Question 1

First we will find the order of this group. We see that if we consider all 2×2 matrices over $\mathbb{Z}_3$, then that is 3^4 possible elements. Since this is $GL_2(\mathbb{Z}_3)$, we know to expect less elements, since we require they be invertible. We recognize invertability can be found with the determinant, and that the determinant is zero only if we have pairing of elements. That is, suppose

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad a, b, c, d \in \mathbb{Z}_3 \,.$$

Then, we know that $det(A) = ad - bc = 0 \iff a = c \,\&\, b = d$ or $a = b \,\&\, c = d$, which is $4 \times 3^2 - 3$ elements (where the 4 comes from overlap of the three matrices with all elements the same). Then,

$$|GL_2(\mathbb{Z}_3)| = 3^4 - 4 \cdot 3^2 + 3 = 48 = 6 \cdot 8 = 3 \cdot 2^4 \,.$$

By the first Sylow theorem, we see that the size of the Sylow 3-subgroups is just 3. By the third Sylow theorem, we know

$$n_3 = 1 \,(\mathrm{mod}\, 3) \qquad \& \qquad n_3 | 2^4 = 16 \,.$$

So, our candidates for n_3 are 1,4 and 16. To actually find the subgroups, we only need find one and then apply the second Sylow theorem, since the rest better be conjugates of the one we find. From inspection, we notice that the three elements of the Sylow 3-subgroup better be the identity, some element of $GL_2(\mathbb{Z}_3)$ and its inverse. That is, we need a group that is generated by a single element and is of order 3. We notice that we can get

$$\{ \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ 2 & 1 \end{smallmatrix}\right) \}$$

as the first Sylow 3-subgroup. The others will just be conjugacy classes of this. In particular,

$$\{ \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix}\right) \} \quad \{ \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right) \} \quad \{ \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ 1 & 2 \end{smallmatrix}\right) \}$$

will be the other Sylow 3-subgroups. This is all there are, since anyother combination of entries will not be cyclic and of order 3.

Question 2

(a) To prove the correspondance theorem, we need to construct the appropriate subgroup structure. Suppose $\bar{H} \leq {}^{G}/_{N}$, and further suppose $a, b \in \bar{H}$. Then, since this is a coset subgroup, we know that $\exists g, h \in G$ such that $a = gN$ and $b = hN$. Since these representatives are in G, we can form a subset of G, call it H which contains all representatives of the cosets in $\bar{H}$. Now, clearly $H \subset G$, and we can see that if $g, h^{-1} \in H$, then

$$(gN)(h^{-1}N) = (gh^{-1})N \in \bar{H}$$

because $\bar{H}$ is a subgroup. Then, by definition of H, $gh^{-1} \in H \; \forall g, h \in H$. Thus, we can conclude that $H \leq G$, $N \subset H$ and $\bar{H} = {}^{H}/_{N}$.

(b) Suppose that $\bar{H} \trianglelefteq {}^{G}/_{N}$. By the correspondance theorem, we know that $\exists H \leq G$ such that $\bar{H} = {}^{H}/_{N}$. To see that H is also normal in G, we use the normality of the quotient. That is, suppose $g \in G$ and $h \in H$, then

$$(gN)(hN)(g^{-1}N) = (gh^{-1}g)N \in \bar{H}$$

by normality of $\bar{H}$. So, this implies that $ghg^{-1} \in H$, and hence H is normal in G, with $N \subset H$.

(c) Since $p \,|\, |G|$, we can suppose $|G| = p^n m$ where $p \nmid m$. Then, we consider the three cases caused by Lagrange's Theorem, ignoring the trivial case of $|N| = 1$, since then we just get back G in the quotient and of course the number of Sylow p-subgroups will be equivalent. Instead:

Case 1: Suppose $|N| \,|p$. Well, then $|N| \nmid m$, and $|N| \leq p^n$. Thus, we get that $|N| = p^k$, for $k \leq n$. So, we have that $|G/N| = |G|/|N| = p^{n-k}$, and clearly p divides this, so we apply our Sylow theory to this new group. In particular, we notice by correspondance, that every p-subgroup of ${}^{G}/_{N}$ will correspond with the quotient of a subgroup of size p^n in G, and hence a Sylow p-subgroup of G. Thus, if $\bar{n}_p$ is the number of Sylow p-subgroups of G/N, we have that $\bar{n}_p \leq n_p$. When $n = k$, we have a quotient group of a size that is not divisible by p, and hence by Lagrange's Theorem can not have Sylow p-subgroups.

Case 2: Suppose $|N| \,|\, m$, then $|N| \nmid p$. Suppose $m/|N| = q < m$, then $|G/N| = |G|/|N| = p^n m/|N| = p^n q$. So, we can again apply our Sylow theory, and we see that

$$\bar{n}_p \,|\, q \quad n_p \,|\, m \quad \& \quad \bar{n}_p = n_p = 1 (\operatorname{mod} p) \,.$$

But, obviously $q \,|\, m$, and $q < m$, so we must have that $\bar{n}_p < n_p$.

Case 3: Suppose $|N| \,|\, p^n m$, that is, $p^n m/|N| = p^k q$, for some $k \leq n$ and $q \leq m$, and $q \,|\, m$. The case when $|N| = |G|$ is ignored, since then we also get the trivial result of the quotient group being just the identity. Further, if $k = n$, we will get **Case 2**, and if $q = m$ we will get **Case 1**, so we can assume that $k < n$ and $q < m$. Then, we see that the quotient group is divisible by p and hence by Sylow theory, we know that there are Sylow p-subgroups. Suppose there are $\bar{n}_p$ of them, then

$$\bar{n}_p \,|\, q \quad n_p \,|\, m \quad \& \quad \bar{n}_p = n_p = 1(\operatorname{mod} p)\,.$$

Like **Case 2**, we must have that $\bar{n}_p < n_p$, as expected.

Question 3

(a)