UNIVERSITY OF BIRMINGHAM | GRAVITATIONAL WAVE ASTRONOMY

(A VERY BRIEF) INTRODUCTION TO NUMERICAL RELATIVITY

MPAGS: BLACK HOLES AND GRAVITATIONAL WAVES - PART 3

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NOVEMBER 2020

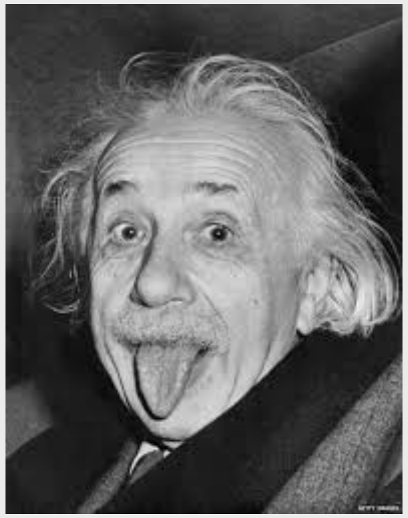
LOGISTICS

- 3 lectures: **12/11, 19/11, 26/11**
- Zoom: https://bham-ac-uk.zoom.us/j/88000872524?pwd=cUR0OURkcWI3UVptSU0yQmZ3MnpPdz09
- Recommended literature:
  - Miguel Alcubierre, “Introduction to 3+1 Numerical Relativity”
  - Thomas Baumgarte, Stuart Shapiro, “Numerical Relativity”
  - Eric Gourgoulhon, “3+1 Formalism and the Bases of Numerical Relativity” (https://arxiv.org/pdf/gr-qc/0703035.pdf)
- Assessed problem sheet

3+1 NUMERICAL RELATIVITY

WHY NUMERICAL RELATIVITY (NR)?

- Note: We use geometric units, i.e. G = c = 1!
- Covariant formulation of General Relativity
  - Einstein field equations:

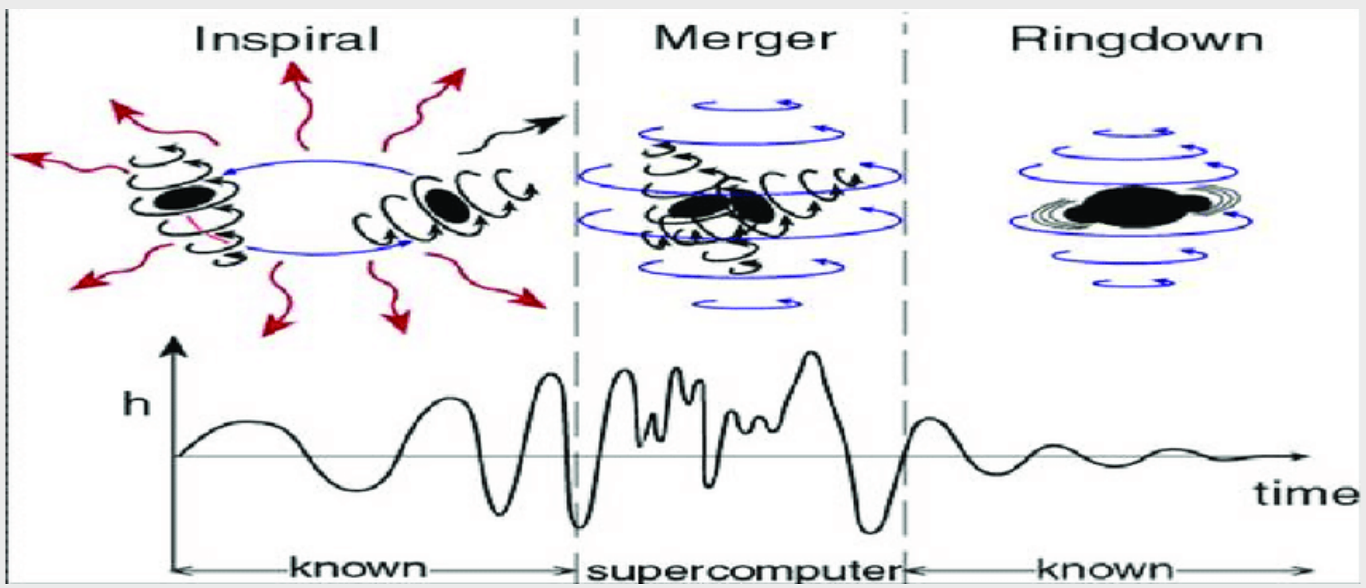


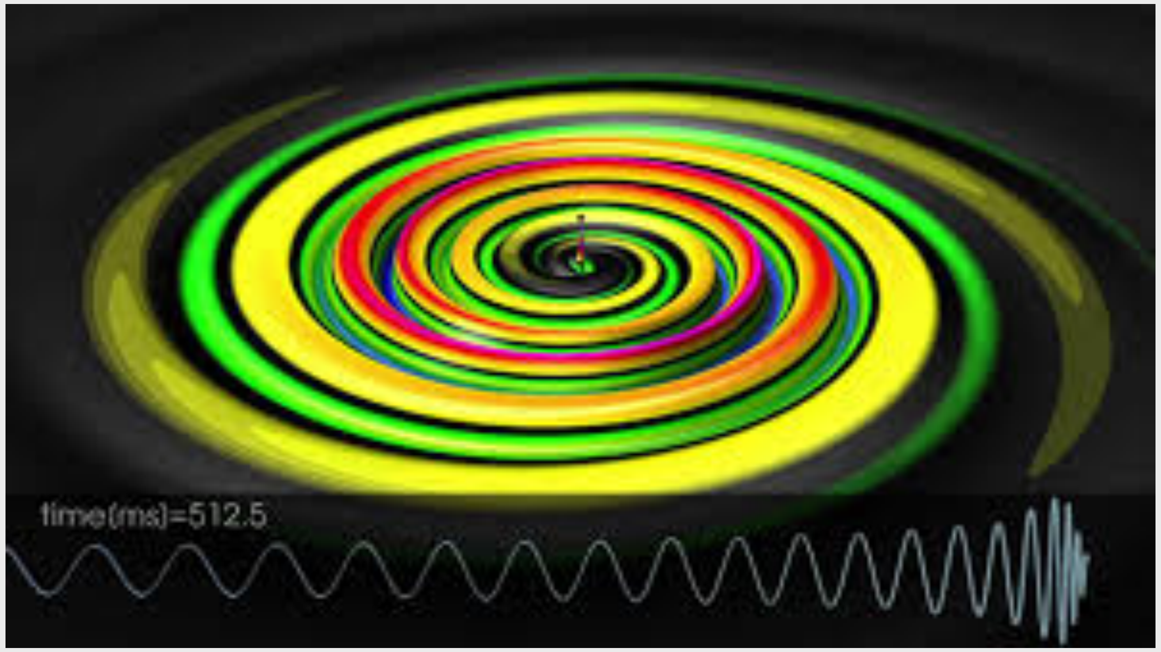
$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi T_{\mu\nu}$$

  - 10 non-linear second-order partial differential equations for the metric tensor $g_{\mu\nu}$
  - Analytic solutions exist only for special cases
    - Examples: Schwarzschild, Kerr, Tolman-Oppenheimer-Volkhov (TOV)
  - No known analytic solutions for more general spacetimes

WHY NUMERICAL RELATIVITY (NR)?

- Examples include:
  - Relativistic two-body problem & gravitational waves ("holy grail")





Credit: K. Thorne & UIB

  - Supernovae explosions
  - Perturbations of isolated stars/BHs
  - Cosmological simulations beyond Newton gravity
  - Critical collapse phenomena (e.g. primordial BH formation)
- … and of course most things beyond GR (not discussed here)

NR INGREDIENTS LIST (FOR BINARY BLACK HOLES)

Reformulate the Einstein field equations (EFE) as initial value problem (IVP)

Prove existence of a well-posed initial value problem

Numerically suitable reformulation of the EFE

"good" coordinates (gauge choices)

Initial data

Deal with singularities

"find" the black hole horizons

Extract gravitational waves

1952 Choquet-Bruhat
IVP

1962 ADM
formulation

1964 Hahn-Linquist
2 wormholes

1984 Unruh
excision

1992,3 Choptiuk
Abrahams, Evans
critical phenomena

1997
Brandt&Brügmann
puncture data

1994-98
BBH grand challenge

1999-00
AEI/PSU
Grazing Collisions

2000-04
AEI/UTB/NASA
Lazarus

~2000 Choptuik,
Brügmann, Schnetter
mesh refinement

2005 Gundlach+
constraint damping

2005 Pretorius
IMR w harmonic

2005-06 Campanelli+,
Baker+; IMR w BSSN &
moving punctures

2006-08 SXS
IMR w spectral

2011 Lousto+
q=100

2011 Lovelace+
a=0.97

2015 Szilagyi+
175 orbits

2014-17 Ruchlin+
high spin puncture ID

1950s — 2005 — 2015+

1975-77 Smarr-Eppley
head on collisions

1979 York
Kinematics &
dynamics of GR

1989-95
Bona-Masso
Modified ADM

1994 Cook
Bowen-York ID

94-95 NCSA/WSU
improved head-on

1999-2005
JW York,
Cornell, Caltech, LSU
hyperbolic formulations

1999
BSSN

1999 York
CTS ID

2000-02
Alcubierre
gauge conditions

2004
Brügmann+
one orbit

2003-08
Cook, Pfeiffer ea
improved ID

2000 Ashtekar
isolated horizons

2006,07 Baker+;
Gonzalez+
non-spinning BBH
kicks

2007-11
RIT; Jena; AEI;...
BBH superkicks

2008
NINJA

2009-11
Bishop+
CCE

2010
Bernuzzi+
C4z

Adapted from slides by C. Lousto & H. Pfeiffer

3+1 FORMALISM

- Goal: Solve for the dynamical evolution of the gravitational field in time
  - Initial value problem (Cauchy problem) formulation of GR

    Given a set of adequate initial (and boundary) conditions, the fundamental equations must predict the future (past) evolution of the system.

  - To rewrite the EFE as an IVP, we first separate spacetime into space *and* time (3+1 formalism)

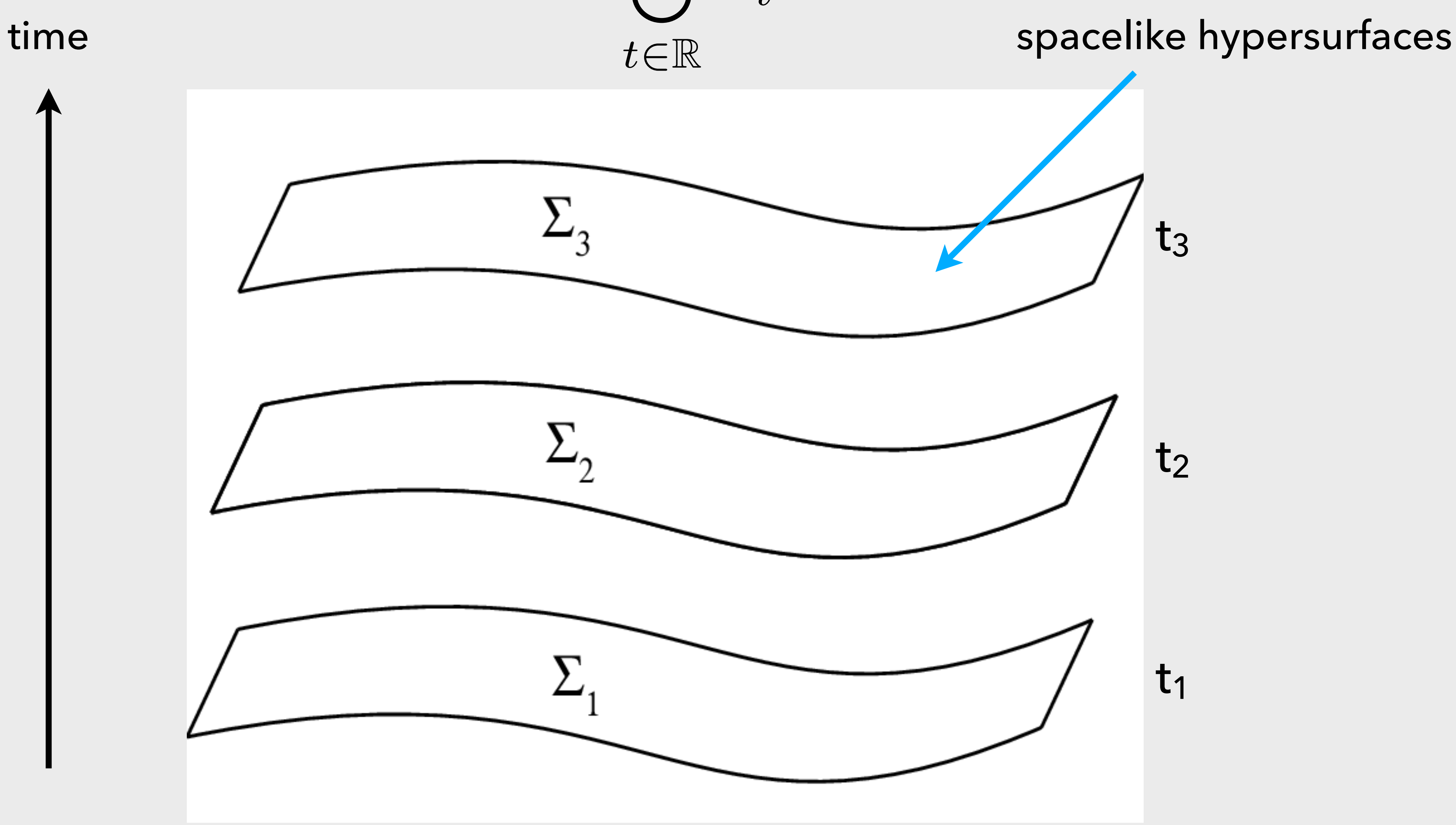
- Assumption: The considered spacetime (M, g) is *globally hyperbolic*, i.e. the spacetime admits a Cauchy surface.

- Definition: A Cauchy surface is a spacelike hypersurface in M such that each causal curve intersects it once and only once.

3+1 FORMALISM

- Any globally hyperbolic spacetime can be completely foliated by space-like hypersurfaces (foliation, slicing). The foliation can be identified as the level sets of a smooth, regular scalar function (e.g. time).

$$\mathcal{M} = \bigcup_{t\in\mathbb{R}} \Sigma_t$$



SPACELIKE HYPERSURFACES

- We define spacelike hypersurfaces Σ_t as the level set of the scalar field t on M. From t, we define the (unnormalised) 1-form

$$\Omega_\mu = \nabla_\mu t$$

- From the 4-metric, we can compute its norm:

$$||\Omega||^2 = g^{\mu\nu}\nabla_\mu t \nabla_v t \equiv -\frac{1}{\alpha^2}$$

    - We assume $\alpha > 0$, then Σ_t is space like and Ω_μ is timelike.

- The unit normal vector to Σ_t is then given by:

$$n^\mu = -\alpha g^{\mu\nu}\Omega_\nu$$

- α denotes the lapse and n^μ can be thought of as the 4-velocity of a normal (Eulerian) observer
- Note: Since Σ_t is spacelike, $g_{\mu\nu}n^\mu n^\nu = -1$. We use the convention $\mathrm{sign}(g_{\mu\nu}) = (-+++)$.

SPACELIKE HYPERSURFACES

- With the definition of a hypersurface normal, we can now construct the spatial metric on the hypersurface:

$$\mathcal{T}_p(M) = \mathcal{T}_p(\Sigma) \oplus \mathrm{span}(n) \quad \forall p \in \Sigma$$

$$\gamma : \mathcal{T}_p(M) \to \mathcal{T}_p(\Sigma)$$

$$\gamma_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu$$

- To "break up" 4D objects into their components parallel and orthogonal to the hypersurface, we require a projection operator

$$\gamma^\alpha{}_\beta = \delta^\alpha{}_\beta + n^\alpha n_\beta$$

- The projection operator allows us to construct purely spatial objects. Together with n^μ we have the tools to related 4D objects in M to 3D objects on Σ_t.

SPACELIKE HYPERSURFACES

- 3D covariant derivative: Let γ_{ij} be the induced non-degenerate metric on Σ_t. Then there exists a unique connection (covariant derivative) D:

$$D_\mu f \equiv \gamma_\mu^\nu \nabla_\nu f$$

- The Riemann tensor associated with this connection, defines the intrinsic curvature of the hypersurface:

$$(D_\mu D_\nu - D_\nu D_\mu) v^\sigma = {}^{(3)}R^\sigma{}_{\mu\nu\rho} v^\rho \quad \forall v \in \mathcal{T}(\Sigma)$$

- The intrinsic (scalar) curvature of the hypersurface (also called Gaussian curvature) is

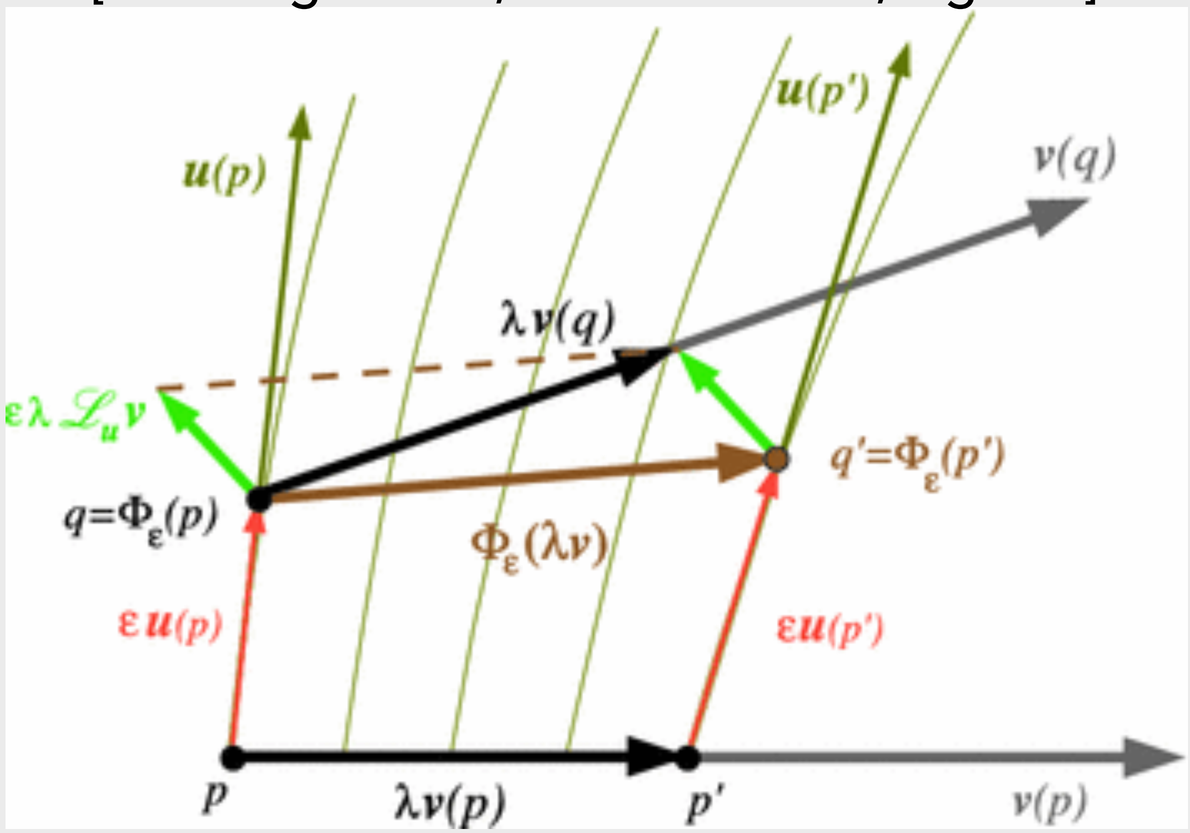
$${}^{(3)}R = \gamma^{\mu\nu}\,{}^{(3)}R_{\mu\nu} = \gamma^{\mu\nu}\,{}^{(3)}R^\sigma{}_{\mu\sigma\nu}$$

LIE DERIVATIVE

- Consider two vector fields u,v on (M, g). What is the variation of the vector field v at the point q in M when transported along the vector field u?

$$\mathcal{L}_u v := \lim_{\epsilon \to 0} \frac{1}{\epsilon}[v(q) - \phi_\epsilon v(p)]$$

[E. Gourgoulhon, 3+1 formalism, Fig. 2.3]

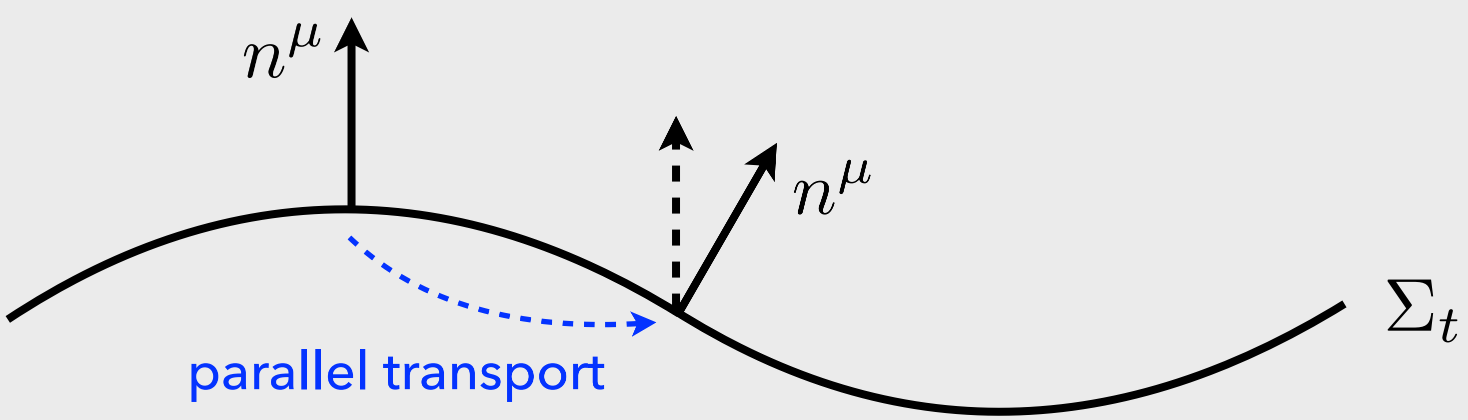


$$\mathcal{L}_u v^\alpha = [u, v]^\alpha = u^\mu \nabla_\mu v^\alpha - v^\mu \nabla_\mu u^\alpha$$

- Note: This definition can be generalised to any tensor field on (M, g)

SPACELIKE HYPERSURFACES

- Extrinsic curvature: Describes the "bending" of (3D) Σ_t in (4D) M, and is measured by the change of direction of n as one moves along the hypersurface via parallel transport.



- The extrinsic curvature tensor (second fundamental form) is defined as:

$$K_{\mu\nu} := -\gamma^{\alpha}{}_{\mu}\gamma^{\beta}{}_{\nu}\nabla_\alpha n_\beta = -(\nabla_\mu n_\nu + n_\mu n^\alpha \nabla_\alpha n_\nu)$$

$$\equiv -\frac{1}{2}\mathcal{L}_n \gamma_{\mu\nu}$$

Geometric generalisation of the "time derivative" the the spatial metric!

- $(\gamma_{\mu\nu}, K_{\mu\nu})$ are the fundamental variables in the 3+1 formulation.

GAUSS, CODAZZI & RICCI

- $\gamma_{\mu\nu}$ and $K_{\mu\nu}$ cannot be chosen arbitrarily - they are related to our 4D manifold (M, g). In particular, we need to relate the 3D and the 4D Riemann tensor. Obtained via contractions with n^μ and $\gamma^\mu{}_\nu$.
- Gauss' equation:

$$\gamma^\alpha_\mu \gamma^\beta_\nu \gamma^\gamma_\sigma \gamma^\delta_\rho \, {}^{(4)}R_{\alpha\beta\gamma\delta} = {}^{(3)}R_{\mu\nu\sigma\rho} + K_{\mu\sigma}K_{\nu\rho} - K_{\mu\rho}K_{\nu\sigma}$$

- Codazzi equation:

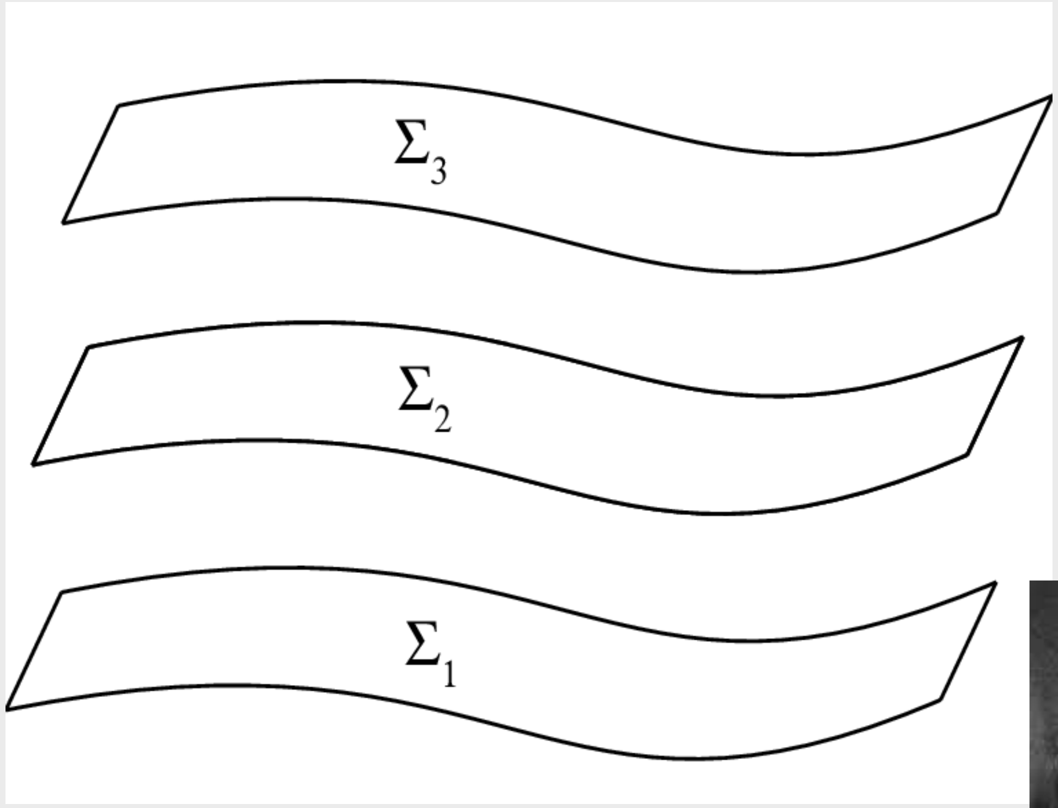
$$\gamma^\alpha_\mu \gamma^\beta_\nu \gamma^\gamma_\sigma n^\delta \, {}^{(4)}R_{\alpha\beta\gamma\delta} = D_\nu K_{\mu\sigma} - D_\mu K_{\nu\sigma}$$

- Ricci's equation:

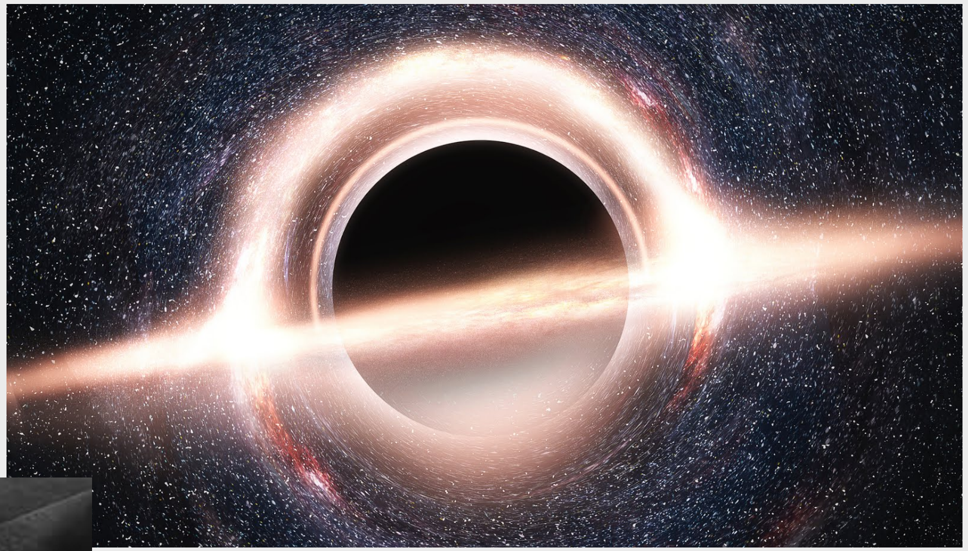
$$n^\alpha n^\gamma \gamma^\beta_\mu \gamma^\delta_\nu \, {}^{(4)}R_{\alpha\beta\gamma\delta} = \mathcal{L}_n K_{\mu\nu} + \frac{1}{\alpha} D_\mu D_\nu \alpha + K^\rho_\nu K_{\mu\rho}$$

THE EINSTEIN CONSTRAINTS

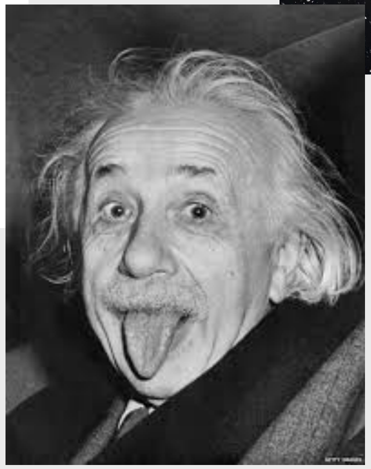
- So far we have only considered the *kinematics of hypersurfaces*. The true gravitational DOF are contained in the Einstein field equations. We now need to link our geometric objects to the physics.



Geometry



Gravity



THE EINSTEIN CONSTRAINTS

- Contracting the Gauss equation twice, we find:

$$\boxed{{}^{(3)}R + K^2 - K_{\alpha\beta}K^{\alpha\beta} = 16\pi\rho}$$

where $\rho \equiv n_\mu n_\nu T^{\mu\nu}$ is the total energy density measured by a normal observer. This equation is commonly referred to as the Hamiltonian constraint.