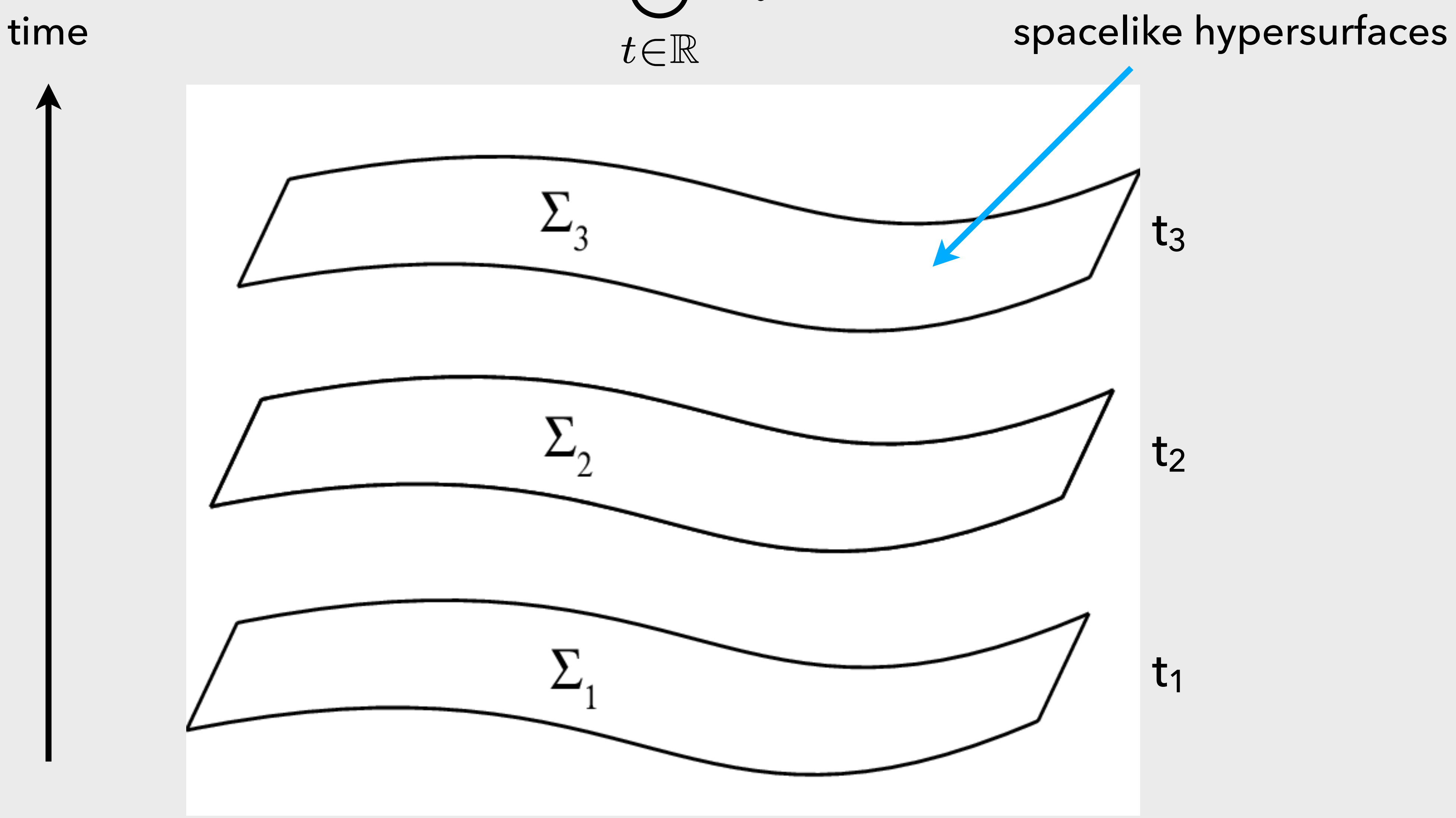
- Assumption: The considered spacetime (M, g) is *globally hyperbolic*, i.e. the spacetime admits a Cauchy surface.

- Definition: A Cauchy surface is a spacelike hypersurface in M such that each causal curve intersects it once and only once.

3+1 FORMALISM

- Any globally hyperbolic spacetime can be completely foliated by space-like hypersurfaces (foliation, slicing). The foliation can be identified as the level sets of a smooth, regular scalar function (e.g. time).

$$\mathcal{M} = \bigcup_{t\in\mathbb{R}} \Sigma_t$$



SPACELIKE HYPERSURFACES

- We define spacelike hypersurfaces Σ_t as the level set of the scalar field t on M. From t, we define the (unnormalised) 1-form

$$\Omega_\mu = \nabla_\mu t$$

- From the 4-metric, we can compute its norm:

$$||\Omega||^2 = g^{\mu\nu}\nabla_\mu t \nabla_v t \equiv -\frac{1}{\alpha^2}$$

  - We assume $\alpha > 0$, then Σ_t is space like and Ω_μ is timelike.

- The unit normal vector to Σ_t is then given by:

$$n^\mu = -\alpha g^{\mu\nu}\Omega_\nu$$

- α denotes the lapse and n^μ can be thought of as the 4-velocity of a normal (Eulerian) observer
- Note: Since Σ_t is spacelike, $g_{\mu\nu}n^\mu n^\nu = -1$. We use the convention $\mathrm{sign}(g_{\mu\nu}) = (-+++)$.

SPACELIKE HYPERSURFACES

- With the definition of a hypersurface normal, we can now construct the spatial metric on the hypersurface:

$$\mathcal{T}_p(M) = \mathcal{T}_p(\Sigma) \oplus \mathrm{span}(n) \quad \forall p \in \Sigma$$

$$\gamma : \mathcal{T}_p(M) \to \mathcal{T}_p(\Sigma)$$

$$\gamma_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu$$

- To "break up" 4D objects into their components parallel and orthogonal to the hypersurface, we require a projection operator

$$\gamma^\alpha{}_\beta = \delta^\alpha{}_\beta + n^\alpha n_\beta$$

- The projection operator allows us to construct purely spatial objects. Together with n^μ we have the tools to related 4D objects in M to 3D objects on Σ_t.

SPACELIKE HYPERSURFACES

- 3D covariant derivative: Let γ_{ij} be the induced non-degenerate metric on Σ_t. Then there exists a unique connection (covariant derivative) D:

$$D_\mu f \equiv \gamma_\mu^\nu \nabla_\nu f$$

- The Riemann tensor associated with this connection, defines the intrinsic curvature of the hypersurface:

$$(D_\mu D_\nu - D_\nu D_\mu) v^\sigma = {}^{(3)}R^\sigma{}_{\mu\nu\rho} v^\rho \quad \forall v \in \mathcal{T}(\Sigma)$$

- The intrinsic (scalar) curvature of the hypersurface (also called Gaussian curvature) is

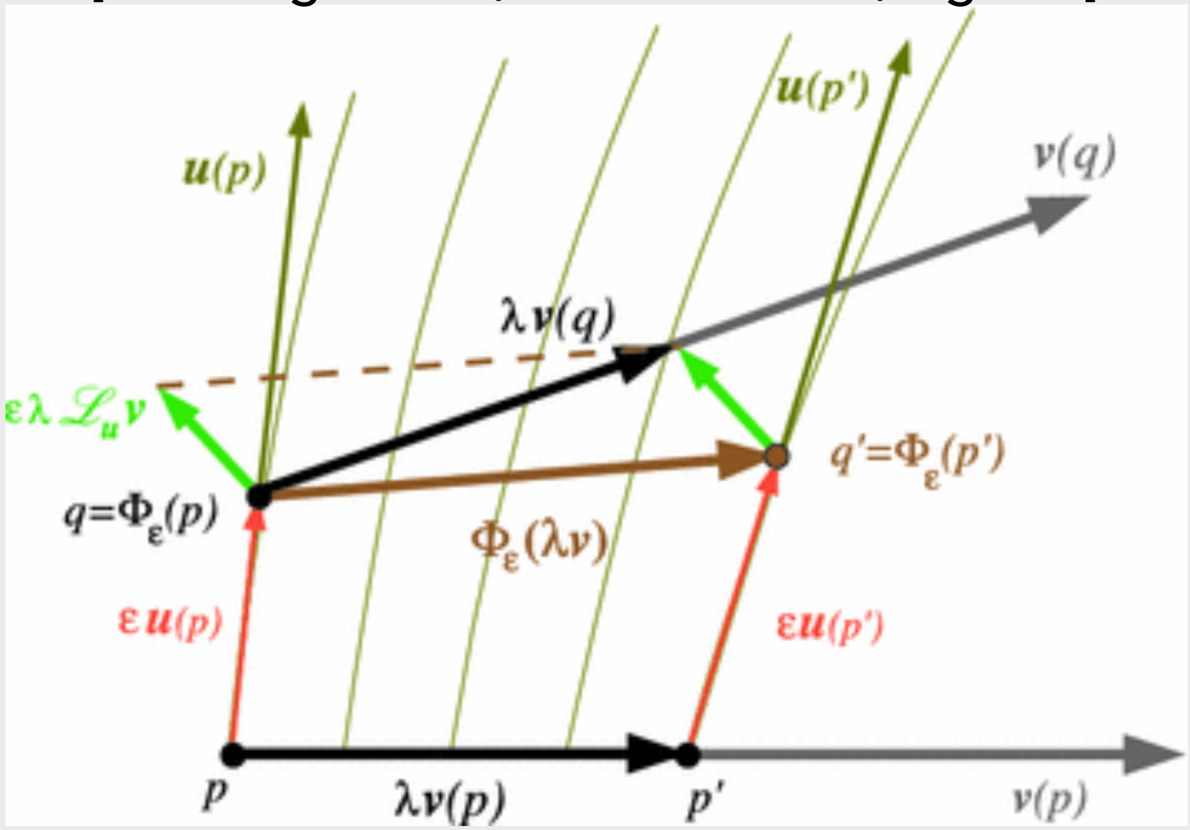
$${}^{(3)}R = \gamma^{\mu\nu}\,{}^{(3)}R_{\mu\nu} = \gamma^{\mu\nu}\,{}^{(3)}R^\sigma{}_{\mu\sigma\nu}$$

LIE DERIVATIVE

- Consider two vector fields u,v on (M, g). What is the variation of the vector field v at the point q in M when transported along the vector field u?

$$\mathcal{L}_u v := \lim_{\epsilon \to 0} \frac{1}{\epsilon}[v(q) - \phi_\epsilon v(p)]$$

[E. Gourgoulhon, 3+1 formalism, Fig. 2.3]

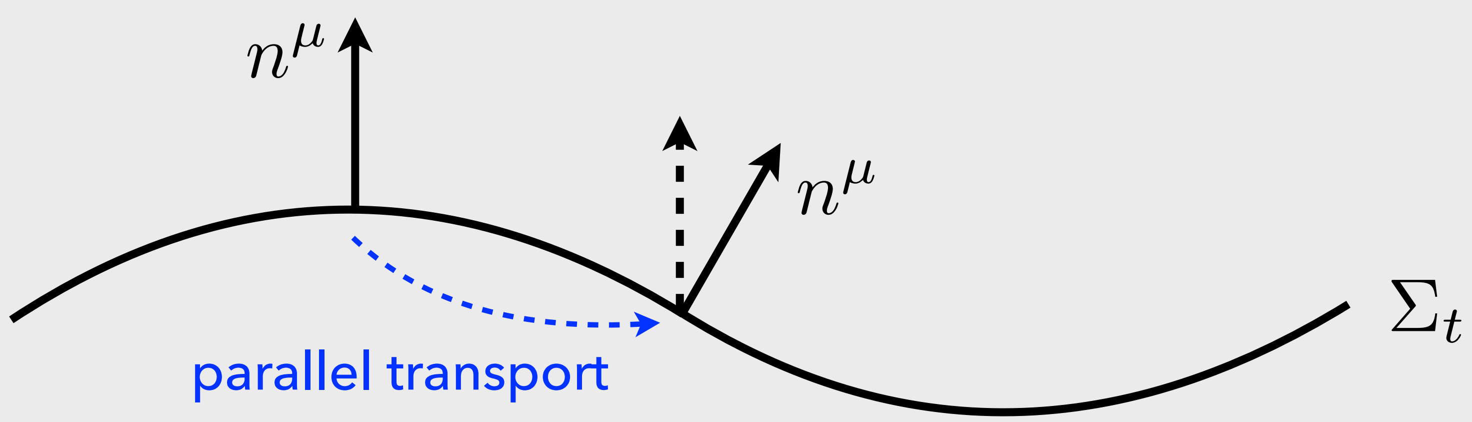


$$\mathcal{L}_u v^\alpha = [u, v]^\alpha = u^\mu \nabla_\mu v^\alpha - v^\mu \nabla_\mu u^\alpha$$

- Note: This definition can be generalised to any tensor field on (M, g)

SPACELIKE HYPERSURFACES

- Extrinsic curvature: Describes the "bending" of (3D) Σ_t in (4D) M, and is measured by the change of direction of n as one moves along the hypersurface via parallel transport.



- The extrinsic curvature tensor (second fundamental form) is defined as:

$$K_{\mu\nu} := -\gamma^{\alpha}{}_{\mu}\gamma^{\beta}{}_{\nu}\nabla_\alpha n_\beta = -(\nabla_\mu n_\nu + n_\mu n^\alpha \nabla_\alpha n_\nu)$$

$$\equiv -\frac{1}{2}\mathcal{L}_n \gamma_{\mu\nu}$$

Geometric generalisation of the "time derivative" the the spatial metric!

- $(\gamma_{\mu\nu}, K_{\mu\nu})$ are the fundamental variables in the 3+1 formulation.

GAUSS, CODAZZI & RICCI

- $\gamma_{\mu\nu}$ and $K_{\mu\nu}$ cannot be chosen arbitrarily - they are related to our 4D manifold (M, g). In particular, we need to relate the 3D and the 4D Riemann tensor. Obtained via contractions with n^μ and $\gamma^\mu{}_\nu$.
- Gauss' equation:

$$\gamma^\alpha_\mu \gamma^\beta_\nu \gamma^\gamma_\sigma \gamma^\delta_\rho \, {}^{(4)}R_{\alpha\beta\gamma\delta} = {}^{(3)}R_{\mu\nu\sigma\rho} + K_{\mu\sigma}K_{\nu\rho} - K_{\mu\rho}K_{\nu\sigma}$$

- Codazzi equation:

$$\gamma^\alpha_\mu \gamma^\beta_\nu \gamma^\gamma_\sigma n^\delta \, {}^{(4)}R_{\alpha\beta\gamma\delta} = D_\nu K_{\mu\sigma} - D_\mu K_{\nu\sigma}$$

- Ricci's equation:

$$n^\alpha n^\gamma \gamma^\beta_\mu \gamma^\delta_\nu \, {}^{(4)}R_{\alpha\beta\gamma\delta} = \mathcal{L}_n K_{\mu\nu} + \frac{1}{\alpha} D_\mu D_\nu \alpha + K^\rho_\nu K_{\mu\rho}$$

THE EINSTEIN CONSTRAINTS

- So far we have only considered the kinematics of hypersurfaces. The true gravitational DOF are contained in the Einstein field equations. We now need to link our geometric objects to the physics.
- Contracting the Gauss equation twice, we find:

$$\boxed{{}^{(3)}R + K^2 - K_{\alpha\beta}K^{\alpha\beta} = 16\pi\rho}$$

where $\rho \equiv n_\mu n_\nu T^{\mu\nu}$ is the total energy density measured by a normal observer. This equation is commonly referred to as the Hamiltonian constraint.

THE EINSTEIN CONSTRAINTS

- Let's now contract the Codazzi equation once. This yields:

$$\boxed{D_\alpha K^\alpha_\mu - D_\mu K = 8\pi S_\mu}$$

where $S_\mu \equiv -\gamma^\alpha_\mu n^\beta T_{\alpha\beta}$ is the momentum density measured by a normal observer. This equation is commonly referred to as the momentum constraint.

The constraint equations only involve the spatial metric, the extrinsic curvature and their spatial derivatives. $(\gamma_{\mu\nu}, K_{\mu\nu})$ are imposed on a timeslice Σ_t and have to satisfy the constraint equations. We need to solve the constraint equations to find suitable initial data.

EVOLUTION EQUATIONS

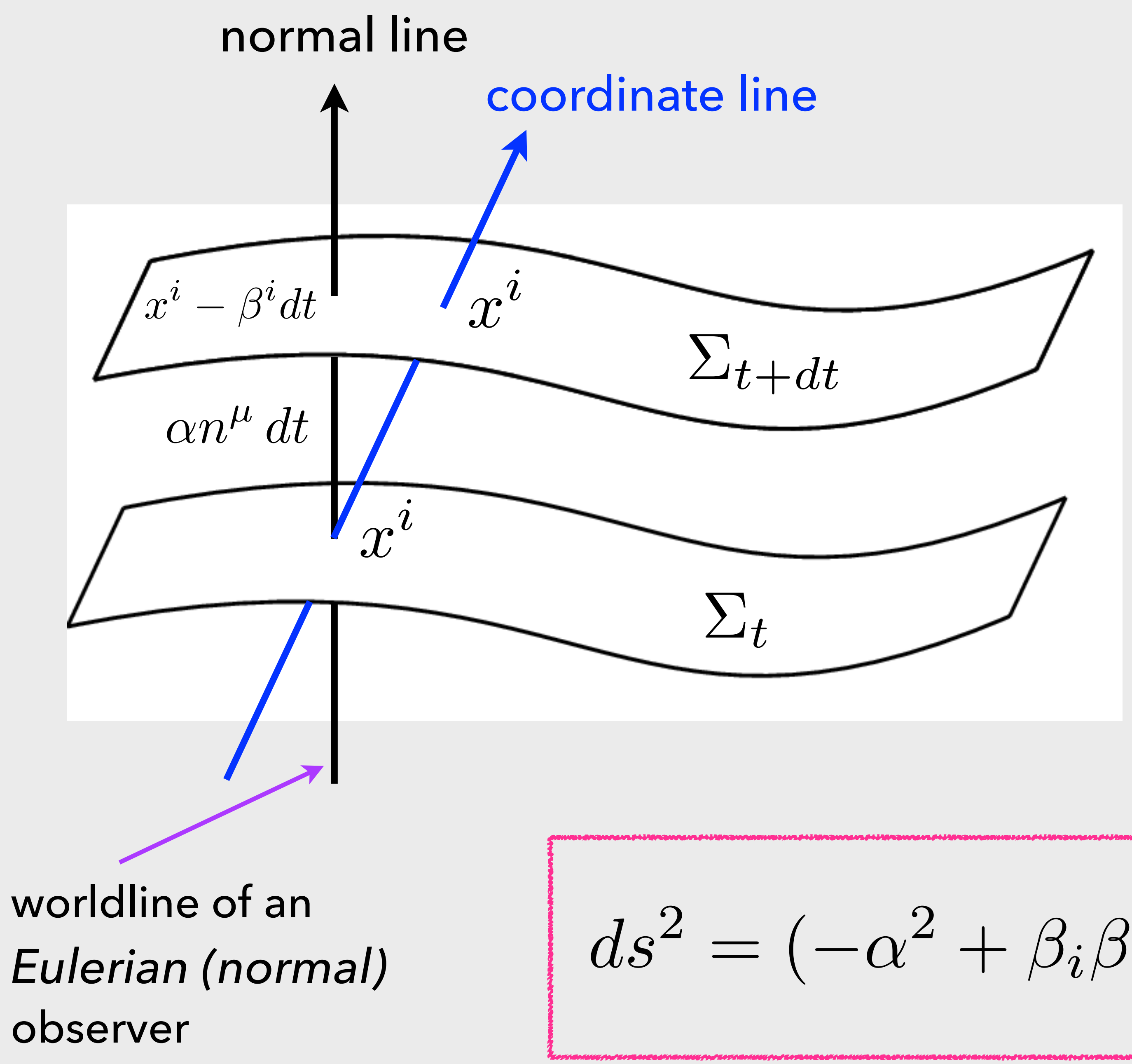
- We already have "evolution equations" for $(\gamma_{\mu\nu}, K_{\mu\nu})$ from the definition of the extrinsic curvature and the Ricci equation.
  - The evolution equations to evolve the data forward in time are given by the Lie derivative of the spatial metric and the extrinsic curvature along the hypersurface normal n^μ.
- However, $\mathscr{L}_n$ is not a natural time derivative since n^μ is not the dual of Ω_μ.
- Instead, consider the following vector:

$$t^\mu = \alpha n^\mu + \beta^\mu$$

  - It is the dual to Ω_μ for any purely spatial vector β^μ - the shift vector.
- t^μ provides a natural congruence along which we can propagate the spatial coordinates from one slice to the next. (α, β^μ) are arbitrary and encode how coordinates evolve in time.

FOLIATION ADAPTED COORDINATES

- Let us now consider coordinates adapted to this 3+1 split:



$$t^\mu = (1,0,0,0)$$
$$\beta^\mu = (0, \beta^i)$$
$$\Rightarrow n^\mu = \left(\frac{1}{\alpha}, -\frac{1}{\alpha}\beta^i\right)$$
$$n_\mu = (-\alpha, 0, 0, 0)$$
$$\gamma_{ij} = g_{ij}$$

$$\boxed{ds^2 = (-\alpha^2 + \beta_i \beta^i) dt^2 + 2\beta_i dt dx^i + \gamma_{ij} dx^i dx^j}$$

ADM EQUATIONS (IN YORK FORM)

- The entire content of the 3+1 decomposed EFE is contained the spatial components. Hence, our final set of equations reads as follows:
- **Constraint equations:**

$$
\begin{aligned}
{}^{(3)}R + K^2 - K_{ij}K^{ij} &= 16\pi\rho \\
D_j\left(K^{ij} - \gamma^{ij}K\right) &= 8\pi S^i
\end{aligned}
$$

- **Evolution equations:**

$$
\begin{aligned}
\partial_t \gamma_{ij} &= -2\alpha K_{ij} + D_i\beta_j + D_j\beta_i \\
\partial_t K_{ij} &= -D_iD_j\alpha + \alpha\Big(R_{ij} + KK_{ij} - 2K_{ik}K^k_j + 4\pi[(S-\rho)\gamma_{ij} - 2S_{ij}]\Big) \\
&\quad + \beta^k D_k K_{ij} + K_{ik}D_j\beta^k + K_{kj}D_i\beta^k
\end{aligned}
$$

- Note: The 3+1 evolution equations are non-unique. We can always add arbitrary multiples of the constraints.

REMARK: WELL-POSEDNESS

- Well-posedness of the system of evolution equations is essential for stable numerical evolution, but it is not enough.
- Original ADM equations are *mathematically* not well-posed. The formulation after York is well-posed.
  - BUT: The ADM equations are not numerically robust (weak hyperbolicity)
- Due to the non-uniqueness of the evolution equations, we can derive "new" evolution equations, that are mathematically well-posed AND numerically robust.
  - **Baumgarte-Shapiro-Shibata-Nakamura (BSSN) formulation**
  - Conformal rescaling of the spatial metric + auxiliary variable Γ^i
  - Obtain a *strongly hyperbolic system of evolution equations*

OTHER FORMULATIONS

- There are alternative approaches to solving the EFE:
  - Characteristic formalism: 2+2 formulation, where ingoing (outgoing) null hypersurfaces emanate from a timelike world tube.
  - Conformal formalism
  - Evolving the full 4D spacetime (generalised harmonic formulation)

INITIAL DATA & GAUGE CONDITIONS

LECTURE 2

INITIAL DATA

- Recall: The spatial metric, the extrinsic curvature and any matter fields have to satisfy the constraint equations on every hypersurface Σ

$$^{(3)}R + K^2 - K_{ij}K^{ij} = 16\pi\rho$$
$$D_j(K^{ij} - \gamma^{ij}K) = 8\pi S^i$$

- We first need to specify (γ_{ij}, K_{ij}) on some initial slice:
  - 12 independent components but only 4 constraint equations
  - 4 components are related to coordinate choices
  - 4 components represent the dynamical DOF (transverse parts)
  - Constraint equations only constrain the longitudinal parts
  - Seek split between the constrained and unconstrained components of the field

CONFORMAL TRANSVERSE-TRACELESS DECOMPOSITION

- Consider a conformal transformation of the spatial metric:

$$\gamma_{ij} = \psi^4 \bar{\gamma}_{ij}$$

- The Hamiltonian constraint yields:

$$8\bar{D}^2\psi - \psi\bar{R} - \psi^5 K^2 + \psi^5 K_{ij} K^{ij} = -16\pi\psi^5\rho$$

- Consider now a conformal transformation of the extrinsic curvature:

$$K_{ij} = A_{ij} + \frac{1}{3}\gamma_{ij} K$$

$$A^{ij} = \psi^\alpha \bar{A}^{ij} \qquad D_j A^{ij} \Rightarrow \alpha = -10$$

$$K = \psi^\beta \bar{K} \qquad \beta = 0$$

$$8\bar{D}^2\psi - \psi\bar{R} - \frac{2}{3}\psi^5 K^2 + \psi^{-7}\bar{A}_{ij}\bar{A}^{ij} = -16\pi\psi^5\rho$$

$$\bar{D}_j\bar{A}^{ij} - \frac{2}{3}\psi^6\bar{\gamma}^{ij}\bar{D}_j K = 8\pi\psi^{10} S^i$$

CONFORMAL TRANSVERSE-TRACELESS DECOMPOSITION

- Any symmetric traceless tensor can split as

$$\bar{A}^{ij} = \bar{A}^{ij}_{TT} + \bar{A}^{ij}_{L}$$

$$\bar{D}_j \bar{A}^{ij} = 0$$

$$\bar{A}^{ij}_{L} = \bar{D}^i W^j + \bar{D}^j W^i - \frac{2}{3}\bar{\gamma}^{ij}\bar{D}_k W^k \equiv (\bar{L}W)^{ij}$$

$$\Rightarrow (\bar{\Delta}_L W)^i - \frac{2}{3}\psi^6 \bar{\gamma}^{ij}\bar{D}_j K = 8\pi\psi^{10} S^i$$

- We see that we can freely choose $\bar{\gamma}_{ij}, K, \bar{A}^{ij}_{TT}$
- Solve for W^i, ψ
- Construct the physical solutions for γ_{ij}, K_{ij}
- Choices for background fields need to be “motivated"

BOWEN-YORK INITIAL DATA

- Example: vacuum case with K=0 ("maximal slicing")

$$\Rightarrow (\bar{\Delta}_L W)^i = 0$$

- Let's also assume conformal flatness, i.e. $\bar{\gamma}_{ij} = \eta_{ij}$, then the vector Laplacian simplifies to:

$$\partial^j \partial_j W^i + \frac{1}{3}\partial^i \partial_j W^j = 0$$

- Solutions to this equations are called Bowen-York solutions
- Well known solutions include a spinning BH and a boosted BH
- A very common type of BH initial data used in particular in the "moving punctures" framework

CONFORMAL THIN-SANDWICH DECOMPOSITION

- If we want to have (quasi-)equilibrium solutions, we require initial data that have a certain time evolution
- CTS: Instead of providing data on one slice Σ, data are provided on two slices with infinitesimal separation
- In summary: The freely specifiable variables are $(\bar{\gamma}_{ij}, \bar{u}_{ij}, K, \bar{\alpha})$. The momentum constraint is solved for β^i

$$(\bar{\Delta}_L \beta)^i - (\bar{L}\beta)^{ij} \bar{D}_j \ln(\bar{\alpha}) = \bar{\alpha} \bar{D}_j (\bar{\alpha}^{-1} \bar{u}^{ij}) + \frac{3}{4} \bar{\alpha} \psi^6 \bar{D}^i K + 16\pi \bar{\alpha} \psi^{10} S^i$$

And then the Hamiltonian constraint is solved for ψ

$$\bar{D}^2 \psi - \frac{1}{8} \psi \bar{R} - \frac{1}{12} \psi^5 K^2 + \frac{1}{8} \psi^{-7} \bar{A}_{ij} \bar{A}^{ij} = -2\pi \psi^5 \rho$$

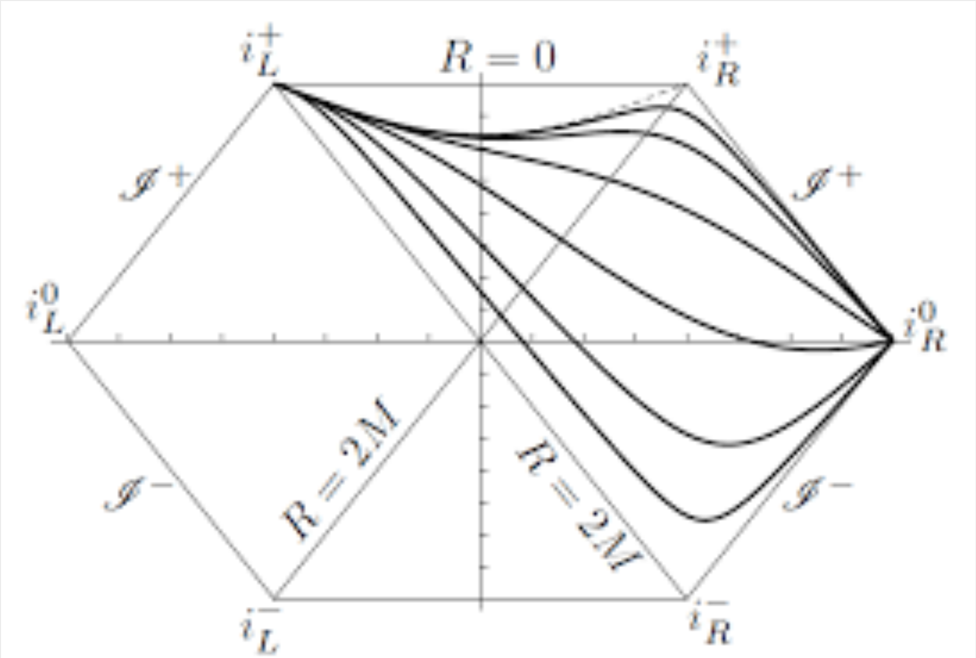
The physical solution is then constructed from:

$$\gamma_{ij} = \psi^4 \bar{\gamma}_{ij}, \quad K_{ij} = \psi^{-2} \bar{A}_{ij} + \frac{1}{3} \gamma_{ij} K, \quad \alpha = \psi^6 \bar{\alpha}$$

where $$\bar{A}_{ij} = \frac{1}{2\bar{\alpha}} \left((\bar{L}\beta)^{ij} - \bar{u}^{ij} \right), \quad \bar{u}_{ij} \equiv \partial_t \bar{\gamma}_{ij}$$

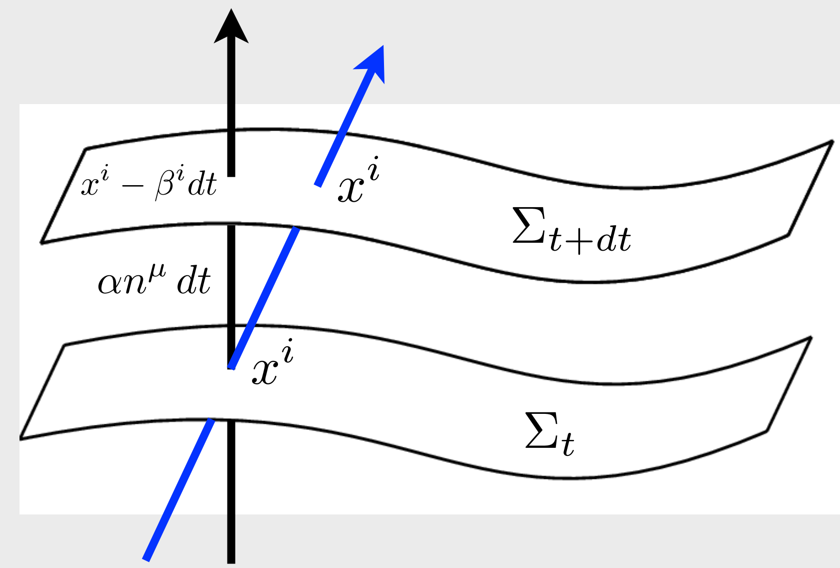
GAUGE CONDITIONS

- 4 gauge functions α and β^i: we need to impose coordinate conditions
- Finding "good" gauge conditions is not trivial but geometric insight combined with numerical experimentation (trial and error) to produce good gauge choices
- Some desired features, especially for BH spacetimes are:
  - Horizon penetrating coordinates
  - Singularity-avoiding gauge conditions
  - Minimal distortion



GEODESIC SLICING

- Simplest possible choice:



$$\boxed{\alpha = 1, \quad \beta^i = 0}$$

- Coordinate observers coincide with normal observers
- Normal observers are freely falling ($a_\mu = 0$) and hence follow geodesics
- Unfortunately, coordinate singularities develop very quickly as geodesics focus near gravitating sources, which can be seen from the expansion:

$$\nabla_\mu n^\mu = -K$$

K grows monotonically in time

$$\partial_t K \geq 0$$

MAXIMAL SLICING

- To control the divergence of normal observers, we need to impose a suitable condition on K
- A common choice is "maximal" slicing (for all times),

$$K = 0 = \partial_t K$$

  - Maximal slicing is volume preserving along congruences of n^μ
- From the contraction of the evolution equation for K_{ij}, we find:

$$D^2\alpha = \alpha(K_{ij}K^{ij} + 4\pi(\rho - S))$$

- Or in the conformal decomposition:

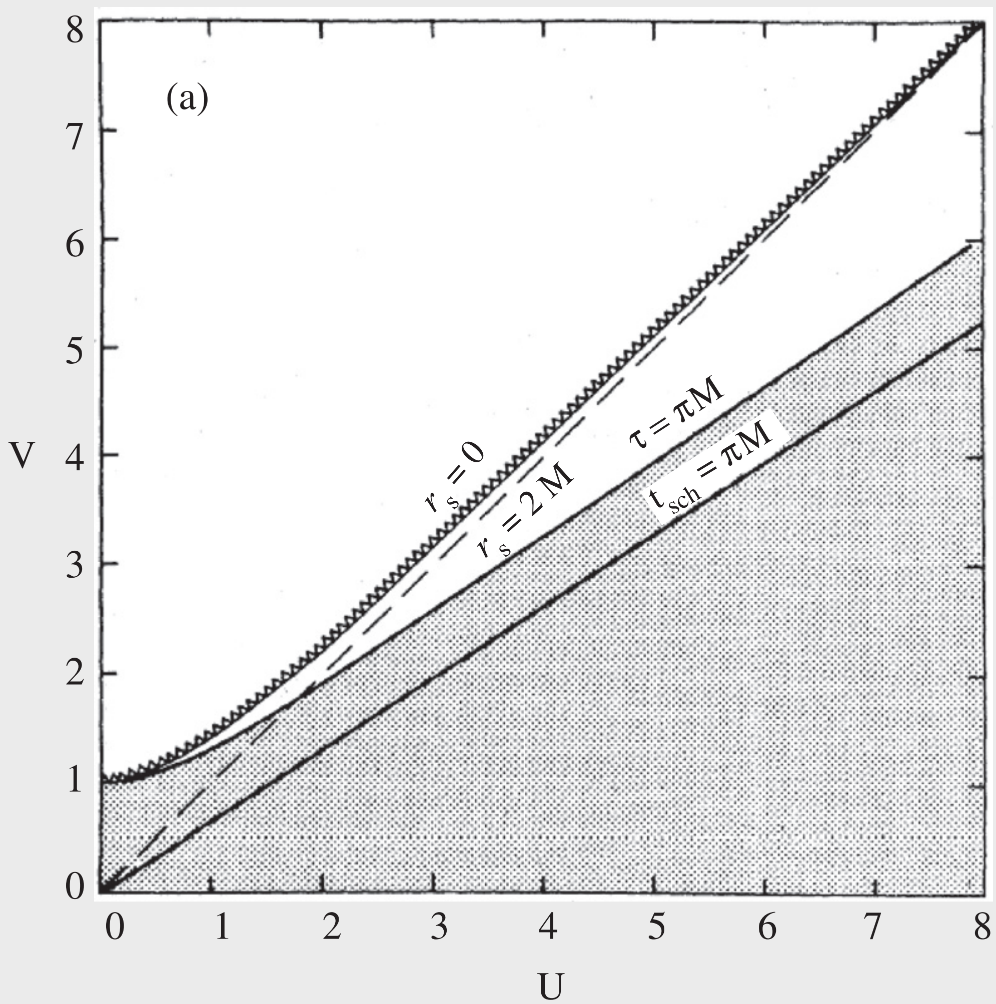
$$\bar{D}^2(\alpha\psi) = \alpha\psi\left(\frac{7}{8}\psi^{-8}\bar{A}_{ij}\bar{A}^{ij} + \frac{1}{8}\bar{R} + 2\pi\psi^4(\rho + 2S)\right)$$

- Note: Elliptic equations are costly. We can recast the equation into a parabolic one via "approximate" maximal slicing, i.e. $\partial_t K = -cK$

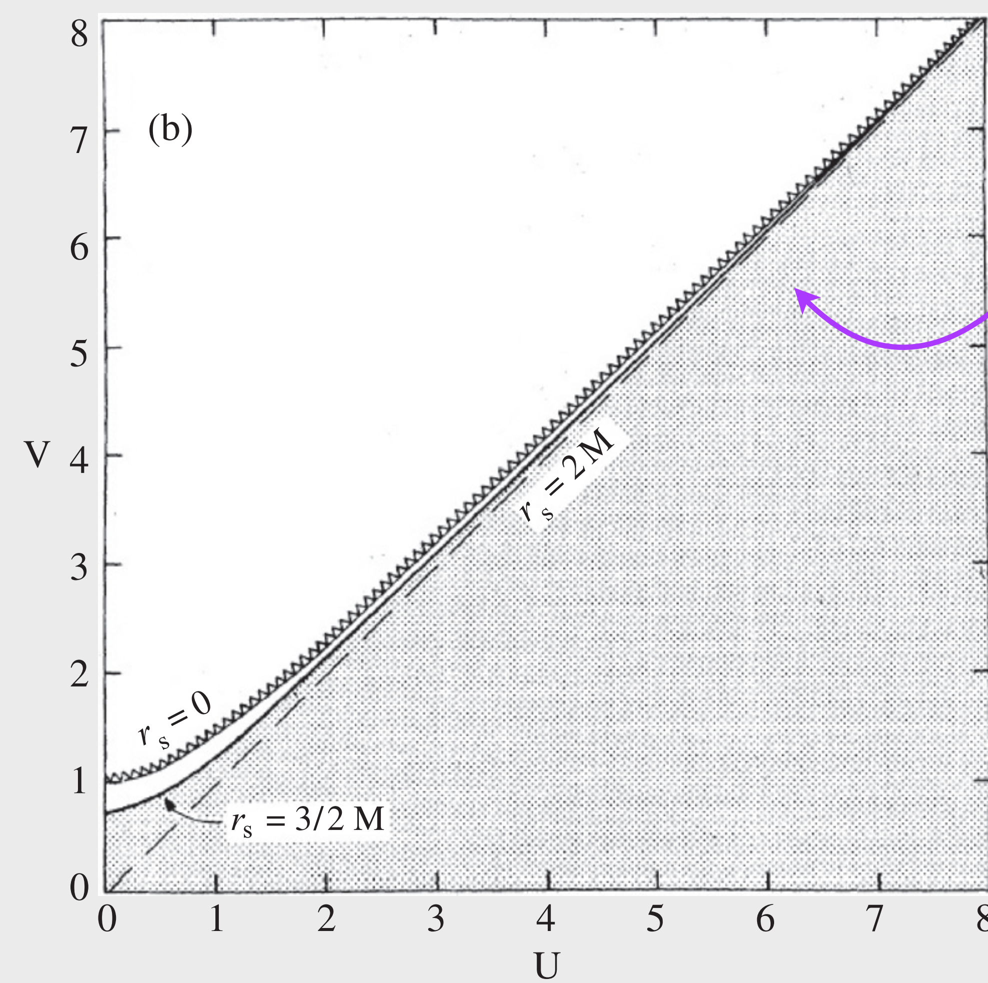
GEODESIC VS MAXIMAL SLICES

- Consider a Schwarzschild spacetime:

geodesic slicing



maximal slicing



Cover the BH exterior fully!

Maximal slices penetrate the BH interior but avoid the singularity asymptoting to the limiting surface at $r_S = 3M/2$.

Taken from Baumgarte & Shapiro, Numerical Relativity

1+LOG SLICING & GAMMA DRIVER

- A popular (algebraic) slicing condition is

$$\alpha = 1 + \ln \gamma \qquad \text{(here } \beta^i = 0 \text{ is assumed)}$$

- Generalised "advective" version (nonzero shift vector) reads as

$$(\partial_t - \beta^j \partial_j)\alpha = -2\alpha K$$

- Shift condition based on "connection functions":

$$\bar{\Gamma}^i \equiv \bar{\gamma}^{kl}\bar{\Gamma}^i_{kl} = -\partial_j \bar{\gamma}^{ij}$$

- Assume $\partial_t \bar{\Gamma}^i = 0$: yields a set of elliptic equations for the shift vector, which can be transformed into parabolic equations through the following approximation: $\partial_t \beta^i = k(\partial_t \bar{\Gamma}^i + \eta \bar{\Gamma}^i)$
- From this we can construct the hyperbolic Gamma-driver:

$$\partial_t \beta^i = \frac{3}{4} B^i, \quad \partial_t B^i = \partial_t \bar{\Gamma}^i - \eta B^i$$