PHYS 444 : A4

Problem 5.1

a) In the trivial representation, we have that we only need one basis vector to represent all of the scalars, that is $\{1\}$. But, this is only possible when $s=0$, so that $m=0$, and hence, since,

$$\hat{S}_z |0,0\rangle = 0 \qquad \hat{S}_\pm |0,0\rangle = 0$$

our trivial representation will just be the zero "matrices" in $\mathbb{R}$.

b) Since $s=1$, $m \in \{-1, 0, +1\}$, which we can represent as $\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}$ and $\begin{bmatrix}0\\0\\1\end{bmatrix}$ respectively. Notice,

$$\langle 1,-1| \hat{S}_z |1,-1\rangle = \langle 1,-1| (-\hbar) |1,-1\rangle = -\hbar,$$

But since the remaining vectors are mutually orthogonal,

$$\hat{S}_z = \hbar \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

As for $\hat{S}_\pm$, we see that,

$$\langle 1,-1| \hat{S}_\pm |1,-1\rangle = \langle 1,-1| \left(\hbar\sqrt{2 + 1(-1 \pm 1)}\right) |1, -1 \pm 1\rangle$$

So, $(\hat{S}_+)_{11} = 0$, $(\hat{S}_-)_{11} = 0$. Diagonally we will have zero's. Instead,

$$\hat{S}_+ = \hbar \begin{bmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{bmatrix} \qquad \hat{S}_- = \begin{bmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{bmatrix}.$$

From this, we can get the remaining generators via the Lie Bracket $[\cdot,\cdot]$. In particular, we get $\hat{S}_x$ and $\hat{S}_y$:

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Problem 5.7

By Taylor expanding, we see

$$f((\mathbb{1}+\theta X)\vec{x}) = f\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \theta\begin{bmatrix} -x_2 \\ x_1 \end{bmatrix}\right) = f(\vec{x}) + \theta x_2 \frac{d}{dx_1} f(\vec{x}) - \theta x_1 \frac{d}{dx_2} f(\vec{x}) + \dots$$

$$= f(\vec{x}) + \theta\left(x_2 \frac{d}{dx_1} - x_1 \frac{d}{dx_2}\right) f(\vec{x}) + \dots$$

$$= \exp\left(\theta\left(x_2 \partial_{x_1} - x_1 \partial_{x_2}\right)\right) f(\vec{x})$$

$$= \exp\left(-\frac{i}{\hbar}\theta \hat{L}_3\right) f(\vec{x}).$$

Problem 5.8

a) We first note that repeating any index gives us zero, which will agree w/ our generators. So, we only need to check indices that don't repeat.

$(X_1)_{23} = 1 = \varepsilon_{123}$, $(X_3)_{12} = 1 = \varepsilon_{312}$, $(X_2)_{31} = 1 = \varepsilon_{231}$

$(X_1)_{32} = -1 = \varepsilon_{132}$, $(X_3)_{21} = -1 = \varepsilon_{321}$, $(X_2)_{13} = -1 = \varepsilon_{213}$

which all agree.

b) $$[X_1, X_2] = X_1X_2 - X_2X_1 = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = -X_3$$

c) These are the same as the J_i's for a spin-1 system, but differ by a factor of $i\hbar$.

d) $$\Omega_1 = X_1X_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \Omega_2 = X_2X_2 = X_3X_3 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Omega_2 X_1 - X_1 \Omega_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

$$\Omega_2 X_2 - X_2 \Omega_2 = X_2X_2X_2 - X_2X_2X_2 = 0 \qquad \Omega_2 X_3 - X_3 \Omega_2 = X_3X_3X_3 - X_3X_3X_3 = 0$$

Problem 5.4

a) $\sigma_1\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\sigma_2\sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\sigma_3\sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

b) This follows naturally from noticing that $\sigma_i\sigma_i = \mathbb{1}$ and $\sigma_i\sigma_j = i\epsilon_{ijk}\sigma_k$, $i\neq j$.
So, $\sigma_i\sigma_j = \delta_{ij} + i\epsilon_{ijk}\sigma_k \quad \forall i,j,k \in \{1,2,3\}$.

c) $(\vec{\sigma}\cdot\vec{u})(\vec{\sigma}\cdot\vec{v}) = (\sigma_i u_i)(\sigma_j v_j) = u_i v_j \sigma_i \sigma_j = u_i v_j (\delta_{ij} + i\epsilon_{ijk}\sigma_k)$

$= u_i v_j \delta_{ij} + i\epsilon_{ijk} u_i v_j \sigma_k$

$\boxed{= \vec{u}\cdot\vec{v} + i(\vec{u}\times\vec{v})\cdot\vec{\sigma}}$

d) $U(\theta) = e^{-i\frac{\theta}{2}(\hat{\theta}\cdot\vec{\sigma})} = \sum_{n=0}^{\infty} \frac{(-i\frac{\theta}{2})^n}{n!}(\hat{\theta}\cdot\vec{\sigma})^n = \sum_{n=0}^{\infty} \frac{(\frac{\theta}{2})^{2n}}{(2n)!}(\hat{\theta}\cdot\vec{\sigma})^{2n} + \sum_{n=0}^{\infty}\frac{(\frac{\theta}{2})^{2n+1}}{(2n+1)!}(\hat{\theta}\cdot\vec{\sigma})^{2n+1}(-1)^{2n+1}$

$\boxed{= \cos(\theta/2) - i(\hat{\theta}\cdot\vec{\sigma})\sin(\theta/2)}$

Problem 5.5

a) Orthogonality of R, and determinant 1 gives us the $SO(2)$ group. Further, since
$R^T(\theta)R(\theta) = e^{\theta X^T} e^{\theta X} = e^{\theta(X^T + X)} \Rightarrow X^T + X = 0 \Rightarrow \boxed{X^T = -X}$.

b) We again get rotations, but no longer in $\mathbb{R}^2$, and instead in $\mathbb{R}^n$ depending on J.
So, J is the generator of rotations. So,

$R^T R = e^{i\theta J^T/\hbar} e^{-i\theta J/\hbar} = e^{i\theta(J^T - J)/\hbar} \Rightarrow \boxed{J^T = J}$

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Chapter 5

Lie Algebra

This is chapter 3 of [1]...

5.1 Generators

From a physics standpoint we don't really care about all of the elements of a Lie group, just what are known as the **generators**, the infinitesimal elements of the group. Because Lie groups have a continuous parameter we can form the Taylor expansion of an infinitesimal group element about the identity transformation. [**Hint:** the remainder of this section follows the text closely]

Consider an element of a Lie group, $g \in G$, that is infinitesimally close to the identity, I,

$$g(\epsilon) = I + \epsilon X,$$

in which X is known as the **generator** and ϵ is an infinitesimal. I can build up a group element that is a finite distance from the identity, θ, as $\left(I + \frac{\theta}{N} X\right)^N$, and take the limit as $N \to \infty$.

By using the binomial theorem this limit can be shown to be exactly the exponential function,[1]

$$g(\theta) = \lim_{N \to \infty} \left(I + \frac{\theta}{N} X\right)^N = e^{\theta X}.$$

It is in this sense that X is termed the **generator** of the finite transformation $g(\theta)$.

If I have an expression for the Lie group element, $g \in G$, then I can find the generator by differentiating with respect to the parameter, θ, and evaluating the resulting expression

[1]Using

$$n! = n(n-1)(n-2)\cdots(n-k+2)(n-k+1)\ (n-k)!$$

the binomial theorem gives me

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} \left(\frac{x}{n}\right)^k = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} \left(\frac{x}{n}\right)^k = \sum_{k=0}^{n} \left(\frac{x^k}{k!}\right) \frac{n(n-1)\cdots(n-k+1)}{n^k}$$

and in the $n \to \infty$ limit the last fraction is 1, leaving $\sum_{k=0}^{n} \left(\frac{x^k}{k!}\right)$ which is equal to the exponential, e^x.

at $\theta = 0$,

$$X = \left.\frac{dg}{d\theta}\right|_{\theta=0}.$$

For SO(2) I have $R(\theta) = \left[\begin{smallmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{smallmatrix}\right]$ so

$$X = \left.\frac{dR}{d\theta}\right|_{\theta=0} = \left.\begin{bmatrix} -\sin\theta & -\cos\theta \\ \cos\theta & -\sin\theta \end{bmatrix}\right|_{\theta=0} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

5.1.1 Exponentiation of Operators

Before we continue we need to understand the exponentiation of operators, $g(\theta) = e^{\theta X}$ (aka the exponential map between group elements, $g(\theta)$ and generators, X, of the Lie algebra). This is best seen with a simple example.

Example 5.1 The Generator of Rotations in 2D.
Consider the matrix $X = \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$ and compute (a) X^k for $k \in \mathbb{N}$, (b) $e^{\phi X}$ by writing the Taylor expansion, and (c) the finite rotation $\lim_{N\to\infty}(1+\Delta\phi X)^N$ made up of the infinitesimal rotation $\Delta\phi = \phi/N$, and (d) $\left.\frac{d}{d\phi}\left[\begin{smallmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{smallmatrix}\right]\right|_{\phi=0}$ ♦

Solution

(a) I have

$$\begin{aligned} X^0 &= \mathbb{1} \\ X^1 &= X \\ X^2 &= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = -\mathbb{1} \\ X^3 &= X^2 X = -\mathbb{1}X = -X \end{aligned}$$

so there are two cases,

$$X^{2k} = (-1)^k\mathbb{1} \quad \& \quad X^{2k+1} = (-1)^k X.$$

(b) I have

$$\begin{aligned} e^{\phi X} &= \sum_{k=0}^{\infty} \frac{(\phi X)^k}{k!} \\ &= \sum_{k=0}^{\infty} \frac{\phi^{2k}}{(2k)!}(-1)^k X^{2k} + \sum_{k=0}^{\infty} \frac{\phi^{2k+1}}{(2k+1)!}(-1)^{k+1} X^{2k+1} \\ &= \mathbb{1}\sum_{k=0}^{\infty} \frac{\phi^{2k}}{(2k)!}(-1)^k - X\sum_{k=0}^{\infty} \frac{\phi^{2k+1}}{(2k+1)!}(-1)^k \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\cos\phi - \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\sin\phi \\ &= \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix}, \end{aligned}$$

the rotation matrix!

(c) I remember this from calculus class!

$$\lim_{N\to\infty}(1+\Delta\phi X)^N = \lim_{N\to\infty}(1+X\frac{\phi}{N})^N = e^{\phi X}.$$

This is probably why X is called a generator.

(d) Easy-peasy!

$$\frac{d}{d\phi}\begin{bmatrix}\cos\phi & \sin\phi\\ -\sin\phi & \cos\phi\end{bmatrix}\Bigg|_{\phi=0} = \begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix} = X.$$

♦

Problem 5.1 Generators of SU(2).
The SU(2) group comes up repeatedly in particle physics so it behooves us to understand its mathematical properties in detail. The standard representation of the Lie algebra $\mathfrak{su}(2)$ is the 2×2 Pauli matrices. We can construct the $n\times n$ matrix representation of $\mathfrak{su}(2)$ by using the S_z basis $\hat{e}_1 = [1,0,0,\cdots]^\intercal$, etc, (*i.e.* the basis in which $\hat{S}_z$ is diagonal) and using the ladder operators, $\hat{S}_\pm \equiv \hat{S}_x \pm i\hat{S}_y$, which have the property (see section 3.4)

$$\hat{S}_\pm\,|s,m\rangle = \hbar\sqrt{s(s+1)-m(m\pm 1)}\,|s,m\pm 1\rangle\,.$$

(a) Construct the trivial 1×1 representation. (b) Construct the 3×3 representation for a spin-1 particle. [**Hint:** Calculate the matrix elements $\langle m|\hat{S}_z|m'\rangle$ and then $\langle m|\hat{S}_\pm|m'\rangle$.] ♦

5.1.2 Generators in Quantum Mechanics

In quantum mechanics the generators often turn out to be the observables and so we want them to be Hermitian. Multiplying the generators by a factor of $\mathring{\imath} = \sqrt{-1}$ will make them Hermitian. However, this destroys the anti-symmetric property of the generators, $X = -X^\intercal$, and replaces it with the Hermitian condition, $X = X^\dagger$. This obscures the fact that the generators have zeros along the diagonal (a property of anti-symmetric matrices).

Note that momentum and displacement are conjugate quantities, and that their product has dimensions of action (the same as $\hbar$).

It is often convenient to find the infinite dimensional representation of the translation operators because these become the more familiar differential operators from quantum mechanics.

Example 5.2 The Generator of Translation.
Consider a function of position, $f(x)$, and what happens when I translate the origin by a by (a) calculating the Taylor expansion of $f(x+a)$, (b) rewriting the expansion as an exponential of $a\frac{d}{dx}$. (c) Because we will want our physics operators to be Hermitian, introduce the familiar momentum operator,[2] $\hat{p}_x \equiv -\mathring{\imath}\hbar\frac{d}{dx}$ so that the Taylor expansion can be written

[2]Here $\hbar$ is Planck's constant which has units of action (aka energy-time or momentum-length or just plain angular momentum).

as $f(x+a) = \hat{P}_a f(x)$. (Note that an exponential of an operator, $\frac{d}{dx}$, is to be understood as a shorthand for the Taylor expansion.) ♦

Solution

$$\begin{aligned} f(x) \to f(x+a) &= f + a\,\frac{d}{dx}f + \frac{(a)^2}{2!}\frac{d^2}{dx^2}f + \cdots \\ &= \sum_{n=0}^{\infty}\frac{1}{n!}\left(a\,\frac{d}{dx}\right)^n f(x) \\ &= \exp\left(a\,\frac{d}{dx}\right)f(x) \\ &= \exp\left(\tfrac{i}{\hbar}a\,\hat{p}_x\right)f(x), \quad \hat{p}_x \equiv -i\hbar\frac{d}{dx} \\ &= \hat{P}_a f(x), \end{aligned}$$

so $\hat{P}_a \equiv \exp\left(\frac{i}{\hbar}a\,\hat{p}_x\right)$ is the operator that takes $f(x) \to f(x+a)$. ♦

This can be generalized to three dimensions,

$$\hat{P}_{\vec{a}} f(\vec{x}) = \exp\left(\tfrac{i}{\hbar}\vec{a}\cdot\hat{\vec{p}}\right)f(\vec{x}) = f(\vec{x}+\vec{a}),$$

where $\hat{\vec{p}} \equiv -i\hbar\vec{\nabla}\cdot$. We say that the momentum operator, $\hat{\vec{p}} = -i\hbar\vec{\nabla}\cdot$, is the generator of space translations. As we shall see, this is only possible because the $\hat{p}_i$ commute with each other.

Problem 5.2 The Generator of Rotation.
The group of rotations in a plane, known as SO(2), has only one degree of freedom and in Example 5.1 we learned that the generator is $X = \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$. In a manner similar to Example 5.2, find the generator as a differential operator using the Taylor expansion of $f(R(\theta)\vec{x}) = f((\mathbb{1} + \theta\, X)\vec{x})$. [**Hint:** Introduce the familiar angular momentum operator, $\hat{L}_3 \equiv i\hbar(x_1p_2 - x_2p_1)$.] ♦

For SO(2) we learned that if we introduce $L_z \equiv -i\hbar X$, then

$$R(\theta) \equiv \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix} = e^{\phi X} = e^{i\phi L_z/\hbar},$$

and L_z is the observable we have in quantum mechanics.

Note that we cannot extend this to three dimensions as with did for the case of translations because the operators in this case (*i.e.* rotations in 3D),

$$\exp\left(-\tfrac{i}{\hbar}\theta_1\,\hat{L}_1\right)\exp\left(-\tfrac{i}{\hbar}\theta_2\,\hat{L}_2\right)\exp\left(-\tfrac{i}{\hbar}\theta_3\,\hat{L}_3\right)f(\vec{x}) \neq \exp\left(-\tfrac{i}{\hbar}\theta_1\,\hat{L}_1 - \tfrac{i}{\hbar}\theta_2\,\hat{L}_2 - \tfrac{i}{\hbar}\theta_3\,\hat{L}_3\right),$$

because the $\hat{L}_i$ do not commute, $\hat{L}_1\hat{L}_2 - \hat{L}_2\hat{L}_1 = [\hat{L}_1, \hat{L}_2] \neq 0$. This is investigated in Problem 5.3.

Example 5.3 The Time-Translation Generator and the Hamiltonian Operator. In a similar manner the generator of time-translation is $\hat{H} = i\hbar\frac{\partial}{\partial t}$, the Hamiltonian. See also the Wikipedia article on translation operators for details. Finish this example. ♦

5.2 Lie Algebra

Definition of an Algebra

An algebra consists of a vector space, $\mathcal{V}$, over a field, $\mathcal{F}$, with a law of composition, $\mathcal{V}\times\mathcal{V} \to \mathcal{V}$ which is distributive (aka bilinearity),

$$\begin{aligned} A_1 \circ (a_2 A_2 + a_3 A_3) &= a_2 A_1 \circ A_2 + a_3 A_1 \circ A_3 \\ (a_1 A_1 + a_2 A_2) \circ A_3 &= a_1 A_1 \circ A_3 + a_2 A_2 \circ A_3, \quad \forall\ A_i \in \mathcal{V} \text{ and } \forall\ a_1 \in \mathcal{F}. \end{aligned}$$

The field, $\mathcal{F}$, for us will be either the real or complex numbers.

See also the Wikipedia articles on Lie algebra representation, and Lie group representation.

Law of Composition for a Lie Algebra

The group operation for combining group elements, $g_1 \circ g_2$, with different generators, $g_1(\theta) = e^{\theta X}$ and $g_2(\theta) = e^{\theta Y}$ tells us what the binary operation for the Lie algebra should be,

$$\begin{aligned} g_1 \circ g_2 &\sim e^X \circ e^Y \\ &= (I + X + \tfrac{1}{2}X^2 + \dots) \circ (I + Y + \tfrac{1}{2}Y^2 + \dots) \\ &= \text{...a lengthy derivation by Baker-Campbell-Hausdorff omitted...} \\ &= \exp\left(X + Y + \tfrac{1}{2}[X,Y] + \tfrac{1}{12}[X,[X,Y]] - \tfrac{1}{12}[Y,[X,Y]] + \dots\right), \end{aligned}$$

which is the Baker-Campbell-Hausdorff formula. This tells us how to combine generators – recall that when X is an operator, e^X is just a shorthand for the Taylor series – which in general won't commute.

(aka a commutator, $[\_,\_]$, that gives the binary operation for the algebra – *i.e.* the Lie algebra does **not** use regular matrix multiplication)

Generators for a Lie Algebra

There will be one generator for each degree of freedom in the group transformations (*e.g.* for SO(3) there are $\frac{1}{2}n(n-1) = 3$ degrees of freedom and so three generators, while for SO(2) there are $\frac{1}{2}n(n-1) = 1$ degree of freedom and so only one generator). For a group, G, the set of generators, X_i, form a basis of corresponding Lie algebra, $\theta_i X_i \in \mathfrak{g}$, such that any element of the group G can be written as the exponential,

$$g(\vec{\theta}) = e^{\vec{\theta}\cdot\vec{X}} = e^{\theta_i X_i} \in G.$$

This, together with the Lie bracket, form the Lie algebra. This algebra is closed under the Lie bracket,

$$[A,B] \in \mathfrak{g}\ \forall\ A,B \in \mathfrak{g},$$

just as G is closed under its group operation,

$$g_1 \circ g_2 \in G \;\forall\; g_1, g_2 \in G.$$

Definition of Lie Algebra

A Lie algebra, $\mathfrak{g}$, is a vector space with a binary operation, $[\_, \_]$, and has the following three properties

Bilinearity

$$[a_1 X_1 + a_2 X_2, X_3] = a_1[X_1, X_2] + a_2[X_2, X_3]$$

and

$$[X_1, a_2 X_2 + a_3 X_3] = a_2[X_1, X_2] + a_3[X_1, X_3]$$

for numbers a_i (real or complex) and $\forall\; X_i \in \mathfrak{g}$.

Anti-commutativity

$$[X_1, X_2] = -[X_2, X_1] \;\forall\; X_i \in \mathfrak{g}.$$

The Jacobi Identity

$$[X_1, [X_2, X_3]] + [X_2, [X_1, X_2]] + [X_2, [X_3, X_1]] = 0 \;\forall\; X_i \in \mathfrak{g}.$$

Because we are using a matrix representation of Lie algebras these conditions are automatically fulfilled.

As we shall see, the generators of a group form the basis for a Lie algebra. Although the things we will be adding are matrices rather than vectors, they still have the above properties of a vector space. We will then be able to write any element of the algebra as a sum over the basis, $a_i X_i$, in which the a_i are (real or complex) constants.

Example 5.4 The Generators of SU(2).
The group SU(2) is the group of unitary 2×2 matrices with

$$U^\dagger U = U U^\dagger = 1 \quad \& \quad \det(U) = +1.$$

(a) With $U = \exp\left(\frac{\mathring{\imath}}{\hbar} J_i\right)$ show that the first condition results in the J_i being Hermition. (b) Use $\det\left(e^A\right) = e^{\mathrm{Tr}(A)}$ and the first condition to show that the J_i are traceless. (c) Starting with the fact that a complex 2×2 matrix has 8 degrees of freedom show that the two conditions above result in 3 remaining degrees of freedom. (d) The **Pauli matrices**[3] are three linearly independent elements of the vector space and so they can be used the basis for the vector space. Show that $[\sigma_i, \sigma_j] = 2\mathring{\imath}\varepsilon_{ijk}\sigma_k$. ♦

[3] Recall that the Pauli matrices are $\sigma_1 = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$, $\sigma_2 = \left[\begin{smallmatrix} 0 & -\mathring{\imath} \\ \mathring{\imath} & 0 \end{smallmatrix}\right]$, & $\sigma_3 = \left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right]$.

Solution

(a) I have

$$1 = U^\dagger U = \exp\Big(-\tfrac{i}{\hbar} J_i^\dagger + \tfrac{i}{\hbar} J_i + \tfrac{1}{2\hbar^2}[J_i^\dagger, J_i] + \cdots\Big),$$

so the J_i must be Hermitian, $J_i^\dagger = J_i$.

(b) The second condition is $\det(U) = +1$ so

$$1 = \det(U) = \det\big(\exp\big(\tfrac{i}{\hbar} J_i\big)\big) = \exp\big(\tfrac{i}{\hbar}\,\mathrm{Tr}(J_i)\big),$$

and the generators, J_i, must be traceless.

(c) The first condition, $UU^\dagger = 1$, imposes four constraints, and the second condition, $\det(U) = +1$, imposes one more constraint, leaving $8 - 4 - 1 = 3$ degrees of freedom.

(d) I have

$$[\sigma_1, \sigma_2] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} - \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} - \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix} = 2i\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = 2i\sigma_3$$

$$[\sigma_2, \sigma_3] = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} - \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix} = 2i\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = 2i\sigma_1$$

$$[\sigma_3, \sigma_1] = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = 2i\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = 2i\sigma_2$$

and the other three combinations follow from the anti-commutation condition, $[X_1, X_2] = -[X_2, X_1]$. I note that in the above I had $\sigma_i\sigma_j = -\sigma_j\sigma_i$ so for the Pauli matrices I have the general expression

$$[\sigma_j, \sigma_k] = \sigma_j\sigma_k - \sigma_k\sigma_j = \sigma_j\sigma_k - (-\sigma_j\sigma_k) = 2\sigma_j\sigma_k = 2i\varepsilon_{jk\ell}\sigma_\ell,$$

where $\varepsilon_{jk\ell}$ is the Levi-Civita antisymmetric tensor (don't be confused by $i = \sqrt{-1}$ not being an index in the above).

Because of the factor of 2 that appears in the commutator of the Pauli matrices it is common to define the generators of SU(2) as $J_i = \frac{1}{2}\sigma_i$, giving the Lie bracket

$$[J_i, J_j] = i\varepsilon_{ijk}J_k,$$

which is exactly that found for SO(3) in Problem 5.3. Consequently these two groups, SO(3) & SU(2), have the same Lie algebra (because the Lie bracket defines the algebra). ♦

Problem 5.3 The Generators of SO(3).
The group SO(3) consists of all rotations in 3D. As matrices, independent elements of SO(3) can be written as

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_1 & -\sin\theta_1 \\ 0 & \sin\theta_1 & \cos\theta_1 \end{bmatrix}, \quad \begin{bmatrix} \cos\theta_2 & 0 & -\sin\theta_2 \\ 0 & 1 & 0 \\ \sin\theta_2 & 0 & \cos\theta_2 \end{bmatrix}, \quad \begin{bmatrix} \cos\theta_3 & -\sin\theta_3 & 0 \\ \sin\theta_3 & \cos\theta_3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and the complete group consists of all products of these matrices.

Using

$$X = \left.\frac{dg}{d\theta}\right|_{\theta=0}.$$

and the above we can find the independent generators,

$$X_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \quad X_2 = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad X_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

which form a **basis** for all of the elements of the corresponding algebra, $\mathfrak{SO}(3)$,

$$X = a_i X_i$$

with the a_i real constants. (a) Show that the generators can be written as $(X_i)_{jk} = \varepsilon_{ijk}$, where ε_{ijk} is the Levi-Civita antisymmetric tensor. (b) Calculate the commutator $[X_1, X_2]$ and compare the result to X_3. (c) Compare the generators X_i to the usual angular momentum operators, J_i, for a spin 1 system. (d) Calculate $\Omega = X_i X_i$ and show that it is a Casimir operator for this algebra (*i.e.* $[\Omega, X_i] = 0$, it commutes with the other operators, X_i). ♦

5.2.1 The Distinguished Group

As we have see in Example 5.4 and Problem 5.3, it turns out that there can be many groups with the same Lie algebra but that only one of them, the distinguished group, forms a simply connected manifold. For SO(3) the distinguished group is SU(2) (recall that there is a two-to-one mapping from SU(2) to SO(3)).

Comment: Nature seems to be telling us that while we may think that SO(3) is the 3D symmetry we observe, the actual symmetry of nature is SU(2). See §3.4.4, p. 47 of [1] for a more in-depth discussion of this important point.

Abstract Definition of a Lie Group

Our textbook[1] gives us an abstract definition of a Lie group on p. 48,

> A Lie group is a group, which is also a differentiable manifold. Furthermore, the group operation, $\circ$, must induce a differentiable map of the manifold into itself. This is a compatibility requirement that ensures that the group property is compatible with the manifold property. Concretely this means that every group element, say A induces a map that takes any element of the group B to another element of the group $C = AB$ and this map must be differentiable. Using coordinates this means that the coordinates of AB must be differentiable functions of the coordinates of B.

Geometrically, this means that the distinguished group is simply connected, any closed curve can be reduced to a point. This is important because if the manifold isn't simply connected then derivatives may not be well defined.

For example, in SO(3) we can define a rotation by an axis (given by a unit vector, $\hat{e}$, specified by the usual polar and axial angles, θ & ϕ), and how much to rotation about his axis, ψ (which together with θ & ϕ form the Euler angles); these can be combined into an Euler vector, $\vec{\alpha} = \psi\hat{e}$.[4] However, the same rotation can be specified by the axis $-\hat{e}$ and angle $-\psi$, so there are two sets of Euler angles, $\{\theta, \phi, \psi\}$ and $\{\pi - \theta, \phi, -\psi\}$, that result in the same rotation – this manifold isn't simply connected! This is a property of SO(3) itself rather than the choice of using Euler angles to specify a 3D rotation. This why we need to use SU(2) (or quaternions) as our group rather than SO(3).

Problem 5.4 The Pauli Matrices.
The Pauli matrices are

$$\sigma_1 = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right], \quad \sigma_2 = \left[\begin{smallmatrix} 0 & -i \\ i & 0 \end{smallmatrix}\right], \quad \& \quad \sigma_3 = \left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right].$$

Show that (a) $\sigma_i\sigma_i = \mathbb{1}, i = 1, 2, 3$, (b) $[\sigma_i, \sigma_j] = \delta_{ij} + i\varepsilon_{ijk}$, (c) for any two vectors, $\vec{u}$ & $\vec{v}$, that $(\vec{\sigma} \cdot \vec{u})(\vec{\sigma} \cdot \vec{v}) = \vec{u} \cdot \vec{v} + i\vec{\sigma} \cdot (\vec{u} \times \vec{v})$, (d) $U(\theta) = \cos(\theta/2) - i(\hat{\theta} \cdot \vec{\sigma} \sin(\theta/2)$. [**Hint:** Here $\vec{\theta}$ is the Euler vector for the rotation and $\vec{\sigma} \equiv \hat{e}_i\sigma_i$ is a vector of 2×2 matrices, so $\vec{\sigma} \cdot \vec{u} = (\hat{e}_i\sigma_i) \cdot (\hat{e}_j u_j) = (\sigma_i u_j)\hat{e}_i \cdot \hat{e}_j = \sigma_i u_i$.] ♦

Problem 5.5 Matrix representation of $\mathfrak{so}(2)$.
(a) Use the conditions $R^\intercal R = \mathbb{1}$ and $\det(R) = +1$, along with the exponential map, $R(\theta) = e^{\theta X}$, to find the 2×2 matrix representation of the generator of $\mathfrak{so}(2)$. (b) Use, instead, $R^\dagger R = \mathbb{1}$ and $R(\theta) = e^{-i\theta J/\hbar}$ and draw a conclusion about the generator, J. [**Hint:** We like to use the second form, $R(\theta) = e^{-i\theta J/\hbar}$, in quantum mechanics.] ♦

[4]The Euler vector isn't really a vector because rotations in 3D don't commute. See the Wikipedia page for more interesting details.