Redistribution with a continuum of agents

1   Introduction

We study optimal affine taxation in an economy with continuum of agents that face (uninsurable) shocks to their idiosyncratic labor productivities. We first specify the problem without aggregate risk.

2   Ramsey planner's problem

Given a joint distribution $\Gamma_0(x_0, m_0, e_0)$ over marginal utilities adjusted assets, scaled marginal utilities and past productivities, the Ramsey allocation solves the following problem from t>0.

$$\max_{c_t^i, l_t^i, x_t^i, m_t^i} \sum_t \beta^t \int \omega^i u(c_t^i, l_t^i) di \tag{1}$$

subject to

$$\frac{x_{t-1}^i u_{c,t}^i}{\beta \mathbb{E}_{t-1} u_{c,t}^i} = u_{c,t}^i (c_t^i - T_t) + u_{l,t}^i l_t^i + x_t^i \tag{2a}$$

$$\alpha_t^1 = m_t^i \mathbb{E}_t u_{c,t+1}^i \tag{2b}$$

$$\alpha_t^2 = m_t^i u_{c,t}^i \tag{2c}$$

$$- u_{l,t}^i = (1 - \tau_t) u_{c,t}^i e_t^i \tag{2d}$$

$$e_t^i = (1 - \nu)\bar{e} + \nu e_{t-1}^i + q\epsilon_t^i \tag{2e}$$

$$\int m_t^i di = 1 \tag{2f}$$

$$\int l_t^i e_t^i di = \int c_t^i di + g \tag{2g}$$

Equations (2b) and (2c) ensure that individual Euler equations hold. The FOCs of this problem are summarized below:

$$\mu_{t-1}^i = \mathbb{E}_{t-1} \frac{u_{c,t}^i}{\mathbb{E}_{t-1} u_{c,t}^i} \mu_t^i \tag{3a}$$

$$\omega^i u_{l,t}^i - \mu_t^i [u_{ll,t}^i l_t^i + u_{l,t}^i] - \phi_t^i u_{ll,t}^i + \lambda_t e_t^i = 0 \tag{3b}$$

$$\rho_{2,t}^i u_{c,t}^i + \rho_{1,t}^i \mathbb{E}_t u_{c,t+1}^i + \eta_t = 0 \tag{3c}$$

$$\omega^i u_{c,t}^i + x_{t-1}^i \frac{u_{cc,t}^i}{\beta \mathbb{E}_{t-1} u_{c,t}^i} \left[\mu_t^i - \mu_{t-1}^i \right] - \mu_t^i [u_{cc,t}^i c_t^i + u_{c,t}^i] + \beta^{-1} \rho_{1,t-1}^i m_{t-1}^i u_{cc,t}^i + \rho_{2,t}^i m_t^i u_{cc,t}^i - \phi_t^i e_t^i (1-\tau_t) u_{cc,t}^i - \lambda_t = 0 \tag{3d}$$

$$\int \phi_t^i u_{c,t}^i e_t^i di = 0 \tag{3e}$$

$$\int u_{c,t}^i \mu_t^i di = 0 \tag{3f}$$

$$\int \rho_{j,t}^i di = 0 \quad j = 1, 2 \tag{3g}$$

The solution to these FOC can be expressed recursively using individual states $\xi_{t-1}^i = [\mu_{t-1}^i, m_{t-1}^i, e_{t-1}^i]$, shocks ϵ_t^i and aggregate state $\Gamma_{t-1}(\xi_{t-1}^i)$.

3   Ricardian Equivalence

Note that, using equation (2b), equation (3f) can be rewritten as

$$\int \frac{\mu_t^i}{m_t^i} = 0$$

Moreover, equation (3a) can be expressed as

$$\frac{\alpha_{t-1}^1 \mu_{t-1}^i}{m_{t-1}^i} = \alpha_t^2 \mathbb{E}_{t-1} \frac{\mu_t^i}{m_t^i}$$

By integrating both sides with respect to i, we obtain

$$\alpha_{t-1}^1 \int \frac{\mu_{t-1}^i}{m_{t-1}^i} di = \alpha_t^2 \int \frac{\mu_t^i}{m_t^i} di$$

This implies that if

$$\int \frac{\mu_{t-1}^i}{m_{t-1}^i} = 0$$

is satisfied for time $t-1$ we have then

$$\int \frac{\mu_t^i}{m_t^i} = 0$$

thus equation (3f) is redundant and we can choose T to be any value. For simplicity we set $T = 0$.

4 Optimal Taxes

In this section we derive a simple expression for taxes and use it to examine how the Ramsey plan uses distortionary taxes to redistribute inview of inequality in earnings and wealth.

The FOC for labor (3b) can be re-written as

$$\omega^i u_{c,t}^i \frac{u_{l,t}^i e_t^i}{u_{ll,t}^i} - \mu_t^i u_{c,t}^i e_t^i [l_t^i + \frac{u_{l,t}^i}{u_{ll,t}^i}] + \frac{\lambda_t u_{c,t}^i (e_t^i)^2}{u_{ll,t}^i} = \phi_t^i u_{c,t}^i e_t^i$$

Suppose $u(c,l) = \frac{c^{1-\sigma}}{1-\sigma} - \frac{l^{1+\gamma}}{1+\gamma}$. The above expression simplifies to

$$\frac{\omega^i u_{c,t}^i y_t^i}{\gamma} - \mu_t^i u_{c,t}^i y_t^i [1 + \frac{1}{\gamma}] - \lambda_t \gamma^{-1} \frac{y_t^i}{1 - \tau_t} = \phi_t^i u_{c,t}^i e_t^i$$

Define $w_{i,t} = u_{c,t}^i [\omega^i - \mu_t^i (1 + \gamma)]$ and $\bar{w}_t = \int u_{c,t}^i \omega^i di$ and $\bar{w}_{i,t} = \frac{w_{i,t}}{\bar{w}_t}$. Integrating over i's we have

$$\frac{1}{1 - \tau_t} = \frac{\hat{Y}_t}{Y_t} \frac{\bar{w}_t}{\lambda_t} \tag{4}$$

Where $Y_t = \int y_t^i di$ and $\hat{Y}_t = \int y_t^i \bar{w}_{i,t} di$

4.1 Special cases

With quasilinear utility and absent non-negativity constraints, we have a simple expression for how taxes depend on the distribution of initial Pareto weights.

$$\frac{1}{1 - \tau} = \frac{\hat{Y}}{Y} \tag{5}$$

where $\bar{w}_t^i$ weights used in computing $\hat{Y}$ are given by $1 + \gamma(\omega^i - 1)$. We can quickly draw some implications from this.

- Taxes are constant through time.

- If $\omega^i = 1$, taxes are zero.

- Suppose productivities are iid over time, taxes will be close to zero.

- Suppose productivities have a permenant component, taxes will be positive if agents with high productivities have low pareto weights and vice versa

4.2 Risk aversion

With risk aversion we have spreads in marginal utilities. This generates a desire to use transfers even if $\omega^i = 1$. To examine this in detail conisder two extreme economies. The first has iid productivity shocks ($\rho = 0$) and the second one has permenant productivity shocks ($\rho = 1$)[1].

The term $\frac{\bar{w}_t}{\lambda_t}$ captures increase in the welfare from transfers relative to the value of resources (note λ is a multiplier on the resource constraint.) Thus a higher value for this term would imply larger role of taxes and the consequent transfers.

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[1]XXX stationarity

The first term $\frac{\hat{Y}_t}{Y_t}$ is increasing in the covariance between $\bar{w}_t^i$ and pre-tax labor earnings y_t^i. Notice that these weights $\bar{w}_t^i$ are decreasing in the multiplier on the budget constraint of agent i : μ_t^i.

Consider the case where shocks to productivities are predominatly transitory. Agents save most of the shock for precautionary reasons. Overtime the agents with a unlucky stream of shocks will have lower wealth and low consumption and lower leisure. This will imply a lower $\bar{w}_t^i$ and higher y_t^i for such agents. The optimal response of government is to lower taxes and possibly give labor tax subsidies.

In the case where productivities have a permenant component, the output y^i will be higher for productive agents. These agents will also have higher wealth shares. This would make y_t^i and $\bar{w}_t^i$ positively correlated and call for higher taxes.

The next two sections describe the algorithm for computing an approximation for the Ramsey allocation for this economy.

5   Deterministic Steady State

We first look at an economy with $q = 0$. Given a joint distribution $\Gamma_0(\xi)$ the non-stochastic steady state is described by an allocation $c(\xi), l(\xi), x(\xi)$ and aggregate labor taxes τ.

Suppose $\Gamma_0(\xi)$ satisfies the following two properties

$$\int m^i di = 1 \tag{6a}$$

$$\int \frac{\mu^i}{m^i} di = 0 \tag{6b}$$

Given this we can conclude that the FOC can be satisfied with $\alpha^1 = \alpha^2$, $\eta = 0$ and $\rho_1^i = -\rho_2^i$. To obtain the deterministic steady state we add a few auxiliary variables $\phi(\xi), \rho_1(\xi)$ and aggregates λ and α. The steady state is given by the following set of equations. The first set solves for individual policies for a guess of (α, λ, τ) and the second set solves the aggregates

$$\alpha = m^i u_c^i \tag{7a}$$

$$- u_l^i = (1 - \tau) u_c^i e^i \tag{7b}$$

$$\gamma \phi^i = l^i [\omega^i - \mu^i (1 + \gamma)] + \frac{\lambda e^i}{u_{ll}^i} \tag{7c}$$

$$u_c^i [\omega^i - \mu^i (1 - \sigma)] + \rho_1^i m^i u_{cc}^i (\beta^{-1} - 1) - \phi_t^i e^i (1 - \tau) u_{cc}^i - \lambda = 0 \tag{7d}$$

The aggregates (α, τ, λ) solve

$$\int l_t^i e_t^i di = \int c_t^i di + g \tag{8a}$$

$$\frac{1}{1 - \tau_t} = \frac{\hat{Y}_t}{Y_t} \frac{\bar{w}_t}{\lambda_t} \tag{8b}$$

$$\int \rho_1^i di = 0 \tag{8c}$$

4

6 Approximation

For the individual variables let y be the control variables, ξ be the state variables, and ϵ be the idiosyncratic shock. The aggregate state of the economy is described by the distribution Γ over ξ, and Y is defined to be the aggregate control variable. The system of equations that defines our model can be separated into two parts

$$F(y_t, \mathbb{E}_{t-1} y_t, Y_t, \mathbb{E}_t y_{t+1}, \epsilon_t | \xi_{t-1}, q, \Gamma_t) = 0 \text{ for all } \xi_{t-1} \tag{9}$$

and

$$\int G(y_t, , Y_t, \epsilon_t | \xi_{t-1}, q, \Gamma_t) d\Gamma_t(\xi_{t-1}) d\epsilon_t = 0 \tag{10}$$

Here F in equation (9) represents the constraints on the individual variables while equation (10) represents the constraints on the aggregate variables. There are two major differences between this a setup and the standard way of linearizing models. The first is the aforementioned separation between aggregate and individual constraints. The second is the inclusion of the term $\mathbb{E}_{t-1} y_t$ which allows us to impose the measurability restrictions associated with incomplete markets. Specifically, x_{t-1} should be independent of the realization of the shock ϵ_t and ρ_{t-1}^1 should be independent of the realization of ϵ_{t-1}. We want to find the policy functions $y_t = y(\epsilon^i | \xi_{t-1}^i, q, \Gamma)$ and $Y_t = Y(q, \Gamma_t)$ that solve those systems. As next period's state ξ_t can be written as a subset of the controls y_t, the function $y(\epsilon^i | \xi_{t-1}^i, q, \Gamma)$ induces a law of motion of the individual states $\xi_t = \xi(\xi_{t-1} | q, \Gamma_t)$. This along with the distribution ϵ_t allows us to construct the law of motion for the aggregate state $\Gamma_{t+1} = \Gamma(\Gamma_t | q)$.

The key insight to begin this analysis is that at the limit where $q = 0$ the law of motion for the idiosyncratic state variable is identity, that is,

$$\xi(\xi_{t-1} | 0, \Gamma) = \xi_{t-1}$$

for all ξ and Γ. This, in turn, implies that $\Gamma(\Gamma_t | 0) = \Gamma_t$ when $q = 0$. All possible distributions Γ are stationary non-stochastic steady states. We will then use a perturbation approach to approximate locally around a choice of steady state Γ.

6.1 Linear Approximation

We will begin with a linear approximation. Choose any candidate $\overline{\Gamma}$ as the steady state around which we will approximate the policy rules. For convenience we will label each agent by $i \in (0, 1)$. Let $\overline{\xi}^i$ be the steady state value of ξ for agent i. Similarly let $\hat{\xi}_{t-1}^i = \xi_{t-1}^i - \overline{\xi}^i$ be the deviation from the steady state for agent i. This describes the current aggregate state Γ_t. We first want to find the linear approximation $\hat{Y}_t = Y_t - \overline{Y}$. This will be the sum of all the linear contributions of $\hat{\xi}_t^i$:

$$\hat{Y}_t = \int dY_\xi^i \hat{\xi}_{t-1}^i di \tag{11}$$

Note that here dY_ξ^i depends on the individual steady state value $\overline{\xi}^i$ as well as the global steady state $\overline{\Gamma}$. The linear approximation $\hat{y}_t^i = y_t^i - \overline{y}^i$ is a combination to the linear contributions of ξ_{t-1}^i, ϵ_t^i and the aggregation of the contributions of ξ_{t-1}^j for $j \neq i$, thus

$$\hat{y}_t^i = dy_\xi^i \hat{\xi}_{t-1}^i + dy_\epsilon^i q \epsilon_t^i + \int dy_\gamma^{i,j} \hat{\xi}_{t-1}^j dj$$

We can simplify this further by realizing that the contribution of $\hat{\xi}_t^j$ to $\hat{\xi}_t^i$ come through GE contributions via the aggregate variables. Thus we can express the linearization of the individual controls as

$$\hat{y}_t^i = dy_\xi^i \hat{\xi}_{t-1}^i + dy_\epsilon^i q \epsilon_t^i + dy_Y^i \hat{Y}_t \tag{12}$$

Given that the policy rule for $\hat{\xi}_t^i$ is stationary at $q = 0$ we know that the policy rules for the state variables will be

$$\hat{\xi}_t^i = \xi_{t-1}^i + d\xi_\epsilon^i q \epsilon_t^i$$

We can use these to forecast the aggregate variables next period

$$\hat{Y}_{t+1} = \int dY_\xi^i \xi_t^i di = \int dY_\xi^i (\hat{\xi}_{t-1}^i + d\xi_\epsilon^i q \epsilon_t^i) di = \hat{Y}_t$$

here use the property that we are integrating over a continuum to integrate out the effects of ϵ_t^i which are mean zero. Thus, the forecast of future controls satisfies

$$\mathbb{E}_t \hat{y}_{t+1}^i = dy_\xi^i \hat{\xi}_{t-1}^i + dy_Y^i \hat{Y}_t + dy_\xi^i d\xi_\epsilon^i q \epsilon_t^i$$

Differentiating the constrain in equation (9) an constructing the projection matrix I_y^ξ such that $\hat{xi}_t^i = I_y^\xi \hat{y}_t^i$ we obtain

$$F_y^i \hat{y}_t^i + F_{Ey}^i \mathbb{E}_{t-1} \hat{y}_t^i + F_Y^i \hat{Y}_t + F_{Ey'}^i \mathbb{E}_t \hat{y}_{t+1}^i + F_\epsilon^i q \epsilon_t^i + F_\xi^i \hat{\xi}_{t-1}^i = 0$$

We can then plug in the formulas described above and combine like terms to derive the expressions

$$dy_\xi^i = -(F_y^i + F_{Ey}^i + F_{Ey'}^i)^{-1} F_\xi^i \hat{\xi}_{t-1}^i \tag{13a}$$

$$dy_\epsilon^i = -(F_y^i + F_{Ey'}^i dy_\xi^i I_y^\xi)^{-1} F_\epsilon^i \tag{13b}$$

$$dy_Y^i = -(F_y^i + F_{Ey}^i + F_{Ey'}^i)^{-1} F_Y^i \tag{13c}$$

Finally, we can use the aggregate constraints

$$\int \left[G_y^i \hat{y}_t^i + G_Y^i \hat{Y}_t + G_\xi^i \hat{\xi}_{t-1}^i \right] di = 0$$

to obtain

$$dY_\xi^i = -\left(\int G_y^i dy_Y^i + G_Y^i di \right)^{-1} \left(G_y^i dy_\xi^i + G_\xi^i \right) \tag{14}$$

6.2 A Quadratic Approximation

Given that the choice of $\overline{\Gamma}$ is arbitrary, it seems unnecessary to compute dY_ξ^i as we can always choose our approximating steady state to be Γ_t and thus set $\hat{xi}_{t-1}^i = 0$. We must bear in mind that our agents our forward looking and thus will be making predictions about future realizations of aggregate variables (such as interest rates). In the linear approximation uncertainty washes due to certainty equivalence, but with a quadratic approximation uncertainty enters. The values for dY_ξ^i are necessary to compute the effects of uncertainty tomorrow on the allocations today.

The precise methods for computing the quadratic approximation of the policy rules are technical and will be left to an appendix. We can express the resulting policy rules as follows. Creating the stacked variable

$$\hat{S}_t^i = \begin{pmatrix} \hat{\xi}_{t-1}^i \\ \int dY_\xi^i \xi_{t-1}^i di \end{pmatrix}$$

we can express $\hat{Y}_t$ as

$$\hat{Y}_t = \int \left[dY_\xi^i \hat{\xi}_{t-1}^i + \frac{1}{2} (\hat{S}_t^i)' dY_{SS}^i \hat{S}_t^i \right] di + \frac{1}{2} dY_{q^2} q^2 \tag{15}$$

and

$$\hat{y}_t^i = dy_\xi^i \hat{\xi}_{t-1}^i + dy_\epsilon^i q \epsilon_t^i + dy_Y^i \hat{Y}_t + \frac{1}{2} \left[(\hat{S}_t^i)' dy_{SS}^i \hat{S}_t^i + (\hat{S}_t^i)' dy_{S\epsilon}^i q \epsilon_t^i + (q\epsilon_t^i)' dy_{\epsilon S}^i \hat{S}_t^i + dy_{\epsilon\epsilon}^i q^2 (\epsilon_t^i)^2 + dy_{q^2}^i q^2 \right] \tag{16}$$

Note that $dy_{q^2}^i$ incorporates the GE effect given by

$$\hat{Y}_{t+1} = \hat{Y}_t + \frac{1}{2} \int \left[dY_\xi^i I_y^\xi (dy_{\epsilon\epsilon}^i + dy_{q^2}^i) + (I_y^\xi dy_\epsilon^i)' dY_{\xi\xi}^i I_y^\xi dy_\epsilon^i \right] di q^2$$

6.3   Implementation

We can implement this algorithm as follows:

1. Begin with a distribution Γ_t, using this as the non-stochastic steady state approximate the policy rules

$$y_t^i = \overline{y}^i + dy_\epsilon^i q \epsilon_t^i + \frac{1}{2} \left[dy_{\epsilon\epsilon}^i q^2 (\epsilon_t^i)^2 + dy_{q^2}^i q^2 + dy_Y^i dY_{q^2} q^2 \right]$$

and

$$Y_t = \overline{Y} + \frac{1}{2} dY_{q^2} q^2$$

2. Use these policy rules to construct a new distribution Γ_{t+1} (in our case approximate Γ_t with N agents and draw Γ_{t+1} via the policy rules)

3. Go back to step 1.

7   Numerical results

We use the methods describe to compute solutions to two extreme economies. The prefernces are CES and separable in consumption and leisure. The risk aversion paramter σ is set to 2, the Frish elaticity of labor supply is 0.5 and $\beta = 0.95$.

In order to make the shocks roughly equivalent we set standard deviation of the productivity shocks in the IID case to be 20 times the size of that with permanent shocks. The aim was to have approximately the same size shock to the present value of lifetime earnings. Assume that labor is approximately constant (not bad for a single small shock). The percent shock to the present value of lifetime earnings is the same as the percent shock to the present value of lifetime productivity. With a discount rate of 0.95 the factor becomes 1/20.

We solve the model and simulate the economy for 1000 agents for 400 periods. The figures 1 and 2 below plots the time series for labor taxes. We see that in the IID case taxes are evenually decreasing and in the unit root case taxes are increasing. Figures 3 and 4 show how the inequality in consumption, earnings and labor supply vary over time for both these cases. The key graph is the covariance between consumption and pre-tax labor earnings that differs across the two cases.
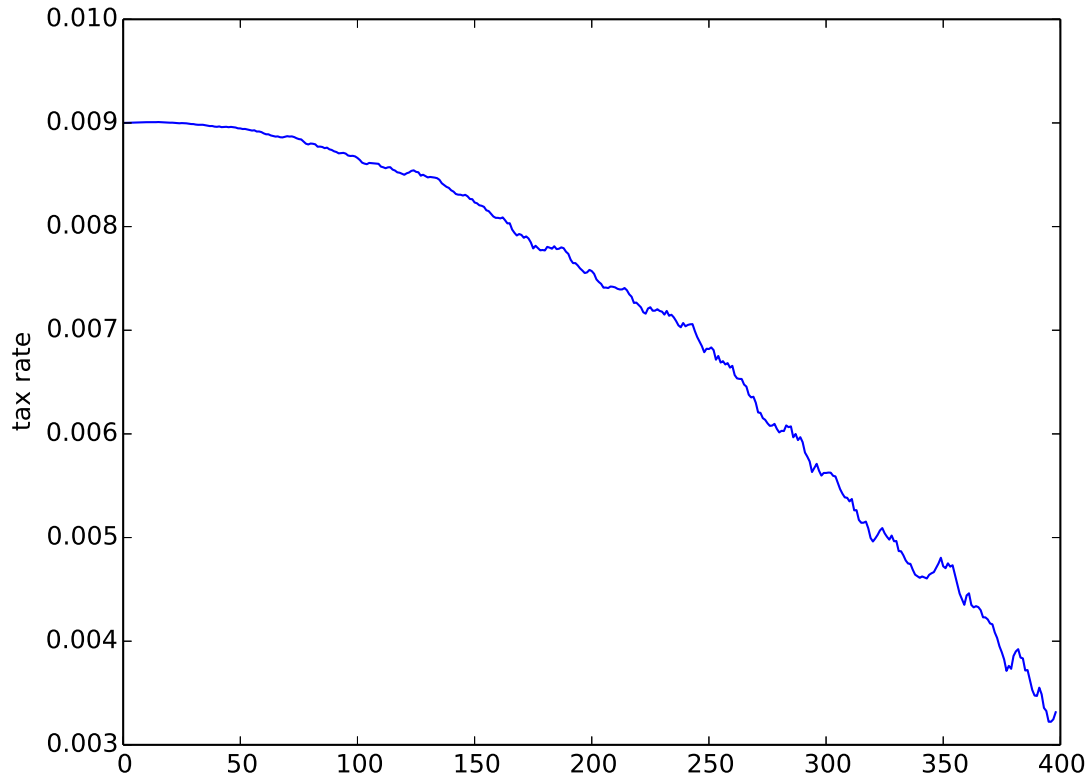

Figure 1: Taxes in the iid economy.

8   Extentions

- More realistic productivity process with common and idiosyncratic components. Something which generates earnings like in Fatih's paper.

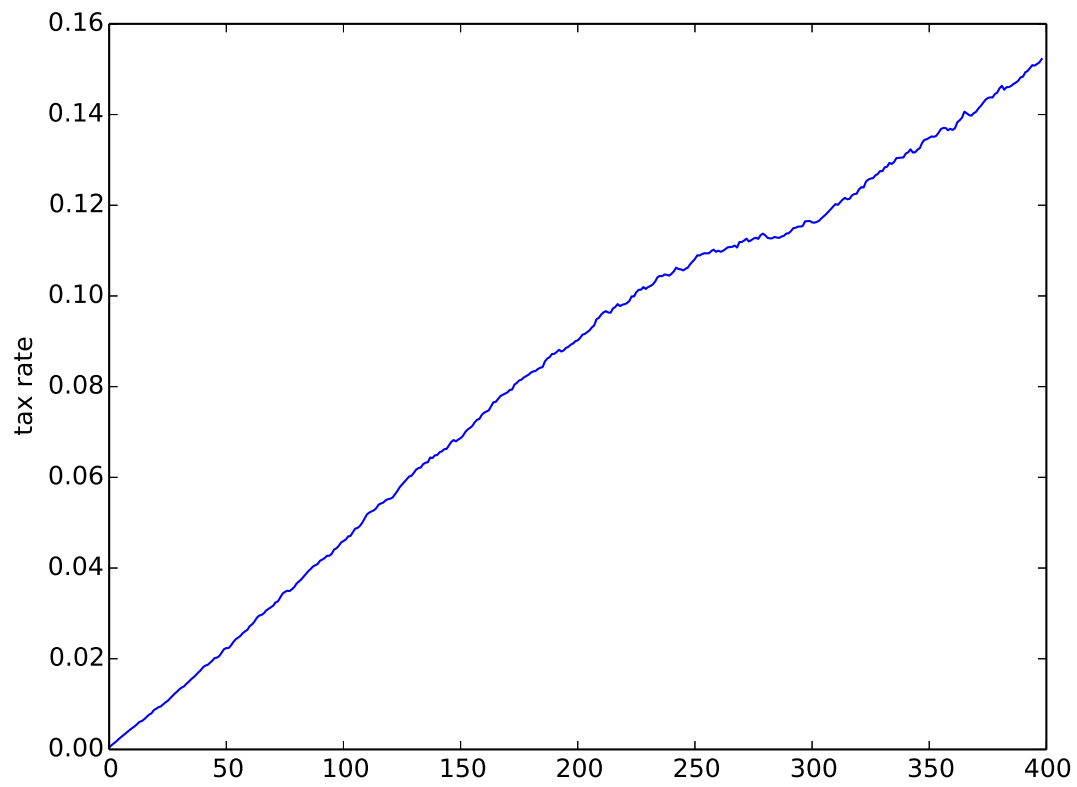- Adding capital

- GHH preferences
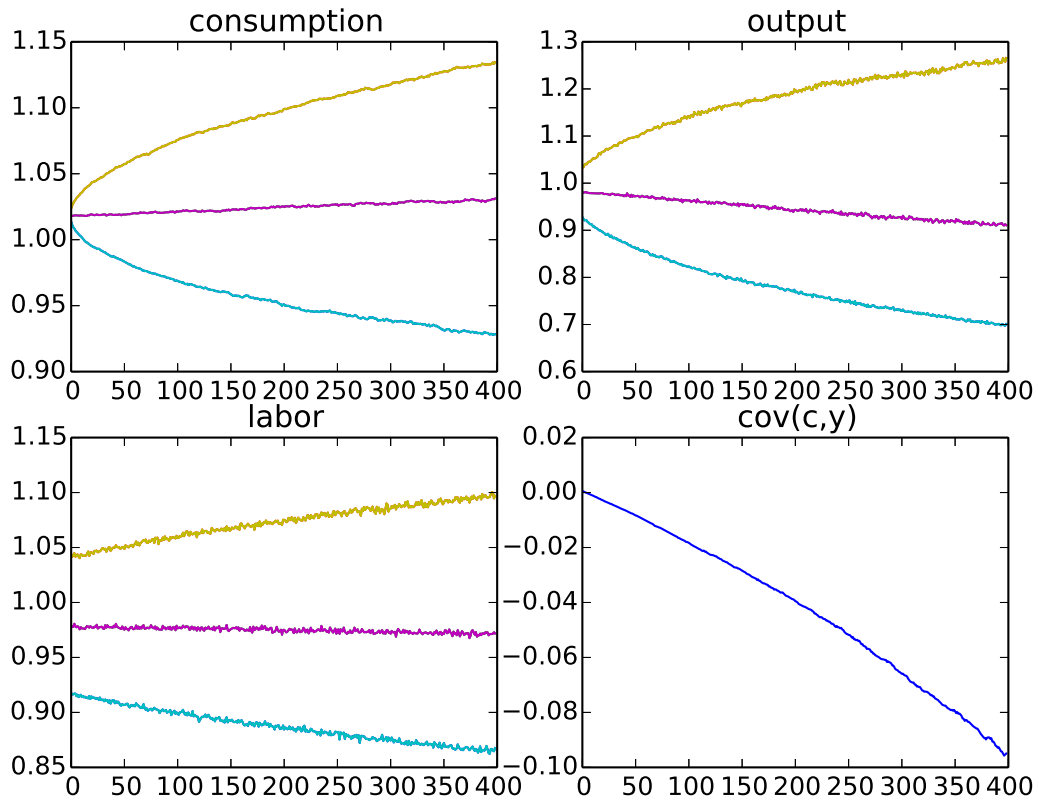
Figure 2: Taxes in the unit root economy.

Figure 3: Inequality in the iid economy. The figure plots the quantiles for consumption, pre-tax labor earnings, labor and the covariance between consumption and pre-tax labor earnings
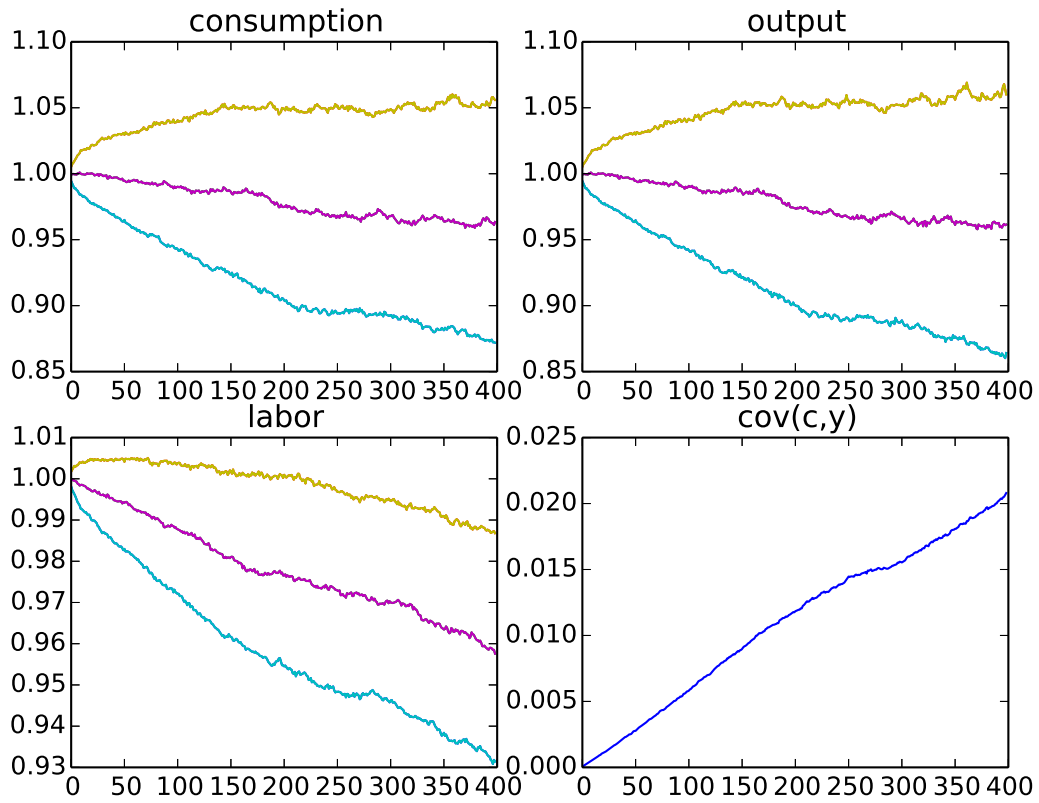
Figure 4: Inequality in the unit root economy. The figure plots the quantiles for consumption, pre-tax labor earnings, labor and the covariance between consumption and pre-tax labor earnings