UNIVERSITY OF BIRMINGHAM | GRAVITATIONAL WAVE ASTRONOMY

(A VERY BRIEF) INTRODUCTION TO NUMERICAL RELATIVITY

MPAGS: BLACK HOLES AND GRAVITATIONAL WAVES - PART 3

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RECAP:

- Introduced foliation-adapted coordinates
- Constraint equations:

$$^{(3)}R + K^2 - K_{ij}K^{ij} = 16\pi\rho$$

$$D_j(K^{ij} - \gamma^{ij}K) = 8\pi S^i$$

- Evolution equations:

$$\partial_t \gamma_{ij} = -2\alpha K_{ij} + D_i\beta_j + D_j\beta_i$$

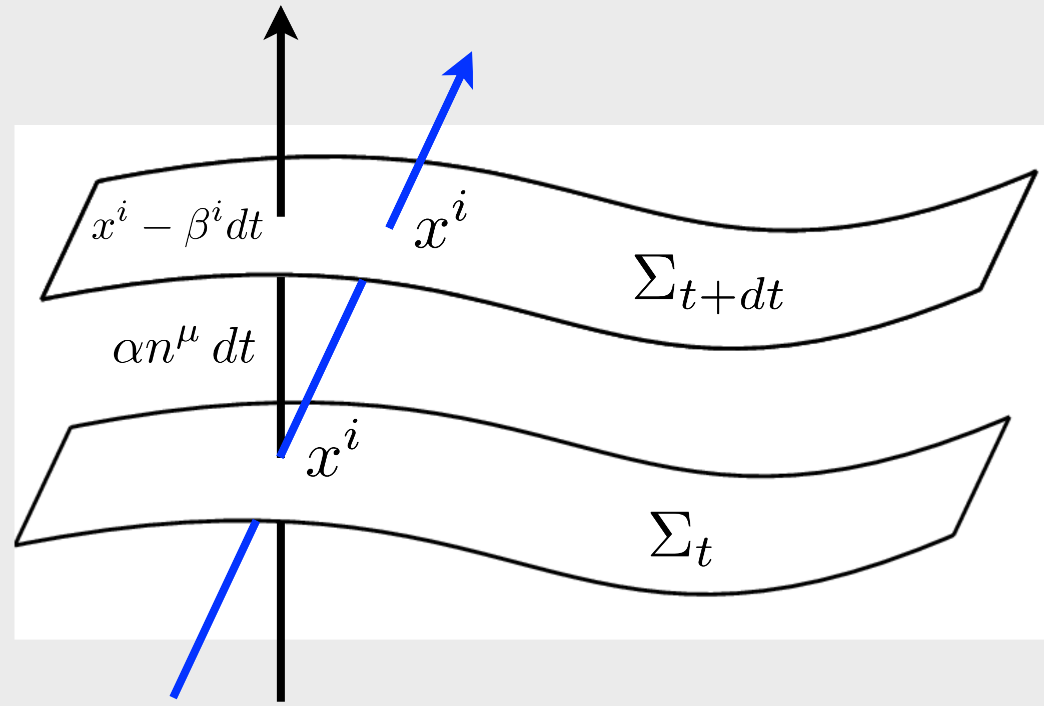
$$\partial_t K_{ij} = -D_iD_j\alpha + \alpha\Big(R_{ij} + KK_{ij} - 2K_{ik}K^k_j + 4\pi[(S-\rho)\gamma_{ij} - 2S_{ij}]\Big) + \beta^k D_k K_{ij} + K_{ik}D_j\beta^k + K_{kj}D_i\beta^k$$

- Initial data: conformal transverse-traceless vs conformal thin sandwich
- Gauge choices

GEODESIC SLICING

- Simplest possible choice:

$$\boxed{\alpha = 1, \quad \beta^i = 0}$$



- Coordinate observers coincide with normal observers
- Normal observers are freely falling ($a_\mu = 0$) and hence follow geodesics
- Unfortunately, coordinate singularities develop very quickly as geodesics focus near gravitating sources, which can be seen from the expansion:

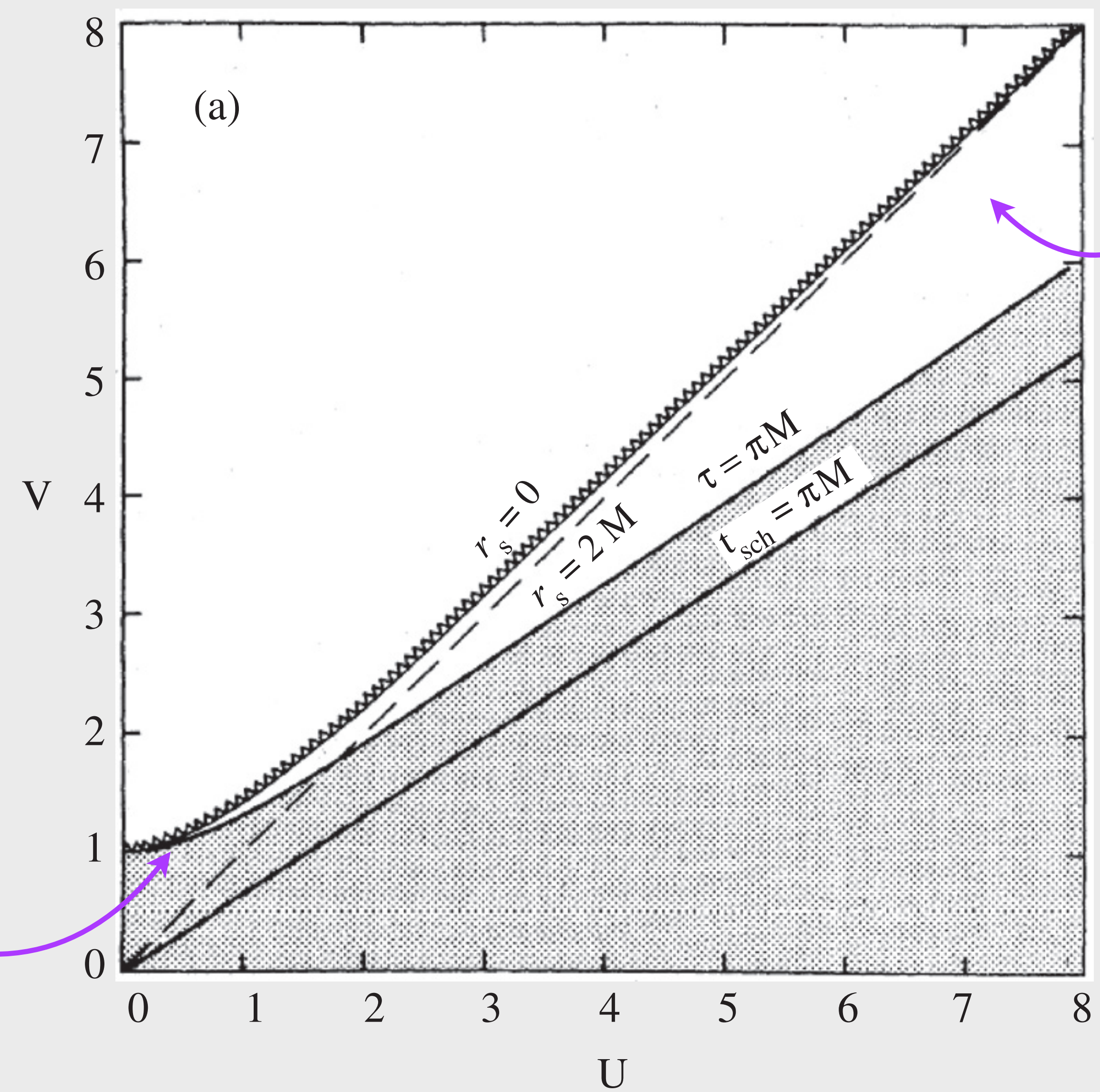
$$\nabla_\mu n^\mu = -K$$

K grows monotonically in time

$$\partial_t K \geq 0$$

GEODESIC SLICING

- Example: Schwarzschild spacetime (in Kruskal coordinates U,V)



Does not fully cover the BH exterior!

Slice "hits" the singularity at coordinate time t = πM

Taken from Baumgarte & Shapiro, Numerical Relativity

MAXIMAL SLICING

- To control the divergence of normal observers, we need to impose a suitable condition on K
- A common choice is "maximal" slicing (for all times),

$$K = 0 = \partial_t K$$

  - Maximal slicing is volume preserving along congruences of n^μ
- From the contraction of the evolution equation for K_{ij}, we find:

$$D^2\alpha = \alpha(K_{ij}K^{ij} + 4\pi(\rho - S))$$

- Or in the conformal decomposition:

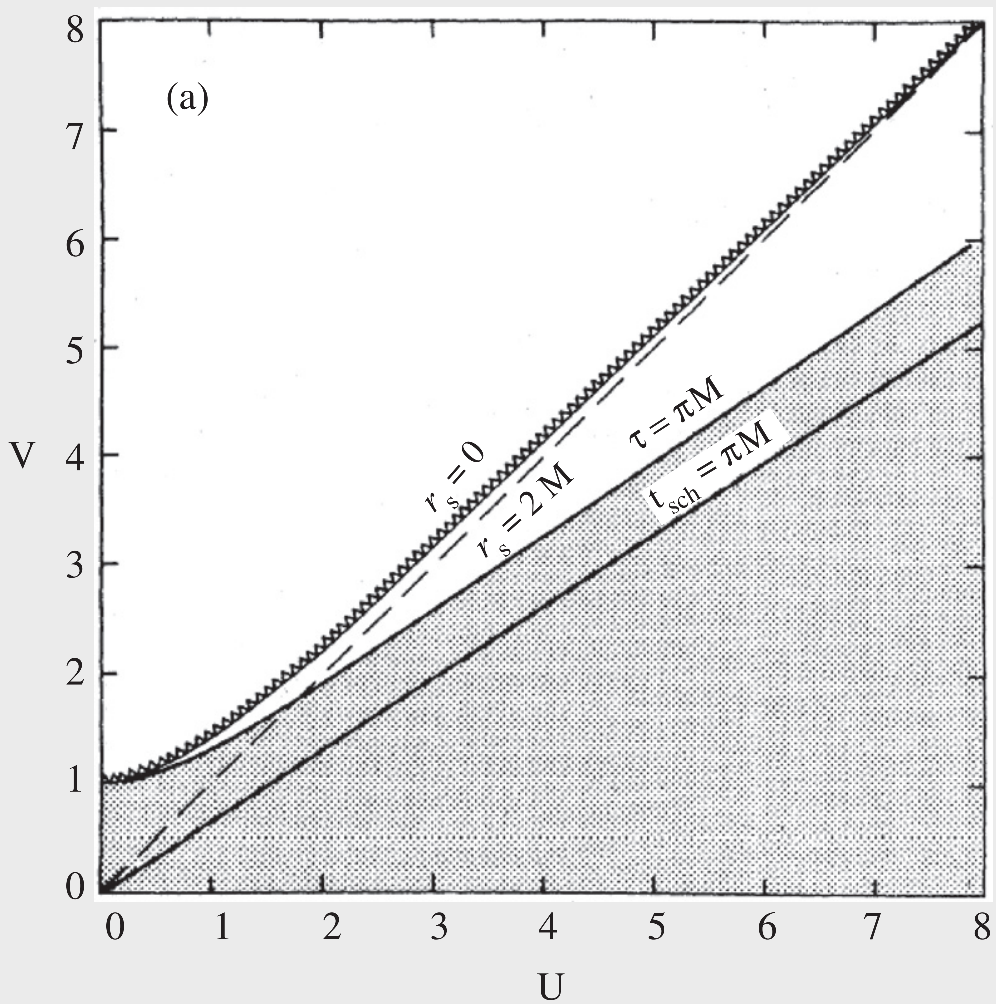
$$\bar{D}^2(\alpha\psi) = \alpha\psi\left(\frac{7}{8}\psi^{-8}\bar{A}_{ij}\bar{A}^{ij} + \frac{1}{8}\bar{R} + 2\pi\psi^4(\rho + 2S)\right)$$

- Note: Elliptic equations are costly. We can recast the equation into a parabolic one via "approximate" maximal slicing, i.e. $\partial_t K = -cK$

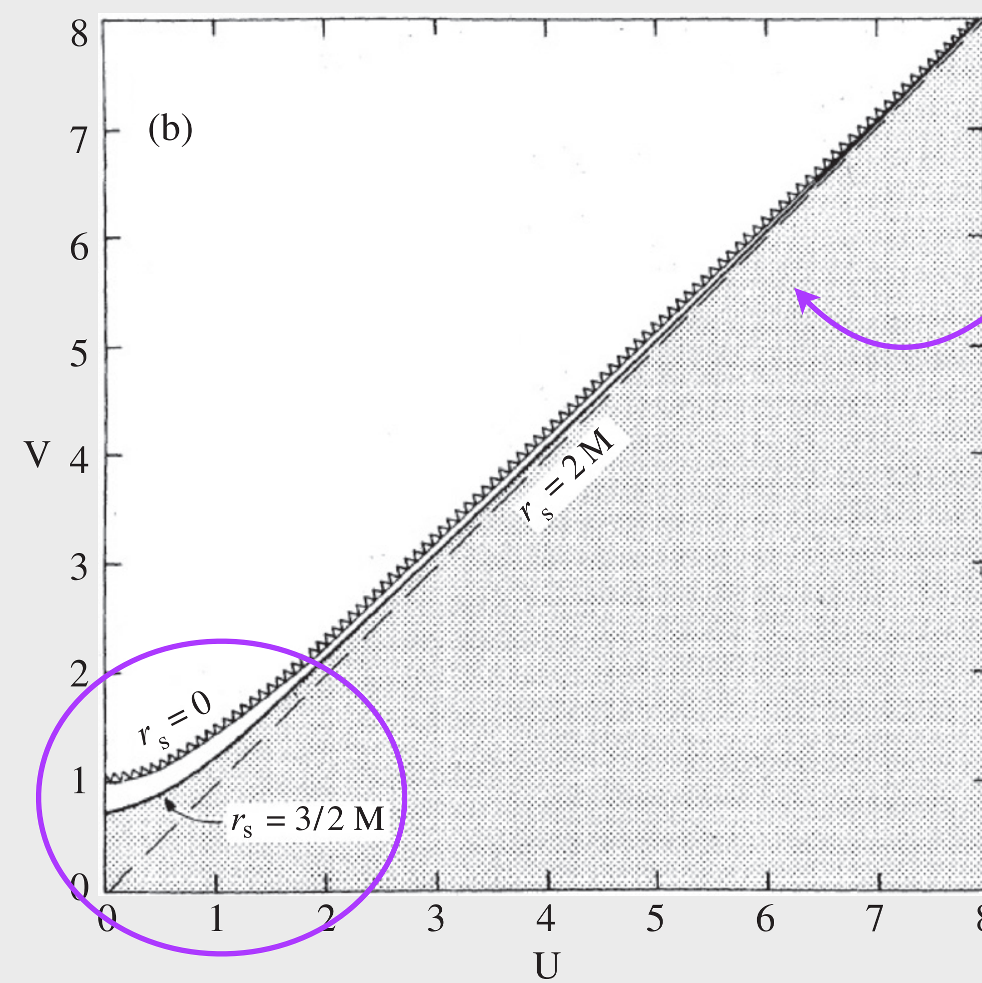
GEODESIC VS MAXIMAL SLICES

- Example: Schwarzschild spacetime (in Kruskal coordinates)

geodesic slicing



maximal slicing



Cover the BH exterior fully!

Maximal slices penetrate the BH interior but avoid the singularity asymptoting to the limiting surface at $r_S = 3M/2$.

Taken from Baumgarte & Shapiro, Numerical Relativity

SOME OTHER GAUGE CHOICES

- Minimal distortion: minimise the time rate of change of $\bar{\gamma}_{ij}$
- Harmonic coordinates: $^{(4)}\Gamma^{\mu} = 0$
- 1+log slicing: Popular (algebraic) slicing condition is

  $$\alpha = 1 + \ln\gamma \quad \text{(here } \beta^i = 0 \text{ is assumed)}$$

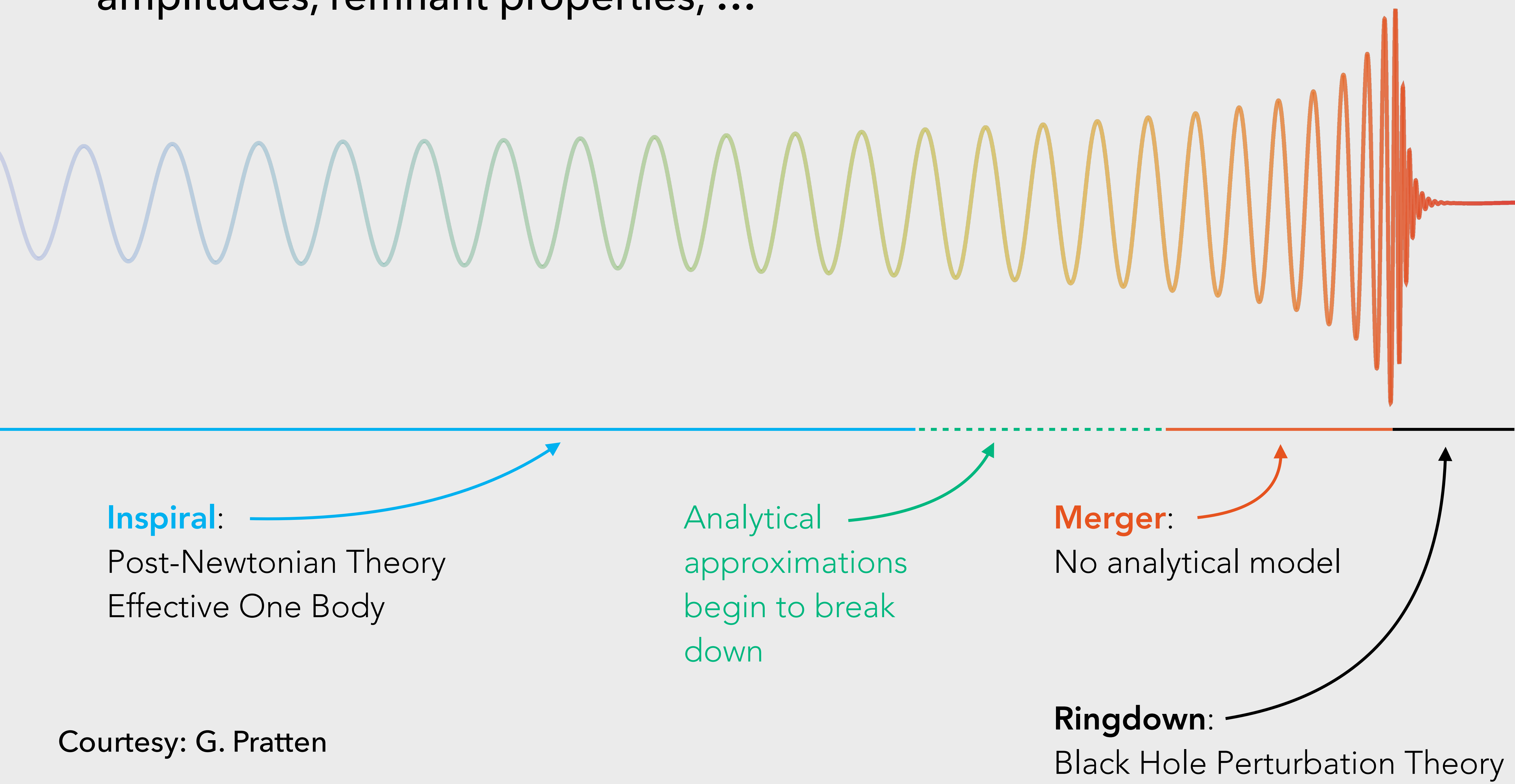
  - Or more general: $(\partial_t - \beta^j \partial_j)\alpha = -2\alpha K$
- (Hyperbolic) Gamma-driver shift condition:

$$\partial_t \beta^i = \frac{3}{4} B^i, \quad \partial_t B^i = \partial_t \bar{\Gamma}^i - \eta B^i$$

SIMULATING BINARY BLACK HOLES

WHY BINARY BLACK HOLES?

- GR in the strong-field regime, gravitational waves, ringdown amplitudes, remnant properties, …



Courtesy: G. Pratten

NR CODES

- About a dozen (proprietary) codes exist
  - Proprietary codes (not exhaustive): SpEC (Caltech-Cornell-AEI), BAM (Jena-Cardiff-Balearic Islands), LEAN (Cambridge), Maya (UTT), LazEv (RIT), …
  - Public code: Einstein Toolkit (https://einsteintoolkit.org/) - community supported by the NSF
  - Note: Many public codes use the framework of the Einstein Toolkit with customised BSSN drivers or alternative evolution schemes

- Some core differences across codes:
  - Formulation: BSSN vs. generalised harmonic
  - Handling of singularities: moving punctures vs. excision
  - Numerical methods: finite differencing vs. spectral methods

HOW TO HANDLE SINGULARITIES

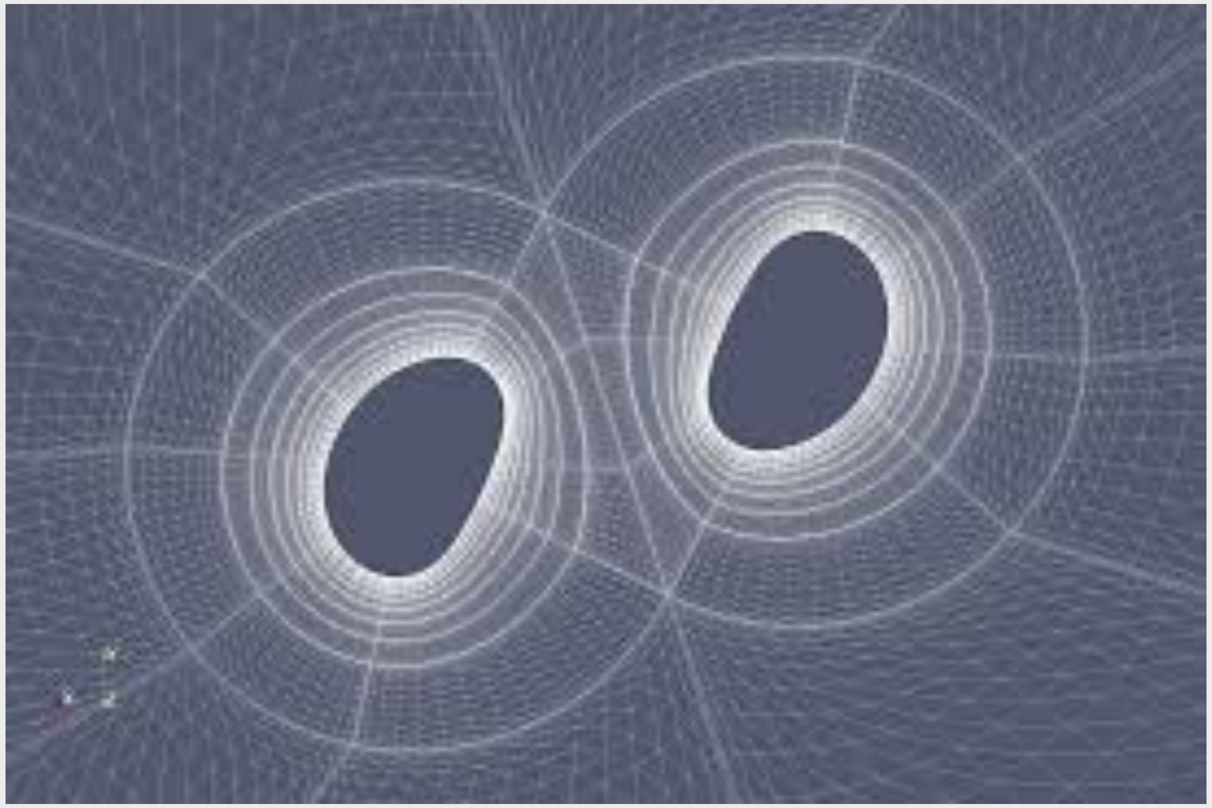
EXCISION

- The black hole horizon is a causal boundary. Anything inside the horizon is causally disconnected from the exterior region and hence cannot affect the physics there.
  - Singularity "hidden" inside the horizon
  - Remove the BH interior, and hence the physical (or coordinate) singularity from the computational domain
- 2 ingredients needed:
  1. A boundary inside the BH horizon such that the BH interior is removed
  2. A non-zero shift vector β^i that keeps the horizon at a roughly constant coordinate location during the evolution

BLACK HOLE EXCISION

- Problem: The BH horizon (event horizon) is a globally defined quantity and to determine it we require the entire future development of the spacetime
  - Proxy: apparent horizon (AH) with $r_{\mathrm{AH}} < r_{\mathrm{EH}}$ defined on every time slice
  - Definition:

An apparent horizon is the outermost 2-surface embedded in a spatial hypersurface Σ, whose outgoing future null geodesics have vanishing expansion everywhere.

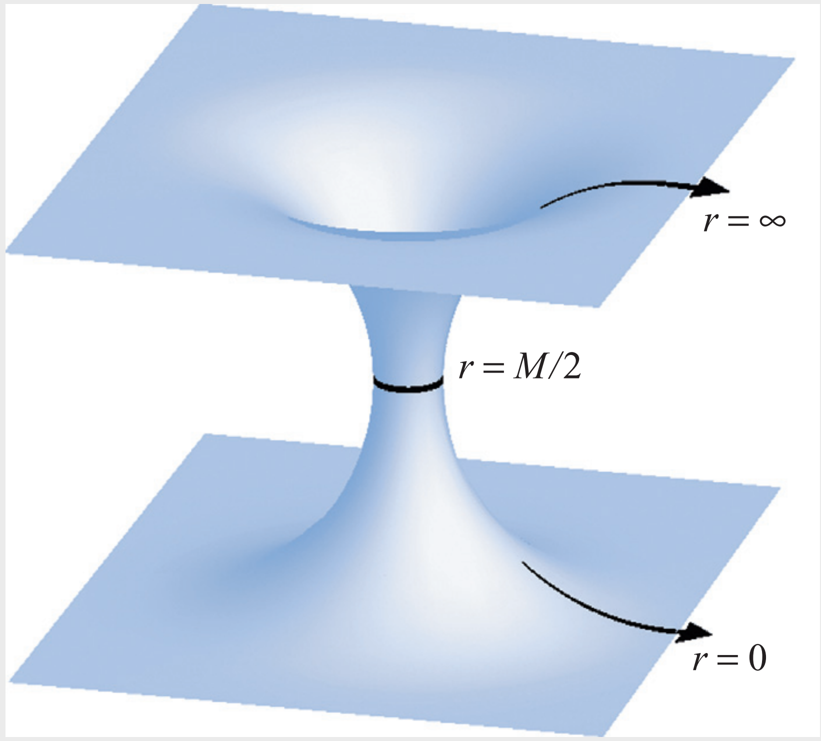


HOW TO HANDLE SINGULARITIES

MOVING PUNCTURES

- Alternatively, we can keep the (coordinate) singularities but make sure that the singularity never coincides with a grid point, where the calculations are performed. This is controlled via appropriate gauge conditions.
  - 1+log slicing & gamma-driver shift

$$\psi_{\mathrm{BL}} = 1 + \sum_{i=1}^{N} \frac{m_i}{2r_i}$$

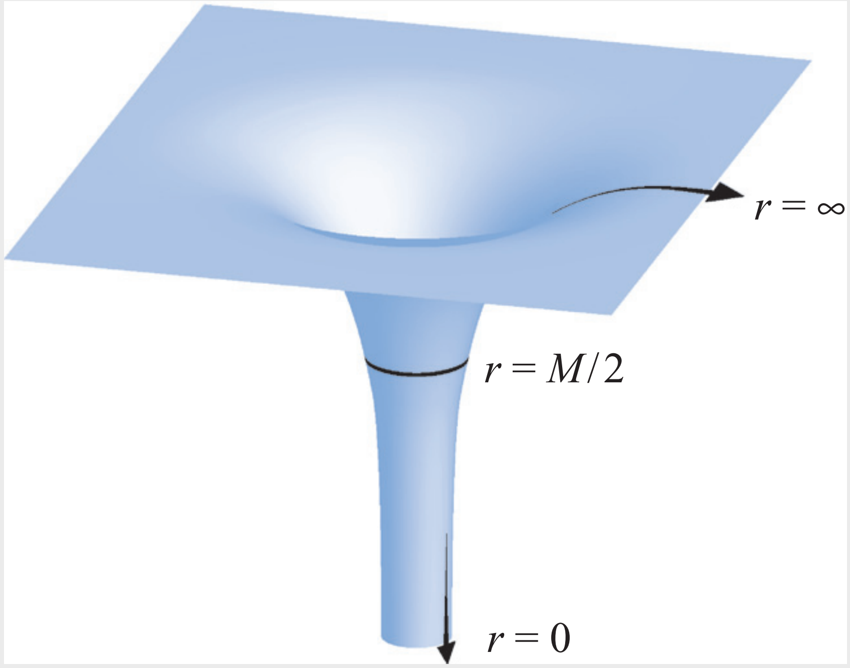


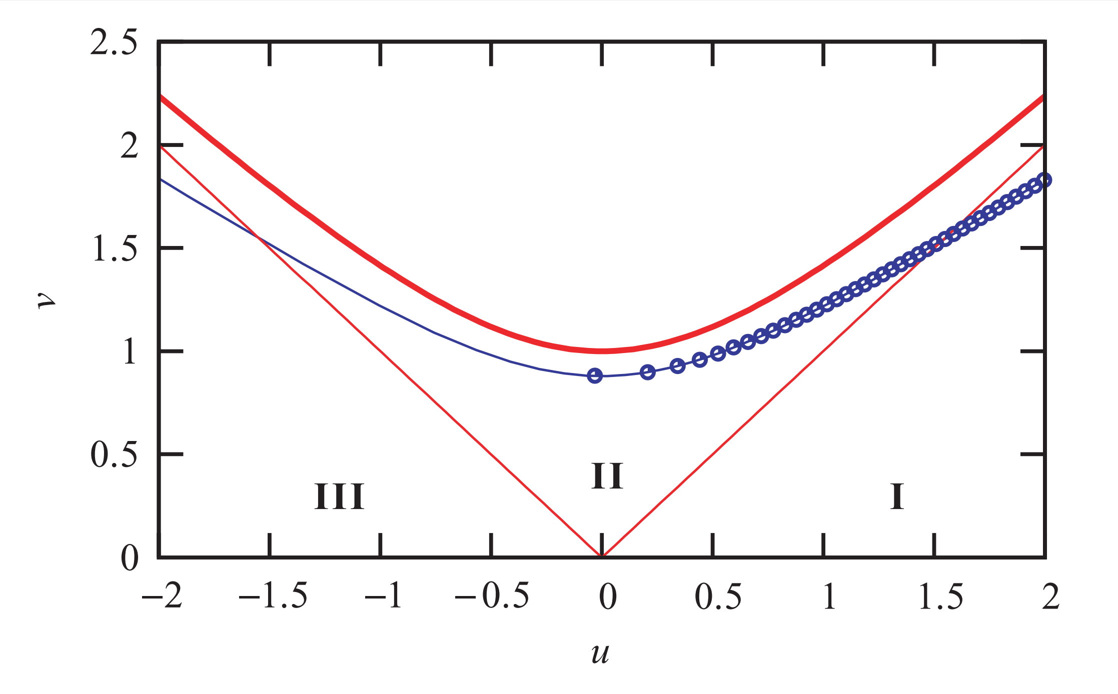
Singularity can be identified with spatial infinity (regular)

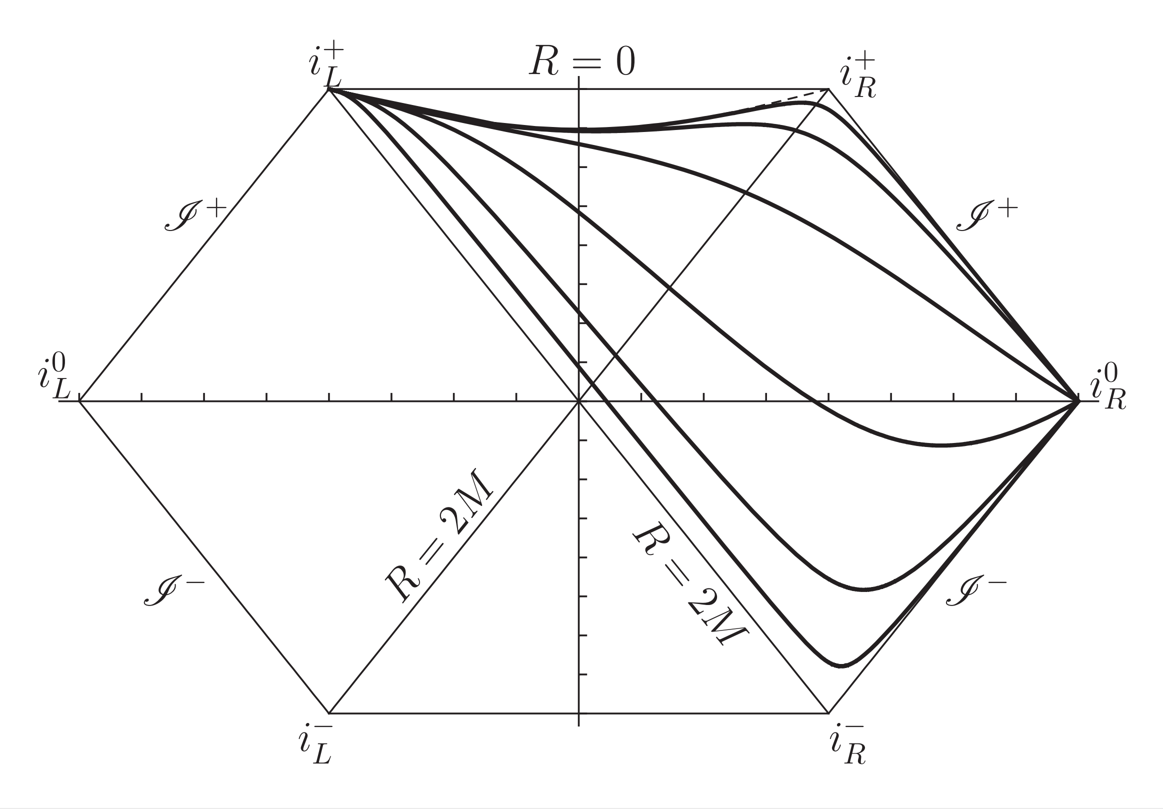
Topology of the initial slice is a wormhole connecting two asymptotically flat copies

MOVING PUNCTURES

- As we evolve the initial data with the 1+log slicing condition, the topology of the slices changes for a short time before the topology settles down.
  - The conformal factor now diverges at $1/\sqrt{r}$. Such a slice terminates on a finite limit surface at $r = 3M/2$ that does not reach the singularity.







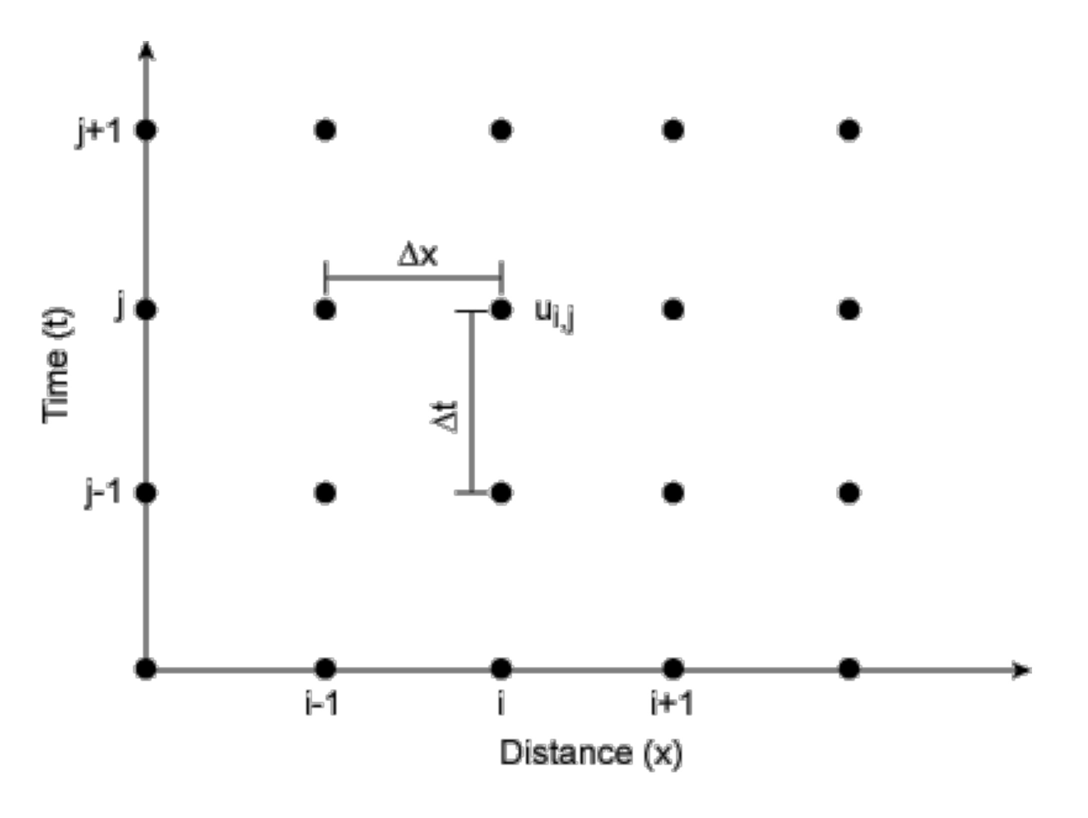
NUMERICAL METHODS

FINITE DIFFERENCING

- Given a continuous function $f(t, x)$, we approximate it by its value at a discrete set of points:

$$f_{ij} = f(t_j, x_i) + \text{truncation error}$$

  - Substitute a continuous spacetime with a set of discrete points (grid or mesh)
  - Replace differential operators by finite differences (algebraic equation)
  - Note: Numerical grids are not infinite - appropriate (outer) boundary conditions have to be chosen



Example 2nd order scheme (centred):

$$\partial_x f\big|_{x_i} = \frac{f_{i+1} - f_{i-1}}{2\Delta x} + \mathcal{O}(\Delta x^2)$$

$$\partial_x^2 f\big|_{x_i} = \frac{f_{i+1} - 2f_i + f_{i-1}}{(\Delta x)^2} + \mathcal{O}(\Delta x^2)$$

FINITE DIFFERENCING

- Example: advective equation (hyperbolic)

$$\partial_t u + v\partial_x u = 0$$

- Solution: $u(t, x) = u(x - vt)$
- FD representation:

$$\partial_x u\big|_{x_i}^{t_j} = \frac{u_{i+1}^j - u_{i-1}^j}{2\Delta x} + \mathcal{O}(\Delta x^2)$$ (centred spatial difference)

$$\partial_t u\big|_{x_i}^{t_j} = \frac{u_i^{j+1} - u_i^j}{\Delta t} + \mathcal{O}(\Delta t)$$ (forward time difference)

- Explicit forward scheme:

$$u_i^{j+1} = u_i^j - \frac{v}{2}\frac{\Delta t}{\Delta x}(u_{i+1}^j - u_{i-1}^j)$$

- Stability criterion:

$$|v|\frac{\Delta t}{\Delta x} \leq 1$$ Courant factor

NUMERICAL METHODS

SPECTRAL METHODS

- Represent the solution to a differential equation, e.g. $u(x)$ as a truncated series in a complete set of basis function $\phi_k(x)$:

$$u(x) \simeq u^{(N)}(x) = \sum_{k=0}^{N} \tilde{u}_k \phi_k(x)$$

spectral coefficients

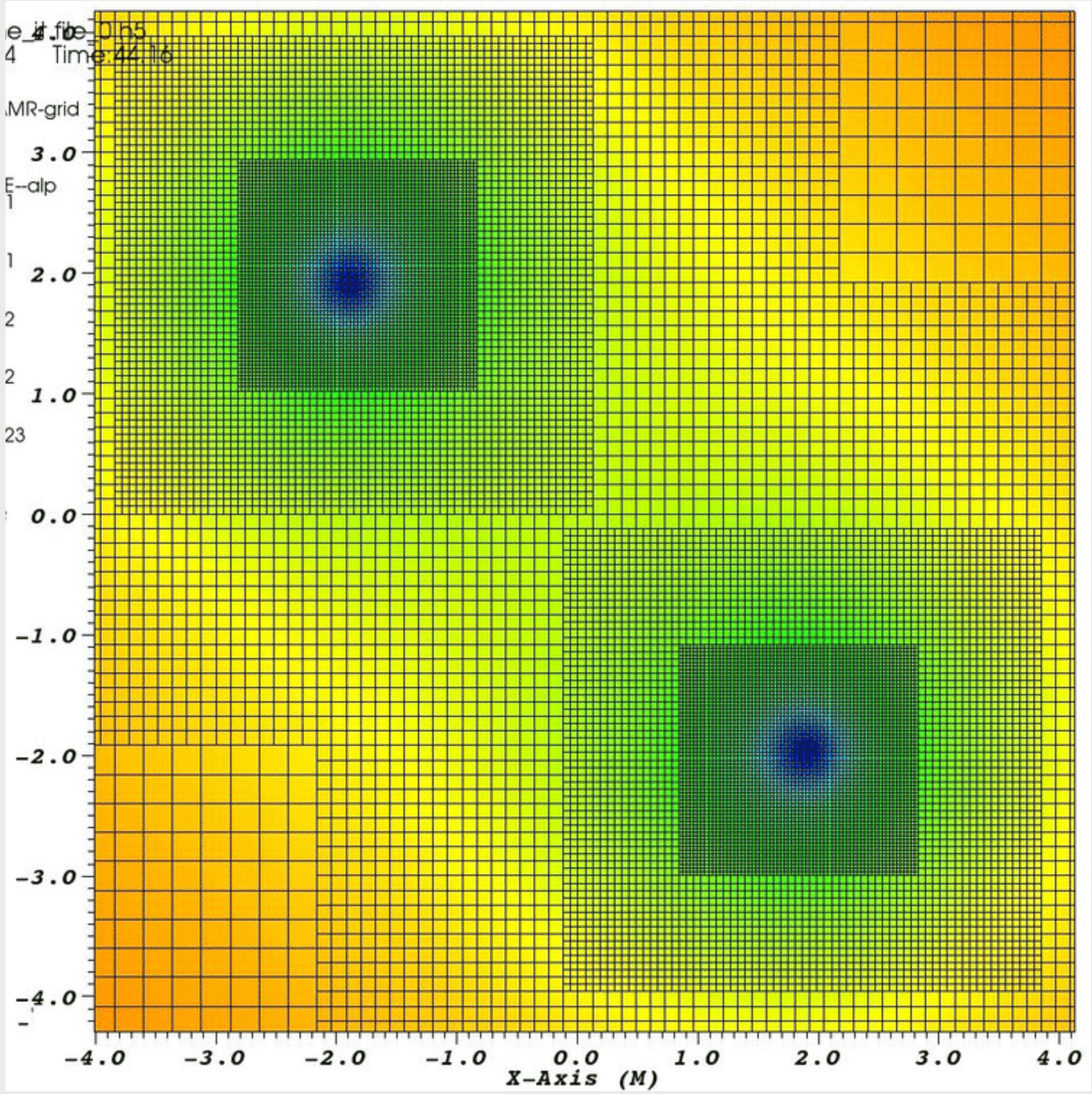
- Derivatives can be expressed analytically:

$$\partial_x u(x) \simeq \partial_x u^{(N)}(x) = \sum_{k=0}^{N} \tilde{u}_k \partial_x \phi_k(x)$$

- Advantage: Numerical error often drops exponentially with N. Often fewer computational resources needed for the same accuracy as FD.
- Disadvantage: More difficult to implement. Discontinuities are a problem.

MESH REFINEMENT

- BBH simulations span a "dynamic range" that we want to resolve, i.e. the dynamics of the black holes in a relatively small area of the grid and GWs in the wave zone far from the black holes.
- BUT: A uniform numerical grid across the dynamic range is very expensive!
- Perform the simulation using a multi-grid structure with a different spatial resolution on each grid as required.
- Note: Special care needs to be taken at grid boundaries.



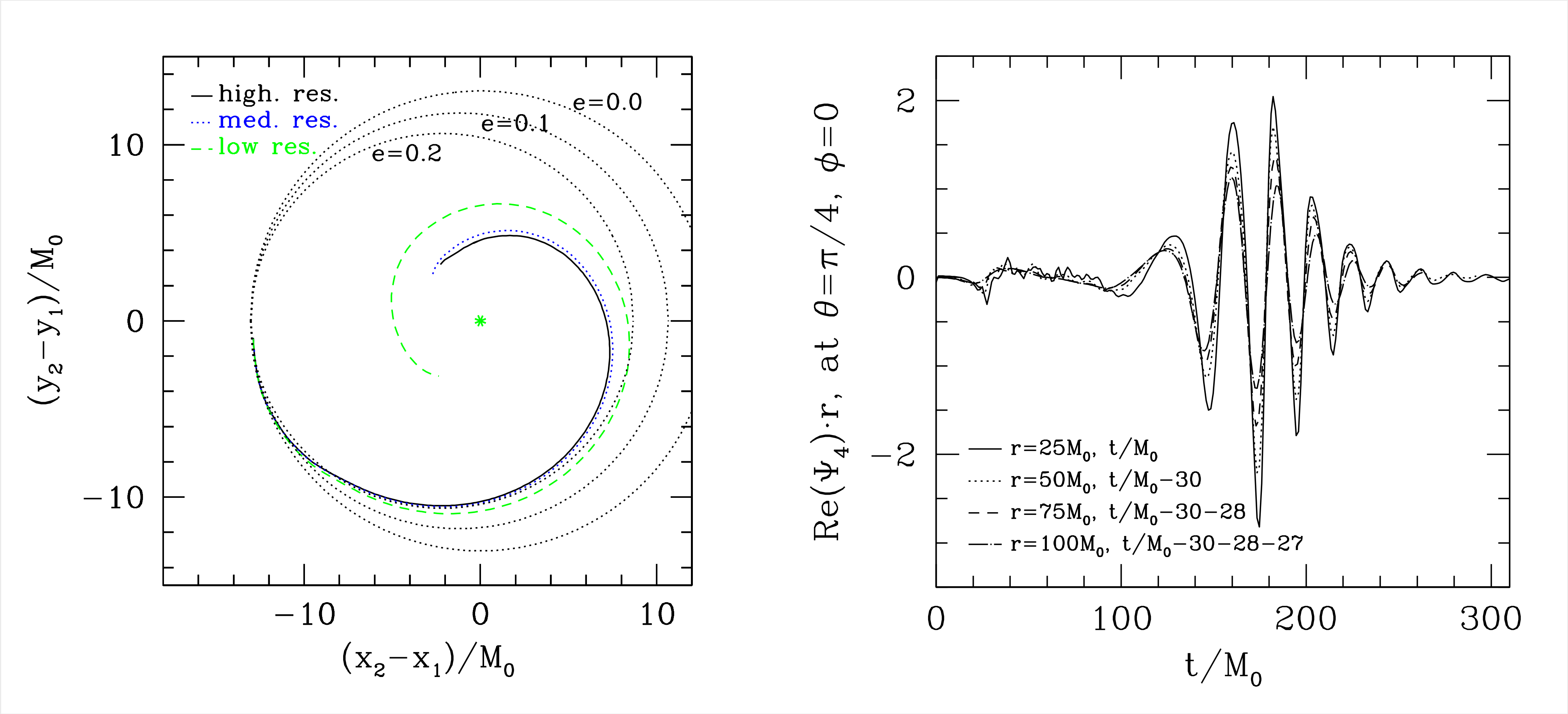
Adaptive mesh: boxes move with the BHs

GRAVITATIONAL WAVE EXTRACTION

- Different methods to extract GWs, e.g. Newman-Penrose formalism
  - 10 independent components of the Weyl tensor can be expressed as 5 scalars $\psi_0, \ldots, \psi_4$, which are formed by contracting the Weyl tensor with a null tetrad.
  - For certain tetrads (quasi-Kinnersley), we can interpret ψ_0 and ψ_4 as ingoing and outgoing null rays:
$$\psi_4 = -{}^{(4)}C_{abcd}k^a\bar{m}^b k^c \bar{m}^d$$
  - Where l and k are real radially outgoing and ingoing null vectors; m is a complex vector such that $-l^a k_a = 1 = m^a \bar{m}_a$.
  - In the TT gauge we find
$$\psi_4 = \ddot{h}_+ - i\ddot{h}_x$$
  - Requires the 4D Riemann tensor, which is constructed from spatial 3D quantities on each slice.

THE BREAKTHROUGH IN 2005

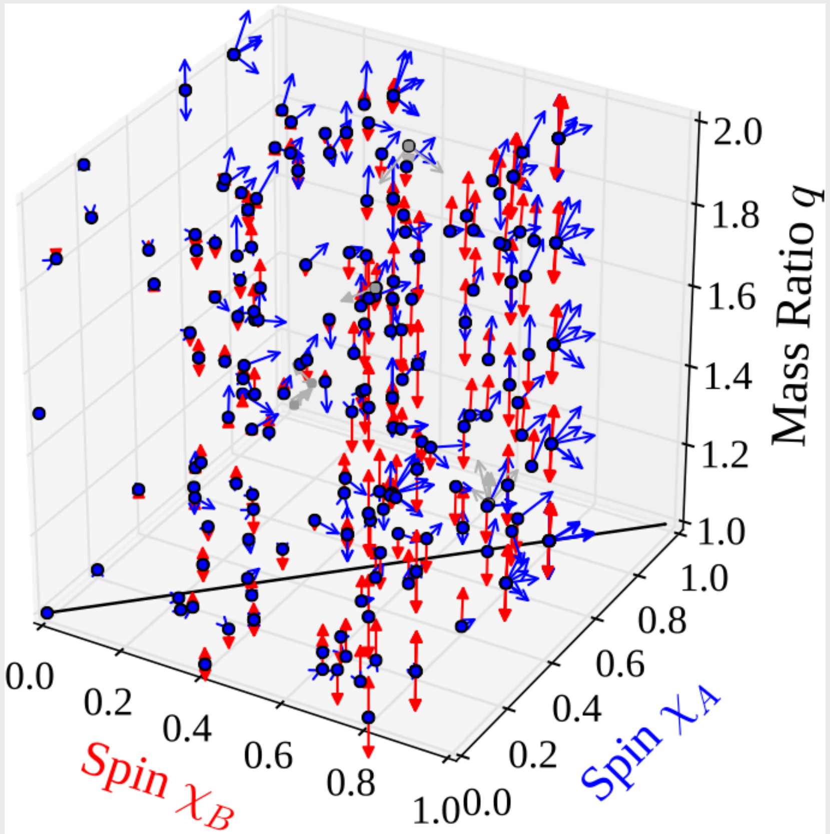
- In 2005, Frans Pretorius produced the first successful numerical relativity simulation of two equal mass black holes
  - Generalised harmonic formulation, excision, finite differencing



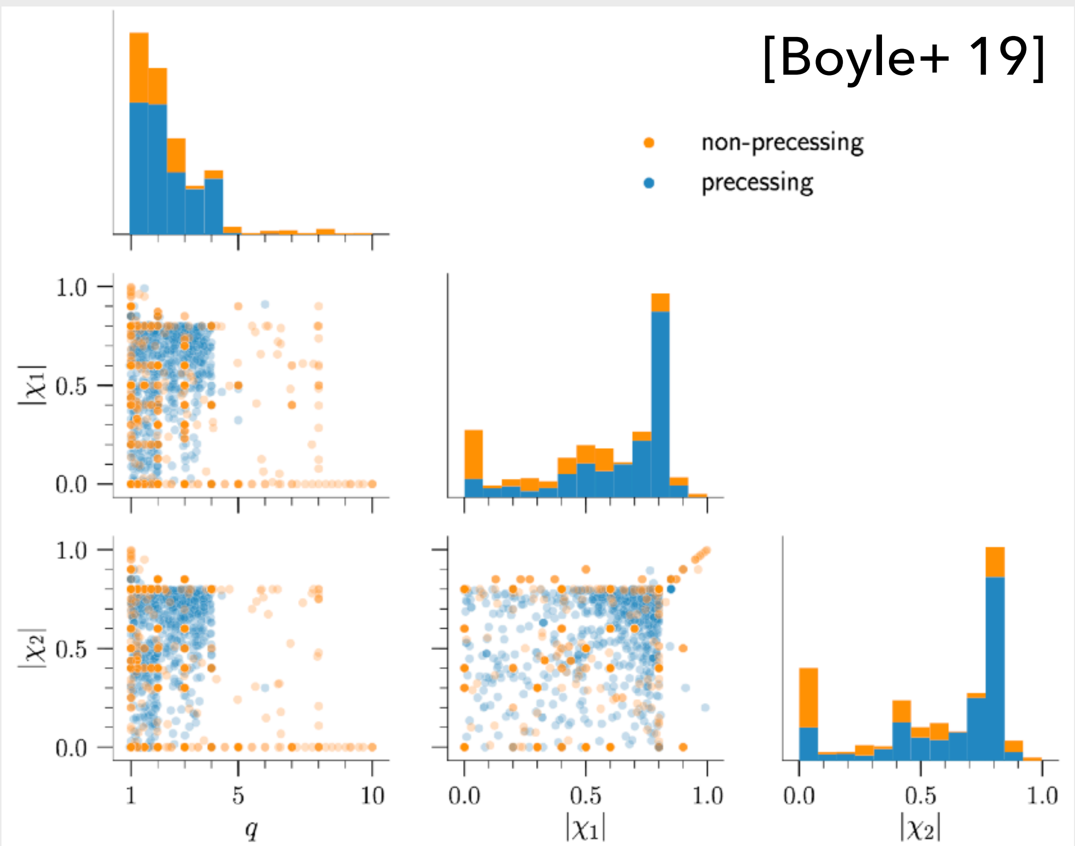
Figures from F. Pretorius, PRL 95, 121101 (2005)

STATE-OF-THE-ART

- Today, several thousands of waveforms from merging black holes have been produced by several different codes
- These waveforms are
  - Crucial for the development of inspiral-merger-ringdown (IMR) waveform models (calibration)
  - Directly used in GW data analysis applications
  - Important to gauge systematics
  - Crucial to derive the remnant properties

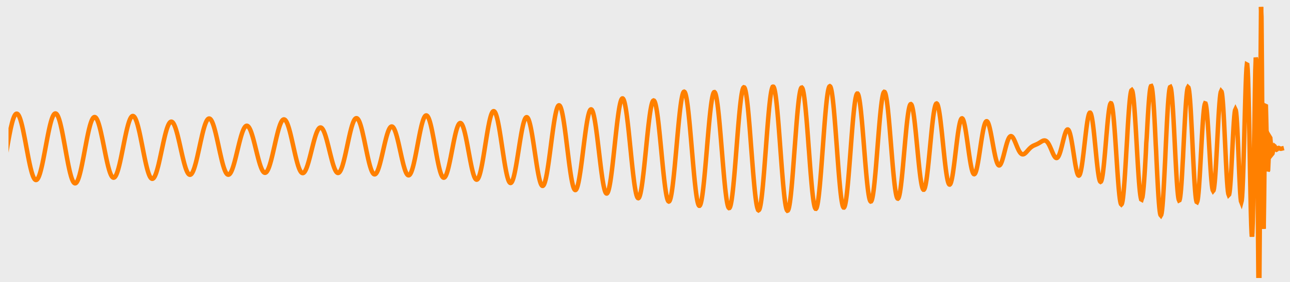


[Blackman+16]

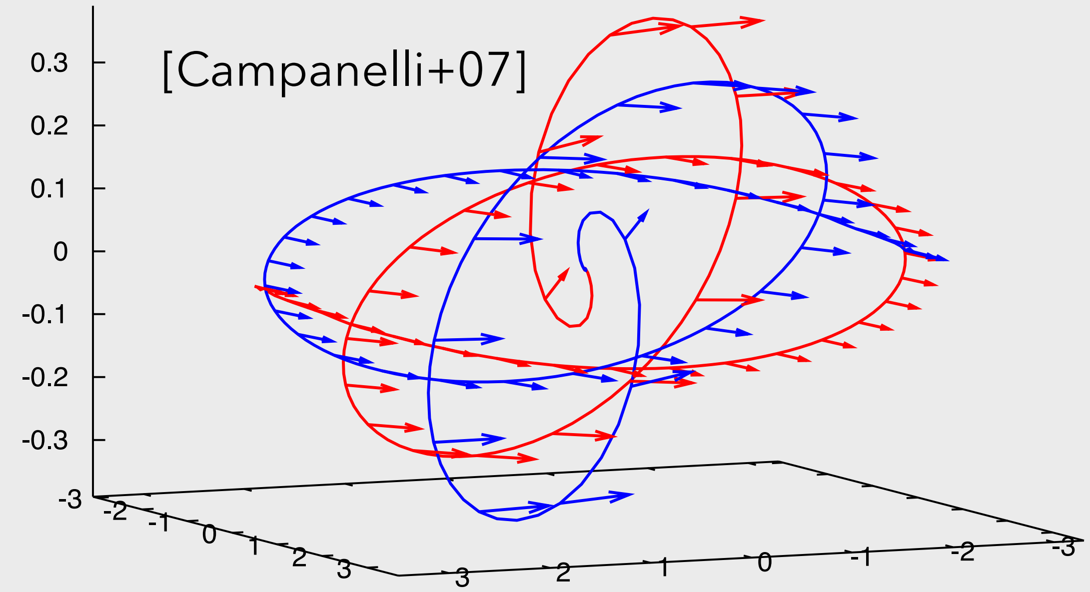


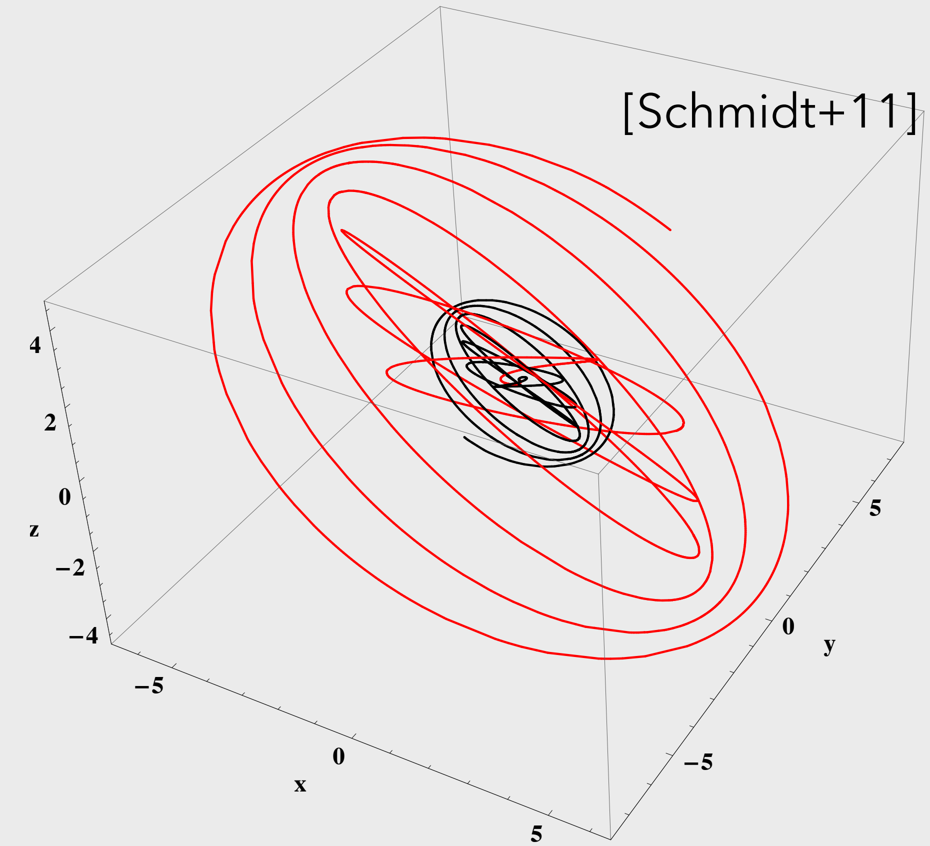
PRECESSION

- Occurs when spins are misaligned with the orbital angular momentum
  - Induces amplitude & phase modulations



- First simulations performed as early as 2006 [Campanelli+]
- 7D parameter space $(q, \vec{a}_1, \vec{a}_2)$
  - Difficult to sample
  - NR simulations concentrated around q=1-4
  - Long simulations required to resolve a complete precession cycle

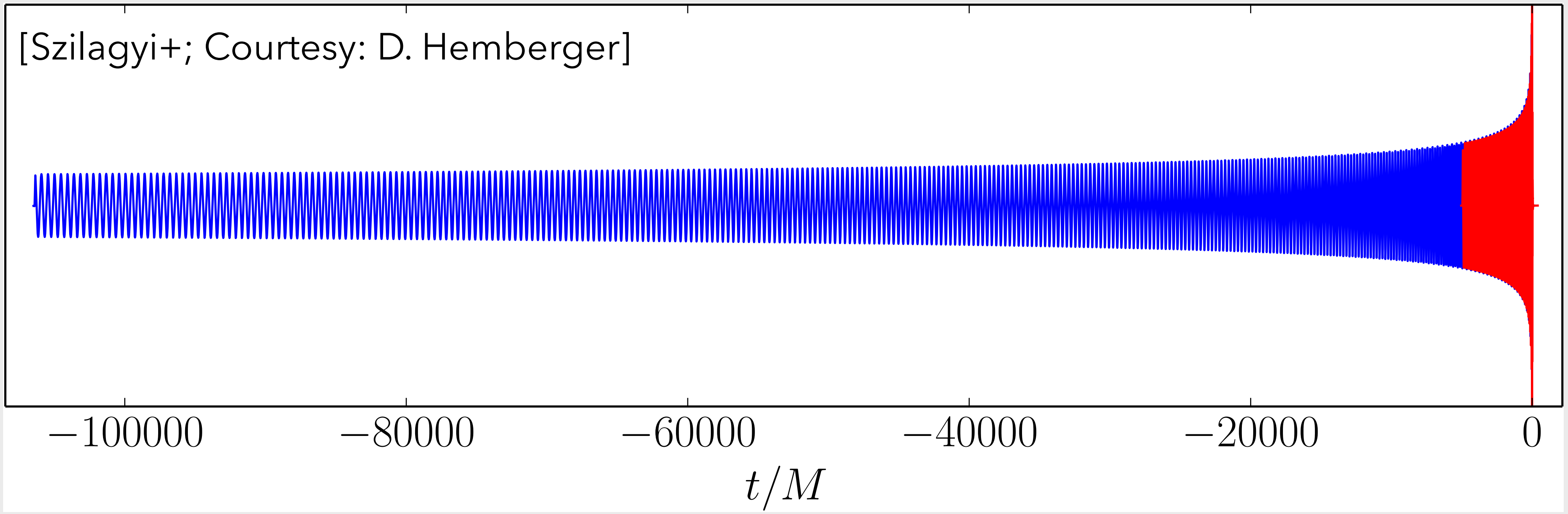




https://www.youtube.com/watch?v=5AkT4bPk-00

LONG SIMULATIONS

- Long simulations desirable to probe the consistency between early inspiral models
- Resolve a precession cycle?
- Stable evolutions longer than ~20 orbits are difficult & expensive
  - Gauge drifts
  - Build-up of numerical errors
- 2015: ~175 orbit long non-spinning simulation with q=7

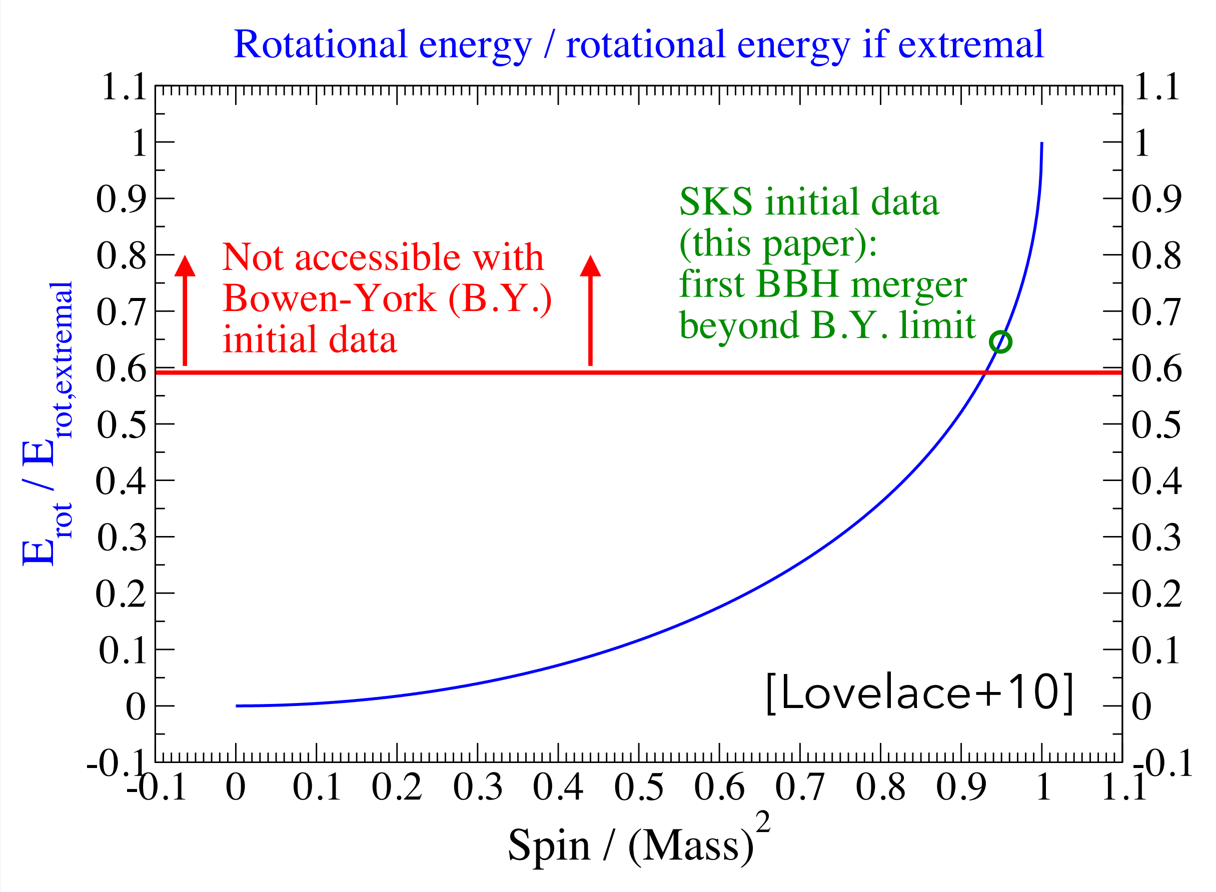


EXTREME SPINS

- Extreme spin: First breakthroughs in 2011 for quasi-equilibrium initial data [Lovelace+11]
  - Fundamental spin limit for Bowen-York (BY) initial data

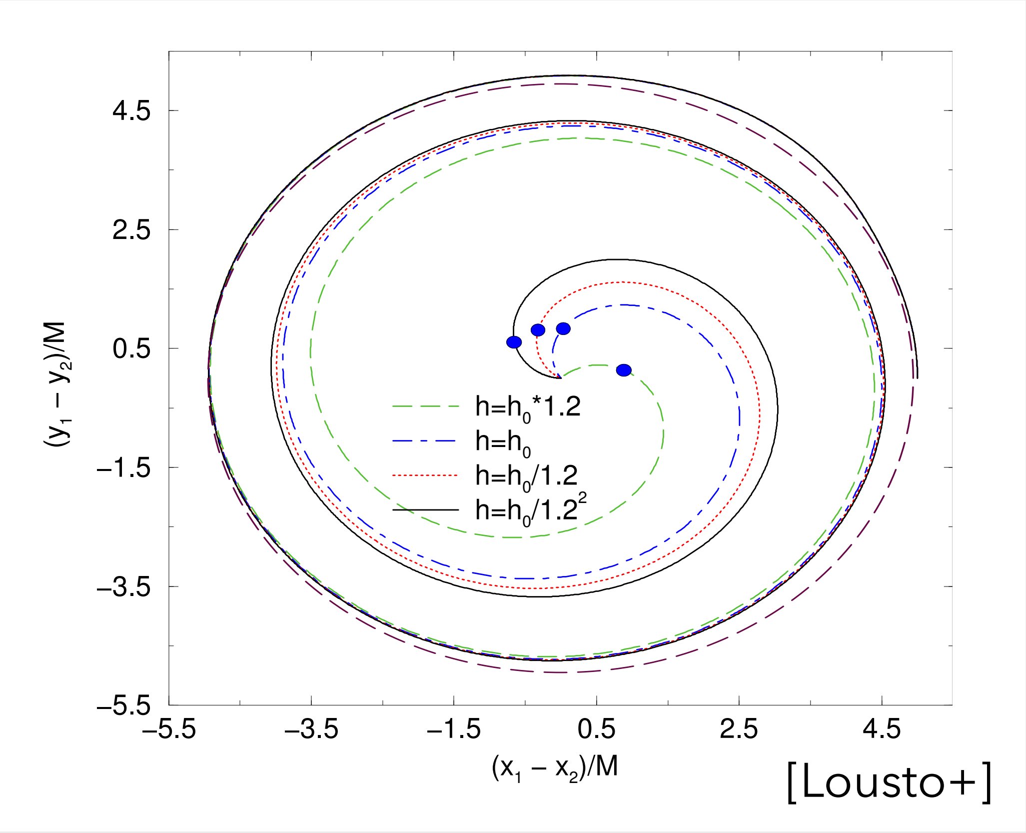
$$a_{\max} \leq 0.93$$ [Dain+]

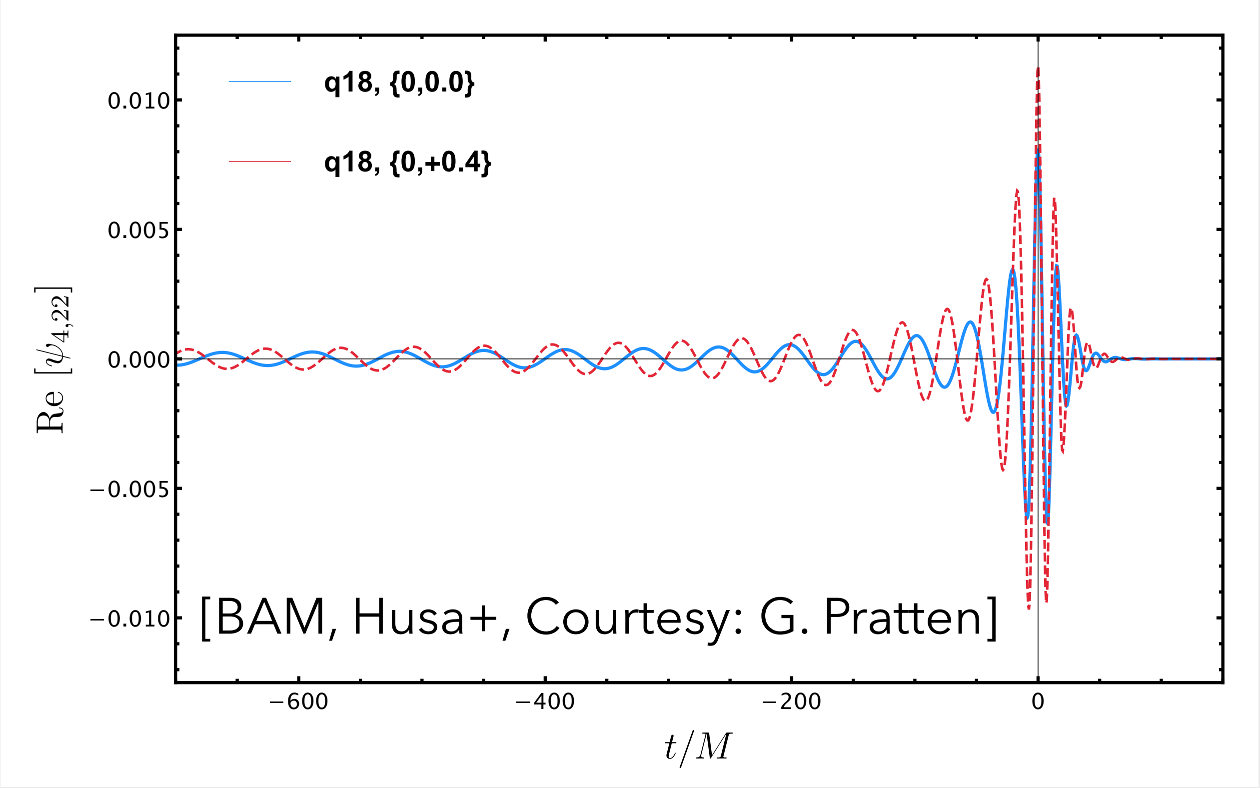
- 2014 onwards: incorporation of non-conformally flat initial data into the moving punctures framework [Ruchlin+14, Zlochower+17] to go beyond the BY limit



HIGH MASS RATIOS

- Very few points beyond mass ratio q=8
  - non-spinning q=100 [Lousto+]
  - non-spinning q=10 [SXS]
  - aligned-spin q=18 [BAM, Husa+15]
- Waveform models are poorly calibrated in the high mass regime due to the lack of NR simulations!
  - Test particle limit information included
- Problems:
  - Large difference in horizon size
  - Difficult to resolve the smaller BH
  - Spins additionally distort the horizon
  - Computationally very expensive



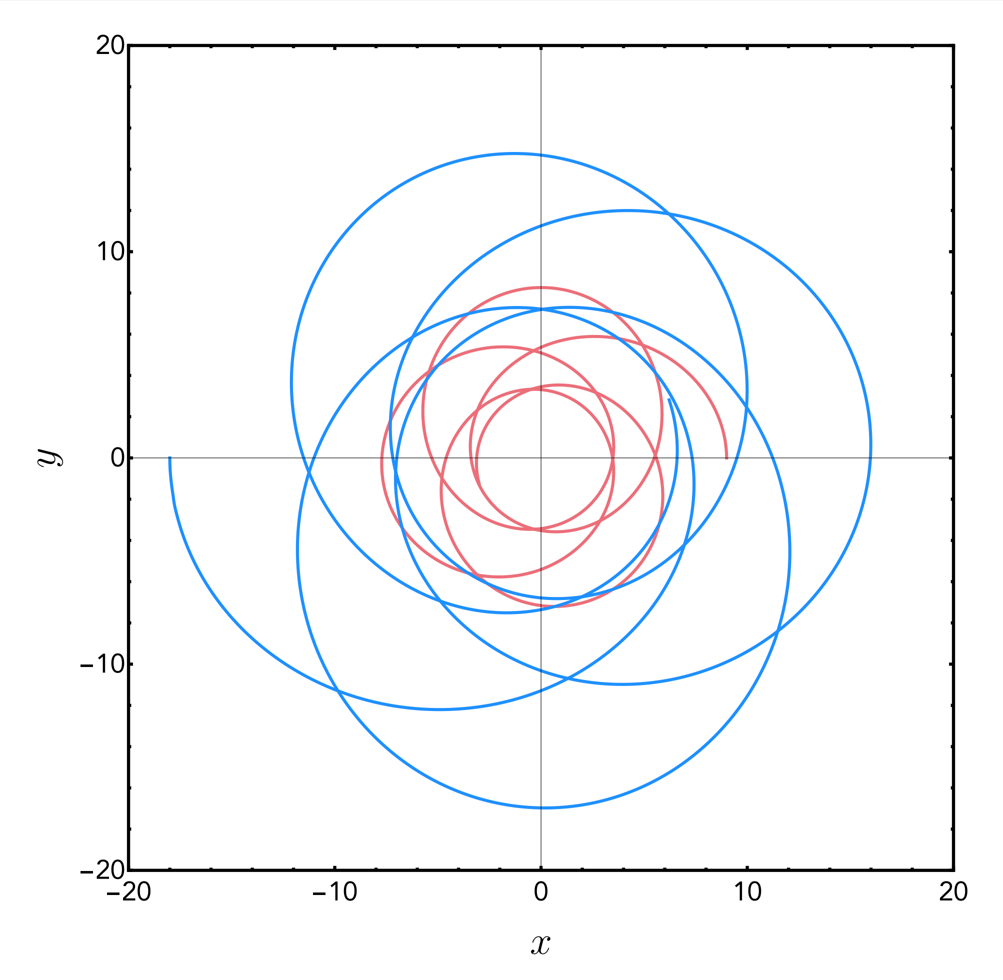


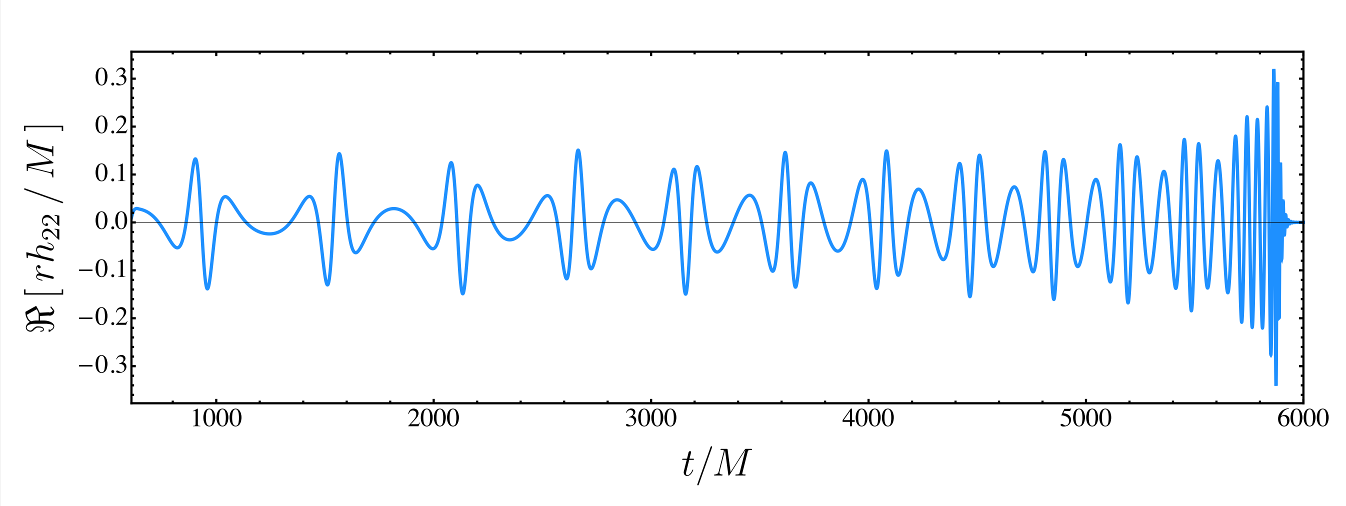
ECCENTRIC BINARIES

- Radiation-reaction circularises orbit
- 3-body encounters, dynamical capture could produce highly eccentric orbits with not enough time to circularise before merger
    - e.g. interactions in globular clusters

- Advance of the periastron (pure GR)
- Bursts of radiation
- Zoom-whirl behaviour

Courtesy: G. Pratten

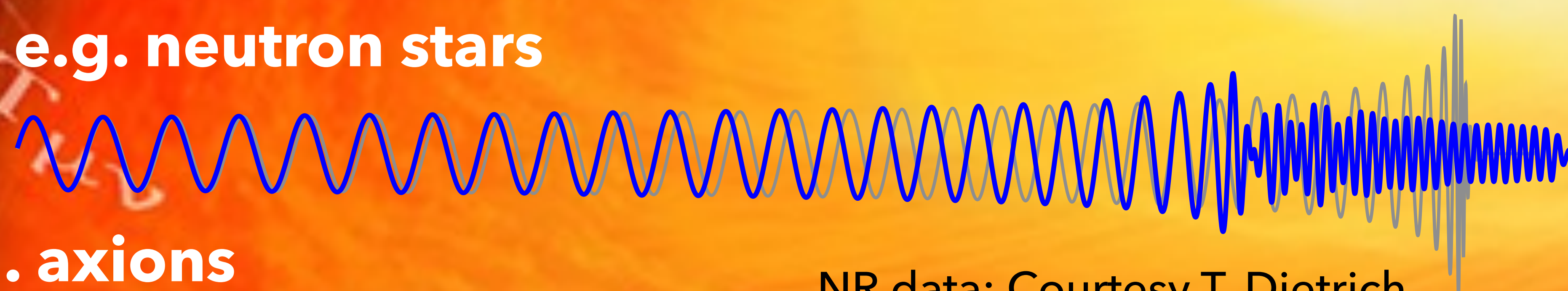




LOTS MORE WORK TO DO

- **Improve**
  - **Accuracy**
  - **Efficiency**
  - **Parameter space coverage**

- **Include more (realistic) physics**
  - **Regular matter, e.g. neutron stars**
  - **Exotic matter**
  - **Extra fields, e.g. axions**



NR data: Courtesy T. Dietrich

- **Going beyond General Relativity**