Question 1

Suppose $\{|0\rangle, |1\rangle\}$ and orthonormal basis for M, then $\exists\, |\eta_1\rangle, |\eta_2\rangle \in R$, with $\alpha, \beta \in \mathbb{R}$ (where the vector space is over the field of real numbers) such that

$$|\psi\rangle_{RM} = \alpha\, |\eta_1\rangle \otimes |0\rangle + \beta\, |\eta_2\rangle \otimes |1\rangle \ .$$

We know that the M state is shared with Alice, so we get Alice and Bob to apply the teleportation protocol on:

$$|\psi\rangle_{RM} |\Phi_0\rangle = \frac{1}{\sqrt{2}}(\alpha\, |\eta_1\rangle \otimes |0\rangle (|00\rangle + |11\rangle) + \beta\, |\eta_2\rangle \otimes |1\rangle (|00\rangle + |11\rangle)) .$$

First, since Alice has the M state, she can apply the CNOT gate with her bit to get

$$\implies \frac{1}{\sqrt{2}}(\alpha\, |\eta_1\rangle \otimes |0\rangle (|00\rangle + |11\rangle) + \beta\, |\eta_2\rangle \otimes |1\rangle (|10\rangle + |01\rangle))$$

and then Alice applies the Hadamard to the M state to get

$$\implies \frac{1}{\sqrt{2}}(\alpha\, |\eta_1\rangle \otimes (|0\rangle + |1\rangle)(|00\rangle + |11\rangle) + \beta\, |\eta_2\rangle \otimes (|0\rangle - |1\rangle)(|10\rangle + |01\rangle)) .$$

It is now convenient to switch our ordering of the tensor product to act on Bob's ket instead, and rearrange to get

$$\implies \frac{1}{\sqrt{2}}(|00\rangle (\alpha\, |\eta_1\rangle \otimes |0\rangle + \beta\, |\eta_2\rangle \otimes |1\rangle) + |01\rangle (\alpha\, |\eta_1\rangle \otimes |1\rangle + \beta\, |\eta_2\rangle \otimes |0\rangle)$$
$$+ |10\rangle (\alpha\, |\eta_1\rangle \otimes |0\rangle - \beta\, |\eta_2\rangle \otimes |1\rangle) + |11\rangle (\alpha\, |\eta_1\rangle \otimes |1\rangle - \beta\, |\eta_2\rangle \otimes |0\rangle)) .$$

To complete the teleportation, Bob will apply a Von Neumann measurement along the M basis and Alice's basis and get his new state ket. Notice, if we relabel Bob's system as D, then the four possible outcomes are

$$\alpha\, |\eta_1\rangle \otimes |0\rangle + \beta\, |\eta_2\rangle \otimes |1\rangle$$

$$\alpha\, |\eta_1\rangle \otimes |1\rangle + \beta\, |\eta_2\rangle \otimes |0\rangle$$

$$\alpha\, |\eta_1\rangle \otimes |0\rangle - \beta\, |\eta_2\rangle \otimes |1\rangle$$

$$\alpha\, |\eta_1\rangle \otimes |1\rangle - \beta\, |\eta_2\rangle \otimes |0\rangle$$

all of which are a phase away from $|\psi\rangle_{RD}$ as required.

Question 2

We prove that this is impossible through contradiction. Suppose there was a process, T, in which Alice was to communicate 1 of the 2^{2n} classical messages to Bob through sending a $[2^{rn}]$-dimensional quantum system consuming some arbitrarily entangles state $|\eta\rangle$. We can't use the teleportation protocol on the $[2^{rn}]$-dimensional qauntum system since the entire process is to send it. We can, however, apply the Super Dense Coding scheme with $|\eta\rangle$ to communicate the messages. So, with Super Dense Coding, we have now communicated 1 of 2^{2n} messages to Bob by sending only $(2^{rn})^2 = 2^{2rn} < 2^{2n}$ messages. This contradicts the no discounted lunch theorem, in the presence of a quantum entangled state. Thus, by contradiction, it is impossible for Alice to communicate even 1 of the 2^{2n} messages to Bob by sending only a 2^{rn} dimensional quantum system.

Question 3

(a) We recall that

$$H = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix} \quad Z = \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix} \quad X = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$$

Then, we see that

$$HXH = \frac{1}{2}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}2 & 0\\ 0 & -2\end{bmatrix} = Z$$

as expected.

(b) To avoid drawing out the diagrams in tikz, I label them as 1, 2, and 3 from top to bottom. So, circuit 1 is the one with 4 hadamard matrices.

(1) We just need to expand the tensor expression, that is the first diagram can equivalently be written as

$$(H \otimes H)(CU)(H \otimes H) = (H \otimes H)\left(|0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes X\right)(H \otimes H)$$

where CU is the $CNOT$ gate. We notice that it will be convenient to use BraKet notation. In that notation, we recognize that $X = |1\rangle\langle 0| + |0\rangle\langle 1|$ and $H = \frac{1}{\sqrt{2}}\left((|0\rangle + |1\rangle)\langle 0| + (|0\rangle - |1\rangle)\langle 1|\right)$. So, first expanding our BraKet version of the circuit using tensor linearity,

$$(H \otimes H)(CU)(H \otimes H) = (H \otimes H)(|0\rangle\langle 0| \otimes I)(H \otimes H) + (H \otimes H)(|1\rangle\langle 1| \otimes X)(H \otimes H)$$

$$= H|0\rangle\langle 0|H \otimes HH + H|1\rangle\langle 1|H \otimes HXH$$

$$= \left(\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\right)\left(\frac{1}{\sqrt{2}}(\langle 0| + \langle 1|)\right) \otimes H^2 + \left(\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\right)\left(\frac{1}{\sqrt{2}}(\langle 0| - \langle 1|)\right) \otimes Z$$

where we recall that $H^2 = I$, so

$$= \frac{1}{2}(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| + |1\rangle\langle 1|) \otimes I + \frac{1}{2}(|0\rangle\langle 0| - |0\rangle\langle 1| - |1\rangle\langle 0| + |1\rangle\langle 1|) \otimes Z$$

$$= \frac{1}{2}(I + X) \otimes I + \frac{1}{2}(I - X) \otimes Z = \frac{1}{2}(I + X) \otimes (|0\rangle\langle 0| + |1\rangle\langle 1|) + \frac{1}{2}(I - X) \otimes (|0\rangle\langle 0| - |1\rangle\langle 1|)$$

$$= \frac{1}{2}(I + X + I - X) \otimes |0\rangle\langle 0| + \frac{1}{2}(I + X - I + X) \otimes |1\rangle\langle 1|$$

$$\implies (H \otimes H)(CU)(H \otimes H) = I \otimes |0\rangle\langle 0| + X \otimes |1\rangle\langle 1|$$

which is exactly what we wanted.

(2) We have

$$(CU)(I \otimes Z)(CU) = (|0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes X)(I \otimes Z)(|0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes X)$$

$$= (|0\rangle\langle 0| I |0\rangle\langle 0|) \otimes Z + (|1\rangle\langle 1| I |1\rangle\langle 1|) \otimes XZX + (|0\rangle\langle 0| I |1\rangle\langle 1| \otimes ZX + |1\rangle\langle 1| I |0\rangle\langle 0| \otimes XZ$$

where we know that $XZX = |1\rangle\langle 1| - |0\rangle\langle 0|$, and the cross terms vanish, since $\{Z, X\} = 0$,

$$= |0\rangle\langle 0| \otimes Z - |1\rangle\langle 1| \otimes Z = (\langle 0| |0\rangle - \langle 1| |1\rangle) \otimes Z = Z \otimes Z$$

as required.

(3) We see

$$(CU)(X \otimes I)(CU) = (|0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes X)(X \otimes I)(|0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes X)$$

$$= (|0\rangle\langle 0| X |0\rangle\langle 0|) \otimes I + (|1\rangle\langle 1| X |1\rangle\langle 1|) \otimes XIX + (|0\rangle\langle 0| X |1\rangle\langle 1| \otimes X + |1\rangle\langle 1| X |0\rangle\langle 0| \otimes X$$

$$= 0 + 0 + |0\rangle\langle 1| \otimes X + |1\rangle\langle 0| \otimes X = X \otimes X$$

as required.

Question 4

(a) We already know that $\{CNOT, H, T\}$ form a universal set of gates, so it would be weird to have just $\{CNOT, T\}$ form a universal set as well. So, we can guess that it will not be universal. To see this, we recall that the T gate and $CNOT$ gate will give us all the Pauli Matrices as gates, since T is just a rotation by $\frac{\pi}{4}$ about the z axis. However, this set does not allow us to produce H. Since H can't be produced, we have a non-universal set.

(b) It suffices to show this is universal by showing that we can produce each of $CNOT, H$ and T from $\{c - Z, K, T\}$. Clearly T is already done by inclusion, so all we need check now is $CNOT$ and H. First, we recall

$$\mathrm{c} - Z = |0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes Z \quad \& \quad T = |0\rangle\langle 0| + e^{i\frac{\pi}{4}}|1\rangle\langle 1| \,.$$

The matrix form can be recovered from the Ket form, this is just more compact. Then, using a computing software, we can find that

$$K^2T^2KT^2K^2 = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} = H \,.$$

So, this shows us that the Hadamard gate can be constructed by the K and T gates. Now we need to find $CNOT$. However, now that we can use H, we recall the relationship $HXH = Z$ which can be written as $HZH = X$ (since H is idempotent of degree 2), and thus

$$(I \otimes H)\mathrm{c} - Z(I \otimes H) = |0\rangle\langle 0| \otimes H^2 + |1\rangle\, 1 \otimes HZH = |0\rangle\langle 0| \otimes H^2 + |1\rangle\, 1 \otimes X = \mathrm{CNOT}\,.$$

Therefore, we have that this set is also universal.