Consider a univariate stochastic process y_t which follows a first-order moving average representation

$$y_t = \epsilon_t - \theta \epsilon_{t-1} \tag{1}$$

where $\{\epsilon_t\}$ is an i.i.d. process distributed $\mathcal{N}(0,1)$ and $\theta > 1$

- 1. Write a program to simulate this process. Simulate 100 periods when $\theta = 2$. (Draw ϵ_{-1} from $\mathcal{N}(0,1)$ and $\theta > 1$)
- 2. Argue that ϵ_t cannot be expressed as a linear combination of y_{t-j} for $j \geq 0$ where the sum of the squares of the weights is finite. **Hint:** Note that

$$\begin{aligned} \epsilon_t &= y_t + \theta \epsilon_{t-1} \\ &= y_t + \theta y_{t-1} + \theta^2 \epsilon_{t-2} \\ &\vdots \end{aligned}$$

if you keep repeating what happens to the coefficients on y_{t-i} ? We call this property non-invertability.

- 3. Represent (1) using as a state space system using our Kalman filter. What are the matrices A, C, and G? How do they depend on θ ? **Hint:** x_t will be a vector.
- 4. Assuming initial beliefs about ϵ_{t-1} are $\mathcal{N}(0,1)$. Write a program to apply this kalman filter to an arbitray sequence $\{y_t\}_{t=0}^T$ for an arbitray θ .
- 5. Write a program to compute the log-likelihood of sequence $\{y_t\}_{t=0}^T$ for an arbitray θ . **Hint:** Using the Kalman filter the uncoditional distribution of y_0 is $\mathcal{N}(G\hat{x}_0), \Omega_0$). The distribution of y_t conditional on the past histories of $y^{t-1} = (y_{t-1}, y_{t-2}, \dots, y_0)$ is $\mathcal{N}(G\hat{x}_t, \Omega_t)$. The likelihood of a sequence of data (y_T, \dots, y_0) can then be constructed from the conditional likelihoods

$$f(y_T, \dots, y_0) = f(y_T | y^{T-1}) f(y_{T-1} | y^{T-2}) \cdots f(y_1 | y_0) f(y_0)$$
(2)

and similarly for the log likelihood.

- 6. For your simulation in part 1. Plot the log likelihood as a function of θ . Where is it maximized? What happens if the simulation length is 1000 periods?
- 7. **Bonus:** Repeat the Kalman filter exercise for $\theta = 0.5$. What happens to Σ_t over time? How does this relate to invertability (can we express ϵ_t as a linear combination of y_{t-j})?