UNIVERSITY OF BIRMINGHAM | GRAVITATIONAL WAVE ASTRONOMY

(A VERY BRIEF) INTRODUCTION TO NUMERICAL RELATIVITY

MPAGS: BLACK HOLES AND GRAVITATIONAL WAVES - PART 3

LECTURER: DR PATRICIA SCHMIDT

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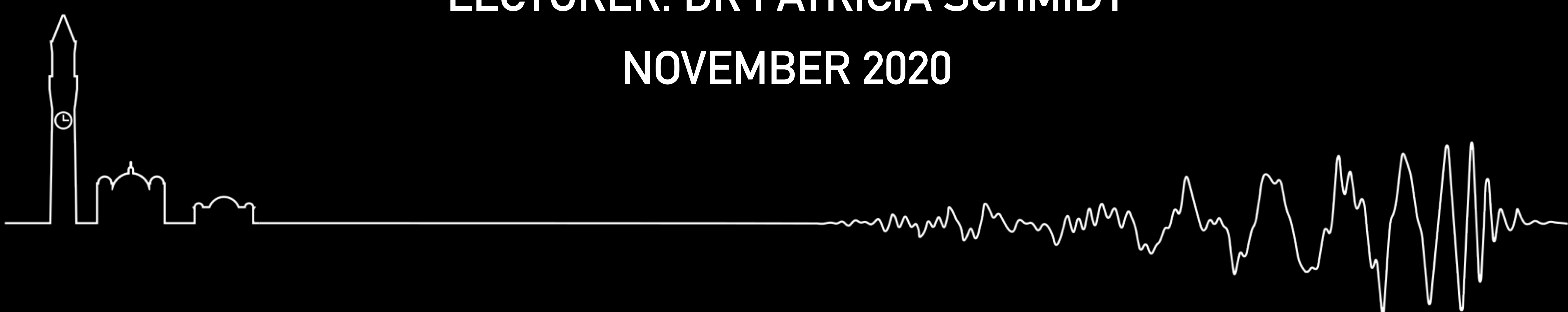
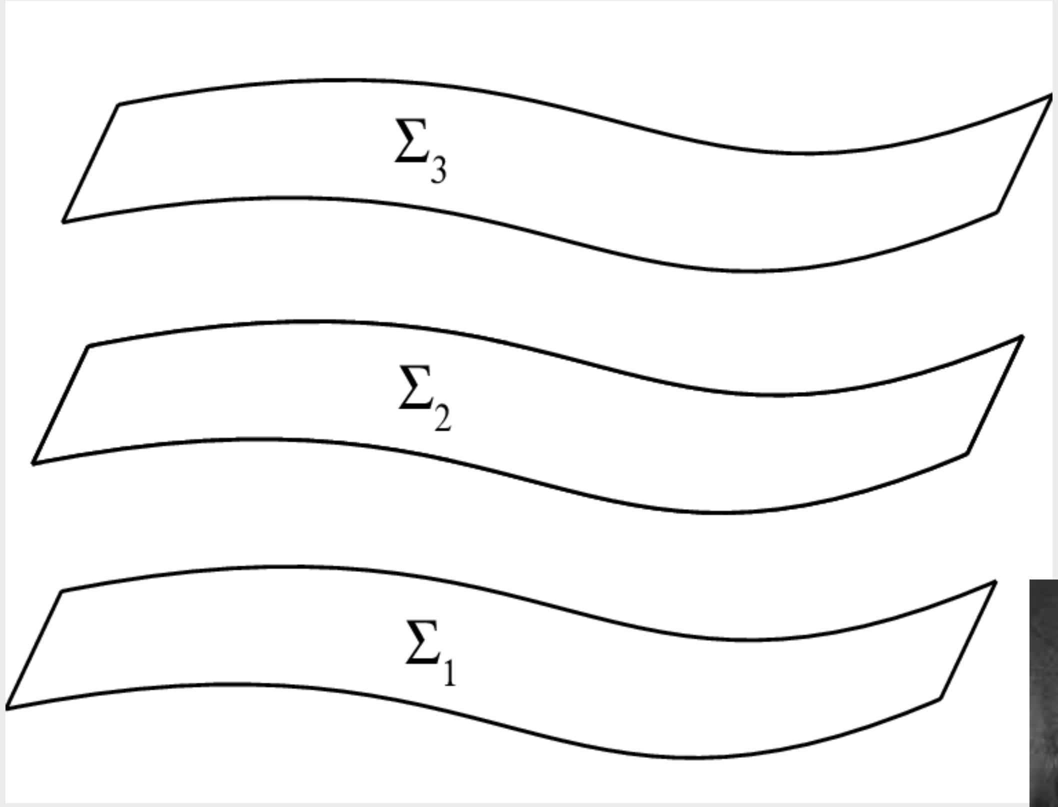
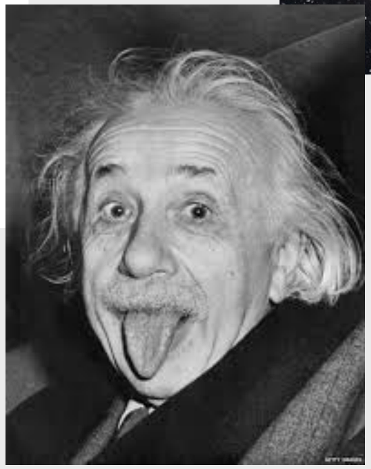
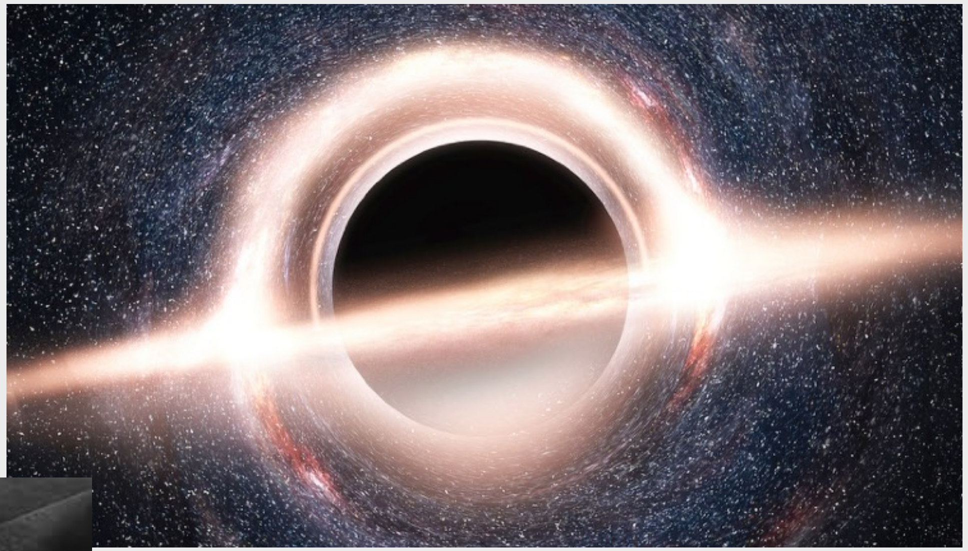


THE EINSTEIN CONSTRAINTS

- So far we have only considered the *kinematics of hypersurfaces*. The true gravitational DOF are contained in the Einstein field equations. We now need to link our geometric objects to the physics.







Gravity

Geometry

HAMILTONIAN CONSTRAINT

Let's contract the Gauss equation twice:

$$\gamma^{ac}\gamma_e{}^b\gamma_f{}^{d\,(4)}R_{abcd} = {}^{(3)}R_{ef} + KK_{ef} - K^s{}_f K_{es}$$

Second contraction:

$$\gamma^{ef}\gamma^{ac}\gamma_e{}^b\gamma_f{}^{d\,(4)}R_{abcd} = {}^{(3)}R + K^2 - K^{ef}K_{ef}$$

$$\begin{aligned}\gamma^{ef}\gamma^{ac}\gamma_e{}^b\gamma_f{}^{d\,(4)}R_{abcd} &= \gamma^{ac}\gamma^{bd\,(4)}R_{abcd}\\ &= (g^{ac}+n^a n^c)(g^{bd}+n^b n^b)^{(4)}R_{abcd}\\ &= {}^{(4)}R + 2^{(4)}R_{ab}n^a n^b\end{aligned}$$

Einstein field equations (finally!):

$$G_{ab} = {}^{(4)}R_{ab} - \frac{1}{2}{}^{(4)}Rg_{ab} = 8\pi T_{ab} \quad |2n^a n^b$$

HAMILTONIAN CONSTRAINT

$$2n^a n^{b\,(4)}R_{ab} - {}^{(4)}R n^a n^b g_{ab} = 16\pi\rho$$

$$\rho := n^a n^b T_{ab}$$

$$n^a n^b g_{ab} = -1$$

$$\Rightarrow 2n^a n^{b\,(4)}R_{ab} + {}^{(4)}R = 16\pi\rho$$

$${}^{(3)}R + K^2 - K_{ab}K^{ab} = 16\pi\rho$$

Hamiltonian constraint

HAMILTONIAN CONSTRAINT

- So far we have only considered the *kinematics of hypersurfaces*. The true gravitational DOF are contained in the Einstein field equations. We now need to link our geometric objects to the physics.

- Contracting the Gauss equation twice, we have found:

$$\boxed{{}^{(3)}R + K^2 - K_{\alpha\beta}K^{\alpha\beta} = 16\pi\rho}$$

where $\rho \equiv n_\mu n_\nu T^{\mu\nu}$ is the total energy density measured by a normal observer. This equation is commonly referred to as the Hamiltonian constraint.

MOMENTUM CONSTRAINT

- Contract the Codazzi equation once yields (Exercise):

$$\boxed{D_\alpha K^\alpha_\mu - D_\mu K = 8\pi S_\mu}$$

where $S_\mu \equiv -\gamma^\alpha_\mu n^\beta T_{\alpha\beta}$ is the momentum density measured by a normal observer. This equation is commonly referred to as the momentum constraint.

The constraint equations only involve the spatial metric, the extrinsic curvature and their spatial derivatives. $(\gamma_{\mu\nu}, K_{\mu\nu})$ are imposed on a timeslice Σ_t and have to satisfy the constraint equations. We need to solve the constraint equations to find suitable initial data.

EVOLUTION EQUATIONS

- We already have "evolution equations" for $(\gamma_{\mu\nu}, K_{\mu\nu})$ from the definition of the extrinsic curvature and the Ricci equation.
  - The evolution equations to evolve the data forward in time are given by the Lie derivative of the spatial metric and the extrinsic curvature along the hypersurface normal n^μ.

- However, $\mathscr{L}_n$ is not a natural time derivative since n^μ is not the dual of Ω_μ:

$$n^\mu \Omega_\nu = -\alpha g^{\mu\nu} \nabla_\mu t \nabla_\nu t = \frac{1}{\alpha} \neq 1 \quad \forall \alpha \neq 1$$

EVOLUTION EQUATIONS

- Instead, let us consider the following vector:

$$t^\mu = \alpha n^\mu + \beta^\mu$$

  - It is the dual to Ω_μ for any purely spatial vector β^μ - the shift vector.

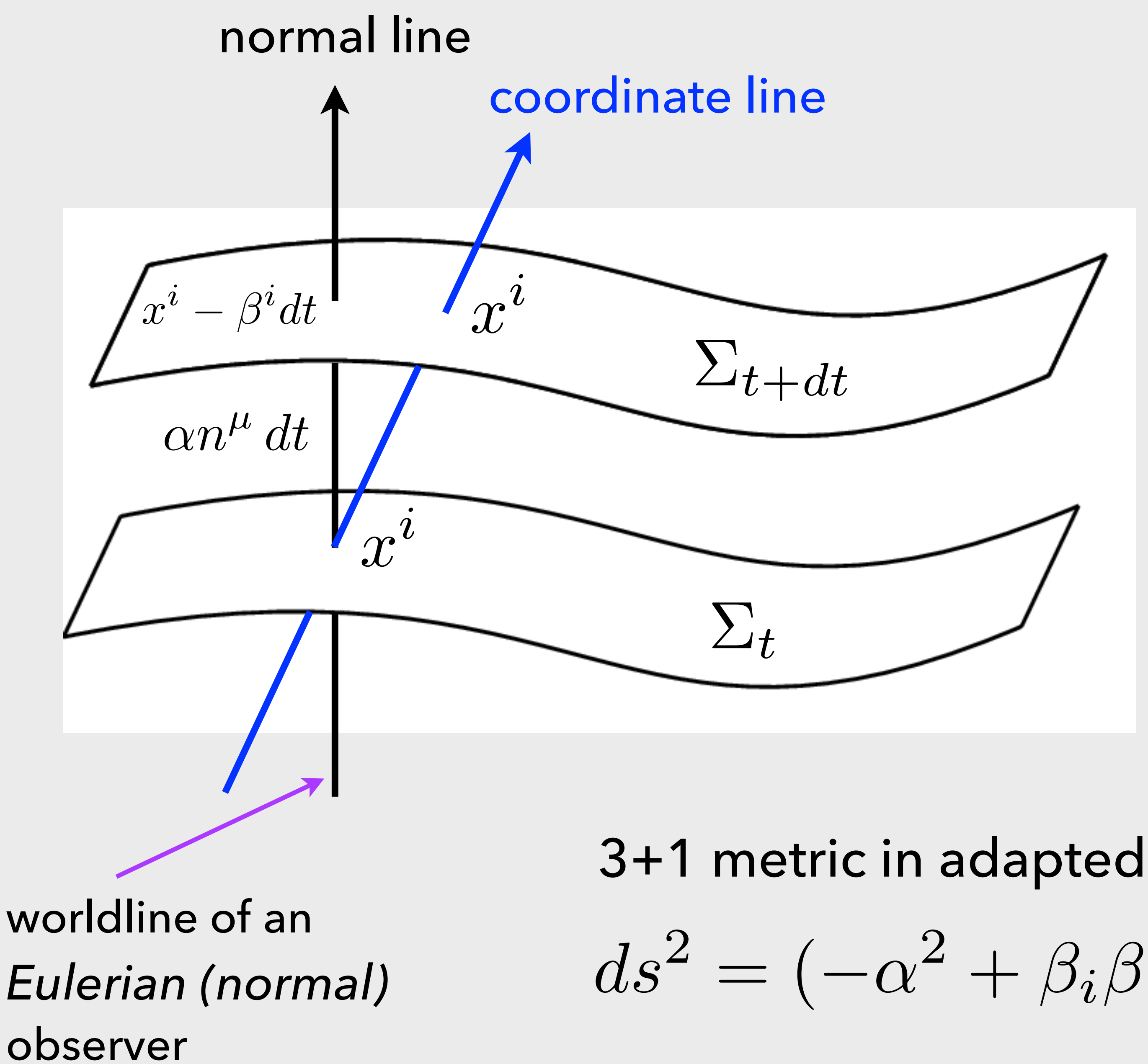
$$t^\mu \Omega_\mu = \alpha n^\mu \Omega_\mu + \beta^\mu \Omega_\mu = 1$$

(where we used that Ω_μ is timelike)

- t^μ provides a natural congruence along which we can propagate the spatial coordinates from one slice to the next. (α, β^μ) are arbitrary and encode how coordinates evolve in time.

FOLIATION ADAPTED COORDINATES

- Let us now consider coordinates adapted to this 3+1 split:



$$t^\mu = (1,0,0,0)$$
$$\beta^\mu = (0,\beta^i)$$
$$\Rightarrow n^\mu = \left(\frac{1}{\alpha}, -\frac{1}{\alpha}\beta^i\right)$$
$$n_\mu = (-\alpha,0,0,0)$$

$$\gamma_{ij} = g_{ij}$$

3+1 metric in adapted coordinates:

$$ds^2 = (-\alpha^2 + \beta_i\beta^i)dt^2 + 2\beta_i dt dx^i + \gamma_{ij} dx^i dx^j$$

ADM EQUATIONS (IN YORK FORM)

- The entire content of the 3+1 decomposed EFE is contained the spatial components. Hence, our final set of equations reads as follows:
- **Constraint equations:**

$$
\begin{aligned}
{}^{(3)}R + K^2 - K_{ij}K^{ij} &= 16\pi\rho \\
D_j(K^{ij} - \gamma^{ij}K) &= 8\pi S^i
\end{aligned}
$$

- **Evolution equations:**

$$
\begin{aligned}
\partial_t \gamma_{ij} &= -2\alpha K_{ij} + D_i\beta_j + D_j\beta_i \\
\partial_t K_{ij} &= -D_iD_j\alpha + \alpha\Big(R_{ij} + KK_{ij} - 2K_{ik}K^k_j + 4\pi[(S-\rho)\gamma_{ij} - 2S_{ij}]\Big) \\
&\quad + \beta^k D_k K_{ij} + K_{ik}D_j\beta^k + K_{kj}D_i\beta^k
\end{aligned}
$$

- Note: The 3+1 evolution equations are non-unique. We can always add arbitrary multiples of the constraints.

REMARK: WELL-POSEDNESS

- Well-posedness of the system of evolution equations is essential for stable numerical evolution, but it is not enough.
- Original ADM equations are *mathematically* not well-posed. The formulation after York is well-posed.
    - BUT: The ADM equations are not numerically robust (weak hyperbolicity)
- Due to the non-uniqueness of the evolution equations, we can derive "new" evolution equations, that are mathematically well-posed AND numerically robust.
    - **Baumgarte-Shapiro-Shibata-Nakamura (BSSN) formulation**
    - Conformal rescaling of the spatial metric + auxiliary variable Γ^i
    - Obtain a *strongly hyperbolic system of evolution equations*

OTHER FORMULATIONS

- There are alternative approaches to solving the EFE:
  - Characteristic formalism: 2+2 formulation, where ingoing (outgoing) null hypersurfaces emanate from a timelike world tube.
  - Conformal formalism
  - Evolving the full 4D spacetime (generalised harmonic formulation)

INITIAL DATA & GAUGE CONDITIONS

INITIAL DATA

- The spatial metric, the extrinsic curvature and any matter fields have to satisfy the constraint equations on every hypersurface Σ

$$^{(3)}R + K^2 - K_{ij}K^{ij} = 16\pi\rho$$
$$D_j(K^{ij} - \gamma^{ij}K) = 8\pi S^i$$

- We first need to specify (γ_{ij}, K_{ij}) on some initial slice:
    - 12 independent components but only <u>4 constraint equations</u>
    - 4 components are related to coordinate choices ("gauge")
    - 4 components represent the <u>dynamical DOF</u> (transverse parts)
    - Constraint equations only constrain the "longitudinal parts"
    - Seek split between the constrained and unconstrained components of the field (no natural way!)

CONFORMAL TRANSVERSE-TRACELESS (CTT) DECOMPOSITION

- Consider a conformal transformation of the spatial metric:

$$\gamma_{ij} = \psi^4 \bar{\gamma}_{ij}$$

- The Hamiltonian constraint yields:

$$8\bar{D}^2\psi - \psi\bar{R} - \psi^5 K^2 + \psi^5 K_{ij}K^{ij} = -16\pi\psi^5\rho$$

- Consider now a conformal transformation of the extrinsic curvature:

$$K_{ij} = A_{ij} + \frac{1}{3}\gamma_{ij}K$$

Ansatz: $A^{ij} = \psi^\alpha \bar{A}^{ij}$

$$K = \psi^\beta \bar{K}$$

Divergence:

$D_j A^{ij} \Rightarrow \alpha = -10$

Simplifying choice: $\beta = 0$

Conformal constraint equations:

$$8\bar{D}^2\psi - \psi\bar{R} - \frac{2}{3}\psi^5 K^2 + \psi^{-7}\bar{A}_{ij}\bar{A}^{ij} = -16\pi\psi^5\rho$$

$$\bar{D}_j\bar{A}^{ij} - \frac{2}{3}\psi^6\bar{\gamma}^{ij}\bar{D}_jK = 8\pi\psi^{10}S^i$$

CONFORMAL TRANSVERSE-TRACELESS (CTT) DECOMPOSITION

- Any symmetric traceless tensor can be split into a divergence-free transverse-traceless (TT) and a longitudinal (L) part:

$$\bar{A}^{ij} = \bar{A}^{ij}_{TT} + \bar{A}^{ij}_{L}$$

$$\bar{D}_j \bar{A}^{ij}_{TT} = 0$$

$$\bar{A}^{ij}_{L} = \bar{D}^i W^j + \bar{D}^j W^i - \frac{2}{3}\bar{\gamma}^{ij}\bar{D}_k W^k \equiv (\bar{L}W)^{ij}$$

Momentum constraint becomes: $\Rightarrow (\bar{\Delta}_L W)^i - \frac{2}{3}\psi^6 \bar{\gamma}^{ij}\bar{D}_j K = 8\pi\psi^{10} S^i$

- We see that we can freely choose: $\bar{\gamma}_{ij}, K, \bar{A}^{ij}_{TT}$
- Solve for the vector potential W^i and the conformal factor ψ
- Then construct the physical solutions for γ_{ij}, K_{ij}
- Choices for background fields $\bar{\gamma}_{ij}, K, \bar{A}^{ij}_{TT}$ need to be motivated

EXAMPLE: BOWEN-YORK INITIAL DATA

- Example: vacuum case with K=0 ("maximal slicing")

$$\Rightarrow (\bar{\Delta}_L W)^i = 0$$

- Let's also assume conformal flatness, i.e. $\bar{\gamma}_{ij} = \eta_{ij}$, then the vector Laplacian simplifies to:

$$\partial^j \partial_j W^i + \frac{1}{3} \partial^i \partial_j W^j = 0$$

- Solutions to this equation are called Bowen-York solutions
- Well known solutions include a spinning BH and a boosted BH
- A very common type of BH initial data used in particular in the "moving punctures" framework

CONFORMAL THIN-SANDWICH (CTS) DECOMPOSITION

- If we want to have (quasi-)equilibrium solutions, we require initial data that have a certain time evolution
- CTS: Instead of providing data on one slice Σ, data are provided on two slices with infinitesimal separation
- Define:

$$u_{ij} \equiv \gamma^{1/3}\partial_t(\gamma^{-1/3}\gamma_{ij})$$

- The time evolution for γ_{ij} then reads as:

$$u^{ij} = -2\alpha A^{ij} + (L\beta)^{ij}$$

- We also define:

$$\bar{u}_{ij} \equiv \partial_t \bar{\gamma}_{ij}$$
$$\bar{\gamma}^{ij}\bar{u}_{ij} \equiv 0$$

CONFORMAL THIN-SANDWICH (CTS) DECOMPOSITION

- Applying the conformation transformation, we find:

$$\bar{A}_{ij} = \frac{1}{2\bar{\alpha}}\left((\bar{L}\beta)^{ij} - \bar{u}^{ij}\right) \qquad \text{where} \quad \bar{\alpha} = \psi^6 \alpha$$

- The momentum constraint then becomes an equation for the shift vector β^i:

$$(\bar{\Delta}_L \beta)^i - (\bar{L}\beta)^{ij} \bar{D}_j \ln(\bar{\alpha}) = \bar{\alpha} \bar{D}_j (\bar{\alpha}^{-1} \bar{u}^{ij}) + \frac{3}{4} \bar{\alpha} \psi^6 \bar{D}^i K + 16\pi \bar{\alpha} \psi^{10} S^i$$

- After solving the Hamilton constraint for ψ, the physical solutions are constructed from:

$$\gamma_{ij} = \psi^4 \bar{\gamma}_{ij}, \quad K_{ij} = \psi^{-2} \bar{A}_{ij} + \frac{1}{3} \gamma_{ij} K, \quad \alpha = \psi^6 \bar{\alpha}$$

- The freely specifiable variables are $\bar{\gamma}_{ij}, \bar{u}_{ij}, K, \bar{\alpha}$
- Note: The lapse & shift did not appear in the CTT initial data construction

GAUGE CONDITIONS

- 4 gauge functions: lapse α and shift vector β^i
  - need to impose coordinate conditions
- Example: Schwarzschild

$$ds^2 = -\left(1 - \frac{r_S}{r}\right)dt^2 + \left(1 - \frac{r_S}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2)$$

- Finding “good” gauge conditions is not trivial but geometric insight combined with numerical experimentation (trial and error)
- Some desired features, especially for BH spacetimes are:
  - Horizon penetrating coordinates
  - Singularity-avoiding gauge conditions
  - Minimal distortion