Problem 1

In order to factor this PDE into two first order PDEs, we notice that the pair of PDEs forces us to factor in such a way that we end up with a nested PDE. In particular, we add and subtract intermediate terms,

$$\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} + D\frac{\partial u}{\partial t} = 0$$
$$\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} + D\frac{\partial u}{\partial t} + \left(-c\frac{\partial^2 u}{\partial t\partial x} + c\frac{\partial^2 u}{\partial t\partial x} - Dc\frac{\partial u}{\partial x} + Dc\frac{\partial u}{\partial x}\right) = 0$$

rearranging so that we can factor out the $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial x}$ independently, we get

$$\left(\frac{\partial^2 u}{\partial t^2} - c\frac{\partial^2 u}{\partial t\partial x} + D\frac{\partial u}{\partial t}\right) + \left(c\frac{\partial^2 u}{\partial t\partial x} - c^2\frac{\partial^2 u}{\partial x^2} + Dc\frac{\partial u}{\partial x}\right) - Dc\frac{\partial u}{\partial x} = 0$$
$$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t} - c\frac{\partial u}{\partial x} + Du\right) + c\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t} - c\frac{\partial u}{\partial x} + Du\right) - Dc\frac{\partial u}{\partial x} = 0$$

Letting v be this embedded PDE, we get our result,

$$\frac{\partial u}{\partial t} - c\frac{\partial u}{\partial x} + Du = v$$
$$\frac{\partial v}{\partial t} + c\frac{\partial v}{\partial x} = Dc\frac{\partial u}{\partial x}$$

As required.

Problem 2

(a) Notice that the terms used to classify this PDE will ignore crossing terms, thus we use $u_{tt} - 6u_{xx} = 0$ to classify the PDE. In particular, the negative sign between the two indicates a hyperbolic PDE.

(b) To factor the differential operator, we pull the function u and look at it in terms of just the operator,

$$u_{tt} + u_{xt} - 6u_{xx} = 0 \quad \rightarrow \quad \partial_{tt} + \partial_{xt} - 6\partial_{xx} = 0$$

and then just factor as we would a quadratic,

$$\partial_{tt} + \partial_{xt} - 6\partial_{xx} = 0 \quad \rightarrow \quad (\partial_t - 2\partial_x)(\partial_t + 3\partial_x) = 0$$

Thus the corresponding characteristic variables are,

$$\alpha = x + 2t \qquad \& \qquad \beta = x - 3t$$

(c) We use the chain rule to incorporate our characteristic variables. First, notice that the second order derivatives will vanish, as both α and β are linear in x and t. Thus, only the cross term remains in the PDE,

$$\frac{\partial^2 u}{\partial x \partial t} = \frac{\partial^2 u}{\partial \alpha^2}\frac{\partial \alpha}{\partial x}\frac{\partial \alpha}{\partial t} = 2\frac{\partial^2 u}{\partial \alpha^2} = 0 \quad \& \quad \frac{\partial^2 u}{\partial x \partial t} = \frac{\partial^2 u}{\partial \beta^2}\frac{\partial \beta}{\partial x}\frac{\partial \beta}{\partial t} = -3\frac{\partial^2 u}{\partial \beta^2} = 0$$

Notice the scalars are irrelevant since the derivative vanishes for both ODEs, thus we end up with

$$\frac{\partial^2 u}{\partial \alpha^2} = 0 \qquad \& \qquad \frac{\partial^2 u}{\partial \beta^2} = 0$$

(d) Working from the previous two ODEs, we see that,

$$\frac{\partial u}{\partial \alpha} = h(\beta) + C \qquad \& \qquad \frac{\partial u}{\partial \beta} = k(\alpha) + D$$

where $k(\alpha), h(\beta)$ are functions and $C, D \in \mathbb{R}$. To solve this, we make the assumption that the solution, $u(x,t)$, is a smooth function $\forall x, t \in S \subseteq \mathbb{R}$, for some subset S. Then a change of variable will retain this smoothness, and will imply,

$$\frac{\partial^2 u}{\partial\alpha\partial\beta} = \frac{\partial^2 u}{\partial\beta\partial\alpha} \quad \Longrightarrow \quad k'(\alpha) = h'(\beta)$$

however, we know this is only possible if $\exists\lambda \in \mathbb{R}$ such that $k'(\alpha) = \lambda = h'(\beta)$. From this we can further deduce that

$$k(\alpha) = \alpha\lambda \qquad \& \qquad h(\beta) = \beta\lambda$$

where the constants of integration are absorbed into the D and C respectively. Plugging this in and integrating we get

$$u = \alpha\beta\lambda + C\alpha \qquad \& \qquad u = \alpha\beta\lambda + D\beta$$
$$\Longrightarrow u(\alpha,\beta) = \alpha\beta\lambda + C\alpha + D\beta$$
$$\Longrightarrow u(x,t) = (x+2t)(x-3t)\lambda + C(x+2t) + D(x-3t)$$

which is the general solution.

(d) To find a specific solution, we apply our ICs.

$$f(x) = u(x,0) = \lambda x^2 + x(C+D) \qquad \& \qquad g(x) = u_t(x,0) = -\lambda x + 2C - 3D$$

expanding our original general solution,

$$u(x,t) = \lambda(x^2 - xt - 6t^2) + x(C+D) + t(2C-3D)$$
$$u(x,t) = (\lambda x^2 + x(C+D)) + t(-\lambda x + 2C - 3D) - 6\lambda t^2$$
$$u(x,t) = f(x) + tg(x) - 6\lambda t^2$$

As required.

Problem 3

(a) Let $u = v(t)w(x)$. Substituting this in, we get,

$$\frac{\partial^2 u}{\partial t^2} = c^4 \frac{\partial^4 u}{\partial x^4} \quad \to \quad w(x)\frac{\partial^2 v(t)}{\partial t^2} = c^4 v(t)\frac{\partial^4 w(x)}{\partial x^4}$$

Assuming non-trivial solution, i.e. $w(x) \neq 0$ and $v(t) \neq 0$, we divide through by $v(t)w(x)$,

$$\frac{1}{v}\frac{\partial^2 v}{\partial t^2} = \frac{c^4}{w}\frac{\partial^4 w}{\partial x^4}$$

which is only possible if $\exists \lambda \in \mathbb{R}$ such that,

$$\frac{1}{v}\frac{\partial^2 v}{\partial t^2} = \lambda = \frac{c^4}{w}\frac{\partial^4 w}{\partial x^4}$$

$$\frac{\partial^2 v}{\partial t^2} = \lambda v \qquad \& \qquad \frac{\partial^4 w}{\partial x^4} = \frac{\lambda w}{c^4}$$

(b) Let $u = v(t)w(x)$. Substituting this in, we get,

$$\frac{\partial u}{\partial t} = c^4 \frac{\partial^4 u}{\partial x^4} \quad \to \quad w\frac{\partial v}{\partial t} = c^4 v\frac{\partial^4 w}{\partial x^4} \implies \frac{1}{v}\frac{\partial v}{\partial t} = \frac{c^4}{w}\frac{\partial^4 w}{\partial x^4}$$

Again, $\exists \lambda \in \mathbb{R}$ such that,

$$\frac{\partial v}{\partial t} = \lambda v \qquad \& \qquad \frac{\partial^4 w}{\partial x^4} = \frac{\lambda w}{c^4}$$

(c) Let $u = v(y)w(x)$. Substituting this in, we get,

$$-\frac{\partial^2 u}{\partial y^2} = c^4 \frac{\partial^4 u}{\partial x^4} \quad \to \quad -w\frac{\partial^2 v}{\partial y^2} = c^4 v\frac{\partial^4 w}{\partial x^4} \implies -\frac{1}{v}\frac{\partial^2 v}{\partial y^2} = \frac{c^4}{w}\frac{\partial^4 w}{\partial x^4}$$

Finally, we use $\lambda \in \mathbb{R}$ such that

$$\frac{\partial^2 v}{\partial y^2} = -\lambda v \qquad \& \qquad \frac{\partial^4 w}{\partial x^4} = \frac{\lambda w}{c^4}$$

Problem 4

Applying implicit differentiation,

$$\frac{\partial u}{\partial t} = \frac{\partial f(x-tu)}{\partial t} = -\left(u + t\frac{\partial u}{\partial t}\right) f'(x-tu) \quad \& \quad \frac{\partial u}{\partial x} = \frac{\partial f(x-tu)}{\partial x} = f'(x-tu)$$

We apply our initial condition to the PDE,

$$\begin{gathered} u_t + uu_x = 0 \quad \rightarrow \quad u_t(x,0) + u(x,0)u_x(x,0) = 0 \\ -\left(u(x,0) + (0)\frac{\partial u}{\partial t}(x,0)\right) f'(x) + u(x,0)f'(x) = 0 \\ -f(x)f'(x) + f(x)f'(x) = 0 \\ 0 = 0 \end{gathered}$$

Thus we see that the PDE is satisfied by this solution at the initial condition provided.