Question 1

First we look at $\tau_1 = \{U \subset X | X \setminus U \text{ is finite or } X\}$, and check the three conditions of a topology. In particular:

- Clearly $X \in \tau_1$, since $X \setminus X = \emptyset$ which is finite, and $\emptyset \in \tau_1$ since clearly $X \setminus \emptyset = X$.
- Suppose $U_i \in \tau_1$ and consider the union of these open sets $\cup_i U_i$. In particular, we notice that

$$X \setminus \bigcup_i U_i = \bigcap_i (X \setminus U_i)$$

However, since $U_i \in \tau_1$, we have an intersection of sets of finite size, which is necessarily finite. Hence, is in τ_1.

- Finally, suppose $U_i \in \tau_1$ and a finite intersection of these sets, $\cap_{i=1}^n U_i$. Then,

$$X \setminus \bigcap_{i=1}^{n} U_i = \bigcup_{i=1}^{n} (X \setminus U_i)$$

but since $U_i \in \tau_1$, we have a finite union of finitely sized sets, which then must be finite, and thus in τ_1.

Now all that is left is to see if τ_1 is Hausdorff. Suppose $p_1, p_2 \in X$ such that $p_1 \neq p_2$, We propose that τ_1 is not Hausdorff, and show this by contradiction. Assuming that τ_1 is Hausdorff, then $\exists\ A, B \in \tau_1$ such that $p_1 \in A$, $p_2 \in B$ and $A \cap B = \emptyset$. We also see that neither of A nor B are $\emptyset$, since then one or both won't contain p_1 or p_2. Then, consider the complement,

$$X \setminus \left(A \bigcap B\right) = (X \setminus A) \bigcup (X \setminus B)$$

but by definition of this topology, $X \setminus A$ and $X \setminus B$ are finite, and a union of finite sets is finite. This is a contradiction, since this should be all of X, which is infinite. Hence, we have that τ_1 can not be Hausdorff.

Next we look at $\tau_2 = \{U \subset X | X \setminus U \text{ is infinite or } \emptyset\}$. Then:

- Clearly both $X, \emptyset \in \tau_2$ since $X \setminus X = \emptyset$ and $X \setminus \emptyset = X$ which is infinite.
- Suppose $U_i \in \tau_2$ for $i \in A$ an indexing set. Then, consider

$$X \setminus \bigcup_{i \in A} U_i = \bigcap_i (X \setminus U_i)$$

but since $U_i \in \tau_2$, $X \setminus U_i$ is either infinite or empty. If it is empty, then the intersection is in τ_2. Otherwise, we have a possibly infinite intersection of infinite sets, which is not necessarily infinite nor empty.

We see that τ_2 can not be a topology because of the second set.

Finally we look at the third possible topology,

- We see that $X \in \tau_3$ since the empty set is countable and $\emptyset \in \tau_3$ by definition.
- Suppose $U_i \in \tau_3$ with $i \in A$ indexing set. Notice,

$$X \setminus \bigcup_{i \in A} U_i = \bigcap_i (X \setminus U_i)$$

  which is the possibly infinite intersection of countable sets or X. If all $U_i = X$ then this is trivially in τ_3, so we suppose otherwise. We know however that the possibly infinite intersection of countable sets will necessarily be countable, and hence $\cup_i U_i \in \tau_3$.
- Suppose $U_i \in \tau_3$ with $i, n \in A$ indexing set. Consider

$$X \setminus \bigcap_{i \in A}^{n} U_i = \bigcup_i^n (X \setminus U_i)$$

  again we suppose that none of the arguments in the union are all of X since then we just get our set trivially in τ_3. Supposing otherwise, we recall that the finite intersection of countable sets is countable, and thus $X \setminus \cup_i^n U_i \in \tau_3$.

We see that τ_3 is indeed a topology.

Now we need to check if τ_3 is Hausdorff .

Question 2

(a) We start by showing that the three properties of a topology hold. Notice:

- Clearly $X, \emptyset \in \tau$ by definition.
- τ only contains X and $\emptyset$ so any combination of unions will be either X or τ necessarily.
- Again, we have only two elements to work with, and in particular the intersection of these elements will be X when $\emptyset$ isn't used in the intersection, and will be $\emptyset$ otherwise.

So we have that this set is indeed a topology on X. Indeed, since τ is finite, it is second countable, but notice that since we only have X or $\emptyset$, if we choose $p_1, p_2 \in X$ with $p_1 \neq p_2$, the only open set that contains either of these is X and obviously $X \cap X \neq \emptyset$, and hence is not Hausdorff.

(b) Again, we look at the three properties of a topology:

- Since $X \subseteq X$ and $\emptyset \subset X$, we have that $X, \emptyset \in \tau$.
- Suppose $U_i \in \tau$ and indeed we see $\cup_i U_i \subseteq X$ which then implies $\cup_i U_i \in \tau$.
- Suppose $U_i \in \tau$, then for finite n we have $\cap_{i=1}^n U_i \subseteq X$ which implies $\cap_{i=1}^n U_i \in \tau$.

Hence we see that τ is a topology on X. Further, suppose $x, y \in X$ with $x \neq y$, then clearly the singletons $\{x\}, \{y\} \in \tau$ since $\{x\} \subset X$ and $\{y\} \subset X$. However, bu supposition, $\{x\} \cup \{y\} = \emptyset$ and hence τ is Hausdorff. To see that τ is not necessarily countable, consider the set $X = \mathbb{N}$, then $\tau = \mathcal{P}(\mathbb{N})$ and by Cantor's Theorem we have that τ is uncountable.

(c) First, suppose (X, τ) is a 0-manifold. Then, we know that τ contains sets that are homeomorphic to singletons, but then since τ is a topology, then the union of these sets is also in τ, and hence τ contains all of the subsets of X, and is the discrete topology. We also have that τ is countable, by second countability of (X, τ), and hence (X, τ) is a countable discrete space.

Now suppose (X, τ) is a countable discrete space. Then, we have already shown that discrete spaces are Hausdorff, and hence all we require is locally homeomorphic to $\mathbb{R}^n$. However, we recall that the simplest way to show the Hausdorff property was to use the fact that all singletons in X are in τ, and we can again apply this to see that each singleton can be associated with a singleton, and is hence a 0-manifold.

Question 3

Suppose X a set and τ a topology on X such that (X, τ) is an n-manifold. We want to show this manifold has a basis consisting of elements in τ that are Euclidean Balls (homeomorphic to open balls in $\mathbb{R}^n$).

In particular, we will build a basis $\mathcal{B}$ by using the locally Euclidean property of (X, τ); $\forall x \in X\ \exists U \subseteq X$ such that U is homeomorphic to some open ball, $B_r(y)$, in $\mathbb{R}^n$, where $y \in \mathbb{R}^n$ and $r > 0$. We let $\mathcal{B}$ be the set of all such $U \subseteq X\ \forall x \in X$, and notice that such a $\mathcal{B}$ contains only Euclidian Balls.

To see that this is a basis of τ, suppose $V \in \tau$, then in particular we see that

Question 4

(a) To verify that the product topology is indeed a topology on $M_1 \times M_2$ we check the three standard properties:

- First, since $\emptyset \in \tau_1$, $\emptyset \in \tau_2$, $M_1 \in \tau_1$ and $M_2 \in \tau_2$ by supposition, we see that naturally $(\emptyset, \emptyset) \in (\tau_1, \tau_2)$ and $(M_1, M_2) \in (\tau_1, \tau_2)$.
- Suppose $(U_i, V_i) \in (\tau_1, \tau_2)$ a family of subsets. Then, notice that $\cup_i (U_i, V_i) = (\cup_i U_i, \cup_i V_i) \in (\tau_1, \tau_2)$ due to (M_1, τ_1) and (M_2, τ_2) being topological spaces.
- Suppose $(U_i, V_i) \in (\tau_1, \tau_2)$ a finite family of subsets. Then, we see that $\cap_{i=1}^n (U_i, V_i) = (\cap_{i=1}^n U_i, \cap_{i=1}^n V_i) \in (\tau_1, \tau_2)$ again due to the topological space assumption.

As a remark, we notice that the family of sets used for the second and third property assume the same number of subsets from τ_1 and τ_2, though we never required uniqueness among the family of each individual subset U_i and V_i and hence we allow for repeating sets and still retain the family in (τ_1, τ_2).

We see that all of the properties are satisfied and hence this product space is a topological space with topology (τ_1, τ_2).

First, suppose that both τ_1 and τ_2 are Hausdorff. Then, in particular, consider two distinct points (a_1, a_2) and (b_1, b_2) in $M_1 \times M_2$. We see that τ_1 and τ_2 being Hausdorff gives us 4 sets, $U_1, V_1 \in \tau_1$ and $U_2, V_2 \in \tau_2$ such that $a_1 \in U_1$, $a_2 \in U_2$, $b_1 \in V_1$, $b_2 \in V_2$ and $U_1 \cap V_1 = \emptyset$ and $U_2 \cap V_2 = \emptyset$. However, then we naturally see that we can just choose such sets to build the disjoint sets in (τ_1, τ_2), in particular $(a_1, a_2) \in (U_1, V_1)$ and $(b_1, b_2) \in (U_2, V_2)$ but by the above findings $(U_1, V_1) \cap (U_2, V_2) = \emptyset$. and hence (τ_1, τ_2) is Hausdorff on the product space.

Second, suppose both τ_1 and τ_2 are second countable. Then $\exists$ bases $\mathcal{B}_1, \mathcal{B}_2$ that are countable for τ_1 and τ_2 respectively. To see that this property extends to the product space, we choose our basis to be $(\mathcal{B}_1, \mathcal{B}_2)$. Clearly this basis will be countable since the direct product of countable sets is countable, but more importantly this is indeed a basis of (τ_1, τ_2). To see this explicitly, take some $(U, V) \in (\tau_1, \tau_2)$, and applying the basis definition to each set U, V respectively, we can conclude $\exists b_{1,i} \in \mathcal{B}_1, b_{2,j} \in \mathcal{B}_2$ such that $U = \cup_i b_{1,i}$ and $V = \cup_j b_{2,j}$ and hence $(U, V) = (\cup_i b_{1,i}, \cup_j b_{2,j})$. Hence (τ_1, τ_2) is second countable.

(b)

(c) In part **(a)** we showed that product spaces of topological spaces are toplogical spaces and also carry the Hausdorff and second countable nature of the paired topological spaces, assuming they both carry the respective property. Hence, the only thing left to show for this case is that the locally euclidean property also carries through to the product space.

In particular, suppose $(p_1, p_2) \in M_1 \times M_2$, then clearly $p_1 \in M_1$ and $p_2 \in M_2$, and by the locally euclidean property of each $\exists\, U \in \tau_1$ and $V \in \tau_2$ such that each is homeomorphic to $B_{r_1}(x_1) \subset \mathbb{R}^{n_1}$ and $B_{r_2}(x_2) \subset \mathbb{R}^{n_2}$ respectively with $p_1 \in U$, $p_2 \in V$, $x_1 \in \mathbb{R}^{n_1}$ and $x_2 \in \mathbb{R}^{n_2}$. Then, assuming that f_1 and f_2 are the homeomorphisms respectively, we see that first $B_{r_1}(x_1) \times B_{r_2}(x_2) \subset \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ is open under the topology $\mu_1 \times \mu_2$ where μ_1, μ_2 are the standard topologies of $\mathbb{R}^{n_1}$ and $\mathbb{R}^{n_2}$ respectively, and this follows from what we showed in **(a)**.

Then, we can propose that the map $F : M_1 \times M_2 \to \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ defined by $F(U \times V) \mapsto f_1(U) \times f_2(V)$ is a homeomorphism. To see that this is true follows directly from the fact that f_1 and f_2 are homeomorphisms and is a simple application of each. In particular, to see that F is continuous, suppose $X \times Y \subset \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ is open, then we see that $F^{-1}(X \times Y) = f_1^{-1}(X) \times f_2^{-1}(Y) \in \tau_1 \times \tau_2$ by continuity of f_1 and f_2. To see that the inverse is continuous, suppose $U \times V \in \tau_1 \times \tau_2$, then $F(U \times V) = f_1(U) \times f_2(V) \in \mu_1 \times \mu_2$, and hence F is a homeomorphism. Further, $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \cong \mathbb{R}^{n_1+n_2}$, and hence we have the locally euclidean property, and we can conclude this is a $(n_1 + n_2)$-topological manifold.

(d) This is an application of all of the technology we developed in the previous parts of this question. In particular, recall that S_1 is a toplogical 1-manifold, and hence, from **(a)** and **(b)** we know $S_1 \times S_1$ is a topological 2 manifold. This motivates an inductive proof of the statement.

The base case is the known result that S_1 is a topological 1-manifold. Suppose $\underbrace{S_1 \times \cdots \times S_1}_{n-1}$ is a topological $(n-1)$-manifold, then consider

$$\underbrace{S_1 \times \cdots \times S_1}_{n-1} \times S_1$$

but by the previous parts we showed that products of topological n-manifolds are topological manifolds of dimension that is the sum of their dimensions, so we have that $\underbrace{S_1 \times \cdots \times S_1}_{n}$ is a topological n-manifold, as required.