Question 1

(a)

i. We first need to verify that D is involutive. To see this, consider the Lie Bracket of the two vector fields:

$$[X,Y] = (y-0)\frac{\partial}{\partial x} + (0-x)\frac{\partial}{\partial y} + (0-0)\frac{\partial}{\partial z} = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}.$$

If we suppose $p \in U$ with $p = (x, y, z)$, then we get

$$[X,Y]_p = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}.$$

now, we need to find $a, b \in \mathbb{R}$ such that

$$[X,Y]_p = aX + bY \iff y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} = bz\frac{\partial}{\partial x} - az\frac{\partial}{\partial y} + (ay - bx)\frac{\partial}{\partial z}$$

$$\implies az = y \qquad bz = x \qquad ay - bx = 0$$

$$\implies a = \frac{y}{z} \qquad b = \frac{x}{z}$$

So, we see that $[X,Y]_p \in D_p \ \forall p \in U$, and hence by definition D is involutive over U.

ii. Start by finding the integral curves generated by each of these vector fields. First, with X we see that we need to solve the following ODEs,

$$\frac{dx}{dt} = 0 \qquad \frac{dy}{dt} = -z \qquad \frac{dz}{dt} = y$$

we first note that $x(t) = a$ for some $a \in \mathbb{R}$. The other two components are a coupled ODE, which we can solve with a simple substitution:

$$\frac{d^2y}{dt^2} = -\frac{dz}{dt} = -y \implies y(t) = A\cos(t) + B\sin(t)$$

$$\implies z(t) = -\frac{dy}{dt} = A\sin(t) - B\cos(t).$$

Thus, we have that our integral curves are

$$\gamma_X(t) = (a, A\cos(t) + B\sin(t), A\sin(t) - B\cos(t))$$

where the coeffiecents are determined depending on the point through which these curves pass at $t = 0$. For Y we see that we need to solve the ODEs

$$\frac{dx}{dt} = z \qquad \frac{dy}{dt} = 0 \qquad \frac{dz}{dt} = -x$$

which we see gives $y(t) = c$ for some constant c. Furthermore

$$\frac{d^2x}{dt^2} = \frac{dz}{dt} = -x \implies x(t) = C\cos(t) + D\sin(t)$$

$$\implies z = \frac{dx}{dt} = -C\sin(t) + D\cos(t)$$

and so we have that the integral curves all take the form of

$$\gamma_Y(t) = (C\cos(t) + D\sin(t), c, -C\sin(t) + D\cos(t)).$$

To see the comparison between the two curves, suppose we considered a point $p = (p_1, p_2, p_3) \in \mathbb{R}^3$ that these curves pass through, both at $t = 0$, for simplicity. Then

$$\gamma_X(0) = (p_1, p_2, p_3) = (a, A, -B) \qquad \gamma_Y(0) = (p_1, p_2, p_3) = (C, c, D)$$

and our new integral curves become

$$\gamma_X(t) = (p_1, p_2\cos(t) - p_3\sin(t), p_2\sin(t) + p_3\cos(t)) \qquad \gamma_Y(t) = (p_1\cos(t) + p_3\sin(t), p_2, -p_1\sin(t) + p_3\cos(t))$$

which are just circles in the yz-plane and xz-plane respectively. If all such integral curves on our integral submanifold for the distribution D are of this form, then we can conclude that the integral submanifold

(b)

i. We approach this by finding an integral curve through the origin such that the tangent to the curve lies in this distribution. Doing this for both X and Y will give us a good idea of what we can get for an integral submanifold.

For X, we know we just need to solve the ODE's

$$\frac{dx}{dt} = 1 \qquad \frac{dy}{dt} = 0 \qquad \frac{dz}{dt} = yz$$

from which we immediatly get

$$x(t) = t + b \qquad y(t) = c$$

for $b, c \in \mathbb{R}$. Then, we see that

$$\frac{dz}{dt} = yz = cz \implies z(t) = de^{ct}$$

But, we want our integral curve to go through the origin, so

$$\gamma(0) = (x(0), y(0), z(0)) = (b, c, d) \implies \gamma(t) = (t, 0, 0).$$

On the other hand, for Y, we see that we need to solve

$$\frac{dx}{dt} = 0 \qquad \frac{dy}{dt} = 1 \qquad \frac{dz}{dt} = 0$$

which we see gives us

$$x(t) = a \qquad y(t) = t + c \qquad z(t) = d$$

for $a, c, d \in \mathbb{R}$. Then, we need this to also go through the origin, so

$$\gamma(0) = (x(0), y(0), z(0)) = (a, c, d) \implies \gamma(t) = (0, t, 0)$$

So our integral curves from the basis vectors are the basis vectors on the $x - y$ plane in $\mathbb{R}^3$. Thus, we can conclude that an integral submanifold of D at the origin is the $x - y$ plane.

ii.

Question 2

First we show that **(a)** $\iff$ **(b)**. To see this, suppose a local chart $(U, \varphi(x_1, \ldots, x_n)$ of M. We need to show that the smoothness of ω as a map is the same as smoothness of the component functions locally, ω^i. We need, then, to consider how the two are related. Notice, we already know that the cotangent bundle is a smooth manifold, so we can use the associated chart on that; set $\tilde{U} = \tilde{\pi}^{-1}(U) \subset T^*M$, with

$$\tilde{\varphi} : \tilde{U} \subset T^*M \to \varphi(U) \times \mathbb{R}^n \subset \mathbb{R}^n \times \mathbb{R}^n$$

where

$$\tilde{\varphi}(p, \omega_p = \sum_{k=1}^{n} \omega_p^k d_p x_k) = (\varphi(p), (\omega_p^1, \ldots, \omega_p^n))$$

with $p \in U$. Consider some $a \in \varphi(u)$, then

$$\tilde{\varphi} \circ \omega \circ \varphi^{-1}(a) = \tilde{\varphi}(\omega_{\varphi^{-1}(a)})$$

and we can unpack this further,

$$\omega_{\varphi^{-1}(a)} = \sum_{k=1}^{n} \omega_{\varphi^{-1}(a)}^k d_{\varphi^{-1}(a)} x_k = \sum_{k=1}^{n} \omega^k(\varphi^{-1}(a)) d_{\varphi^{-1}(a)} x_k$$

so we can see that

$$\tilde{\varphi}(\omega_{\varphi^{-1}(a)}) = (\varphi(\varphi^{-1}(a)), (\omega^1 \circ \varphi^{-1}(a), \ldots, \omega^n \circ \varphi^{-1}(a))).$$

With this, we see that smoothness in ω will hold if and only if the component functions are also smooth ω^i $\forall i \in \{1, \ldots, n\}$.

Now, consider some smooth atlas $\{(U_\alpha, \varphi_\alpha)\}_\alpha$, giving the smooth structure to M. Then

$$\tilde{\varphi_\alpha} \circ \omega \circ \varphi_\beta^{-1} = \tilde{\varphi_\alpha} \circ \tilde{\varphi}^{-1} \circ \tilde{\varphi} \circ \omega \circ \varphi^{-1} \circ \varphi \circ \varphi_\beta^{-1}$$

which is still smooth since each part is smoothly compatible.

Now, we show that **(b)** $\iff$ **(c)**. Suppose first that $\omega(Y) \in C^\infty(M)$ for any $Y \in \mathfrak{X}(M)$. Then, if $(U, \varphi = (x_1, \ldots, x_n)$ is a local chart of M, we get

$$\omega^i = \omega\left(\frac{d}{dx_i}\right) \in C^\infty(M)$$

since $\frac{d}{dx_i} \in \mathfrak{X}(M)$.

Conversely, we suppose the component functions smooth, in particular $\omega^i \in C^\infty(U_\alpha)$, $\forall U_\alpha$ in our smooth atlas $\{(U_\alpha, \varphi_\alpha)\}$. Suppose $a \in \mathbb{R}^n$ and $Y \in \mathfrak{X}(M)$, then

$$\omega(Y) \circ \varphi^{-1}(a) : \varphi(U) \subset \mathbb{R} \to \mathbb{R}$$

$$\omega(Y) \circ \varphi^{-1}(a) = \omega_{\varphi^{-1}(a)}(Y_{\varphi^{-1}(a)}) = \sum_{i=1}^{\infty} \omega^i(\varphi^{-1}(a)) d_{\varphi^{-1}(a)} x_i(Y).$$

By supposition, we already know that ω^i is smooth, so all we need to show being smooth is $d_{\varphi^{-1}(a)} x_i(Y)$ $\forall i \in \{1, \dots, n\}$. We see that

$$d_{\varphi^{-1}(a)} x_i(Y) = Y_{\varphi^{-1}(a)}(x_i)$$

but $Y \in \mathfrak{X}(M)$, so we know that Y is a smooth vector field, and hence, $Y^i = Y(x_i)$ is a smooth function, and thus $d_{\varphi^{-1}(a)} x_i(Y)$ is a smooth function. Thus, we can conclude that $\omega(Y) \in C^\infty(M)$ for any $Y \in \mathfrak{X}(M)$.

Question 3

Suppose M, N smooth manifolds and $F : M \to N$ a smooth map. Further, we will need , $f \in C^\infty(N)$ and $\omega \in \Lambda^1(N)$.

(a) Let $X \in T_pM$ for some $p \in M$, $\omega, \eta \in \Lambda^1(N)$ and $a, b \in \mathbb{R}$. Then, we see that

$$F^*(a\omega + b\eta)(X) = (a\omega + b\eta)(F_{*,p}(X)) = a\omega(F_{*,p}(X)) + b\eta(F_{*,p}(X)) = aF^*(\omega)(X) + bF^*(\eta)(X)$$

and hence we have $\mathbb{R}$-linearity in the pullback.

(b) Let $X \in T_pM$ for some $p \in M$ and $f \in C^\infty(N)$. Then

$$F^*(df)(X) = df(F_{*,p}(X)) = d_{F(p)}f(F_{*,p}(X)) = F_{*,p}(X)(f) = X_p(f \circ F) = d_{F(p)}(f \circ f)(X) = d(f \circ F)(X)$$

as required.

(c) Let $X \in T_pM$ for some $p \in M$, $f \in C^\infty(N)$ and $\omega \in \Lambda^1(N)$. Then,

$$F^*(f\omega)(X) = (F_{*,p})^*(f(F(p))\omega_{F(p)})(X) = f(F(p))(F_{*,p})^*(\omega_{F(p)})(X) = (f \circ F)F^*(\omega)(X).$$

(d) Let $X \in T_pM$ for some $p \in M$, $(U, \varphi = (y_1, \dots, y_n)$ local chart of N, $\omega \in \Lambda^1(N)$ where $\omega = \sum_{i=1}^n \omega^i dy_i$ and $F = (F_1, \dots, F_n)$. Then,

$$F^*(\omega)(X) = (F_{*,p})^*(\omega_{F(p)})(X) = \omega_{F(p)}(F_{*,p}(X)) = \sum_{i=1}^n \omega^i_{F(p)} d_{F(p)}(y_i)(F_{*,p}(X)).$$

Notice that

$$d_{F(p)}(y_i)(F_{*,p}(X)) = (F_{*,p}(X))_{F(p)}(y_i) = X_p(y_i \circ F) = X_p(F_i) = d_{F(p)}F_i(X)$$

which then gives us that

$$F^*(\omega)(X) = \sum_{i=1}^n \omega^i_{F(p)} d_{F(p)}(y_i)(F_{*,p}(X)) = \sum_{i=1}^n (\omega^i \circ F)(p) d_p F_i(X)$$

as expected.

(e) We use the previous result. Supposing the same conditions as **(d)**, we see that we know

$$F^*(\omega) = \sum_{i=1}^n (\omega^i \circ F) dF_i$$

but we also know that ω is smooth, and thus ω^i is smooth and $(\omega^i \circ F) \in C^\infty(M)$ since F is also smooth. Moreover, since y_i is smooth as well, we get that $dF_i \in \Omega^1(M)$. Thus, since the right hand side of the equations is just a sum of smooth 1-forms on M, we get that

$$F^*(\omega) = \sum_{i=1}^{n} (\omega^i \circ F) dF_i \in \Omega^1(M)$$

as expected.

Question 4

(a)