Question 1

(a) We know where the basis vectors are being sent under this map, so we know exactly how this matrix should look. Specifically, since we know that basis vectors just pull out coloumns of matrices, we see that the coloumns of the matrix should be the output vectors. Thus,

$$U = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad \& \qquad V = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The braket notation will look like

$$U = |00\rangle\langle 00| + |01\rangle\langle 01| + |11\rangle\langle 10| + |10\rangle\langle 11| \quad \& \quad V = |00\rangle\langle 00| + |10\rangle\langle 01| + |01\rangle\langle 10| + |11\rangle\langle 11|$$

as required.

(b) We first note that V is symmetric and real valued, also known as hermitian, thus $V^\dagger = V$. The actions are

$$|00\rangle \to |00\rangle \quad |01\rangle \to |11\rangle \quad |10\rangle \to |10\rangle \quad |11\rangle \to |01\rangle$$

which tells us that the matrix representation better be

$$VUV^\dagger = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

The braket notation will look like

$$VUV^\dagger = |00\rangle\langle 00| + |11\rangle\langle 01| + |10\rangle\langle 10| + |10\rangle\langle 11|$$

as required.

(c) If we call I the identity matrix in $\mathbb{C}^2$, we are asked to find $T = H \otimes I$. To get the matrix representation, we first notice that

$$H|0\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle \qquad H|1\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle.$$

With this information, we can get the matrix representation to be

$$T = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}.$$

For UT, we get an action of

$$|00\rangle \to \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \quad |01\rangle \to \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$$

$$|10\rangle \to \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle) \quad |11\rangle \to \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$$

and a basis representation of

$$UT = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \end{bmatrix}$$

as required.

Question 2

(a) This is proven as a direct computation and application of the multilinearity of the tensor. Notice

$$(M \otimes I)|\Phi\rangle = (M \otimes I)\frac{1}{\sqrt{d}}\sum_{n=0}^{d}|nn\rangle = \frac{1}{\sqrt{d}}(M \otimes I)\sum_{n=0}^{d}(|n\rangle \otimes |n\rangle) = \frac{1}{\sqrt{d}}\sum_{n=0}^{d}(M|n\rangle \otimes I|n\rangle)$$

but since the kets $|n\rangle$ form the basis for M, $\exists a_{ij}$ such that $M = \sum_{ij=0}^{d} a_{ij}|i\rangle\langle j|$. Thus

$$(M \otimes I)|\Phi\rangle = \frac{1}{\sqrt{d}}\sum_{n=0}^{d}\left(\sum_{i,j=0}^{d} a_{ij}|i\rangle\langle j|n\rangle \otimes |n\rangle\right) = \frac{1}{\sqrt{d}}\sum_{n=0}^{d}\left(\sum_{i=0}^{d} a_{in}|i\rangle \otimes |n\rangle\right).$$

Applying the multilinearity of the tensor, we can move the sums around *and* the coefficient a_{in}, so

$$(M \otimes I)|\Phi\rangle = \frac{1}{\sqrt{d}}\sum_{i=0}^{d}\left(\sum_{n=0}^{d}|i\rangle \otimes a_{in}|n\rangle\right) = \frac{1}{\sqrt{d}}\sum_{i=0}^{d}\left(\sum_{n,m=0}^{d}|i\rangle \otimes a_{mn}|n\rangle\langle m|i\rangle\right)$$

$$= \frac{1}{\sqrt{d}}\sum_{i=0}^{d}\left(\sum_{n,m=0}^{d}|i\rangle \otimes a_{mn}|n\rangle\langle m|i\rangle\right) = \frac{1}{\sqrt{d}}\sum_{i=0}^{d}(I|i\rangle \otimes M^T|i\rangle)$$

$$= \frac{1}{\sqrt{d}}(I \otimes M^T)\sum_{i=0}^{d}(|i\rangle \otimes |i\rangle) = (I \otimes M^T)|\Phi\rangle$$

as required.

Question 3

(a) We know that the probabilities under a complete von Neumann measurement along the basis will simply give the basis vectors each with a probability given by the coefficient of the state. That is,

$$Prob(|00\rangle) = |a|^2 \quad Prob(|01\rangle) = |b|^2 \quad Prob(|10\rangle) = |c|^2 \quad Prob(|11\rangle) = |d|^2 \,.$$

(b) (i) We see that the projectors better just be the projectors in the qubit that Alice is using, and then leave Bob's qubit untouched. So,

$$P_{|0\rangle_A} = |0\rangle\langle 0| \otimes I \quad \& \quad P_{|1\rangle_A} = |1\rangle\langle 1| \otimes I$$

where we know $I = |0\rangle\langle 0| + |1\rangle\langle 1|$.

(ii) We expect the probability of measuring a "0" and "1" to be

$$||P_{|0\rangle_A}|\psi\rangle||_2^2 = ||a|00\rangle + b|01\rangle||_2^2 = |a|^2 + |b|^2 \quad \& \quad ||P_{|1\rangle_A}|\psi\rangle||_2^2 = ||c|10\rangle + d|11\rangle||_2^2 = |c|^2 + |d|^2$$

respectively. The post measurement states better be

$$|\psi\rangle_{|0\rangle_A} = \frac{1}{\sqrt{|a|^2 + |b|^2}}(a|00\rangle + b|01\rangle) \quad \& \quad |\psi\rangle_{|1\rangle_A} = \frac{1}{\sqrt{|c|^2 + |d|^2}}(c|10\rangle + d|11\rangle)$$

respectively.

(iii) First, we suppose the measurement outcome $|\psi\rangle_{|0\rangle_A}$. Then, we get the probability of a "0" and "1" to be

$$||P_{|0\rangle_B}|\psi\rangle_{|0\rangle_A}||_2^2 = \frac{1}{|a|^2 + |b|^2}||a|00\rangle||_2^2 = \frac{|a|^2}{|a|^2 + |b|^2} \quad \& \quad ||P_{|1\rangle_B}|psi\rangle_{|0\rangle_A}||_2^2 = \frac{1}{|a|^2 + |b|^2}||b|01\rangle||_2^2 = \frac{|b|^2}{|a|^2 + |b|^2}$$

respectively. That is

$$\text{Prob}(|0\rangle_B \,|\, |0\rangle_A) = \frac{|a|^2}{|a|^2 + |b|^2} \quad \text{Prob}(|1\rangle_B \,|\, |0\rangle_A) = \frac{|b|^2}{|a|^2 + |b|^2} \,.$$

In a similar way, if we suppose the measurement outcome $|\psi\rangle_{|1\rangle_A}$, then the probabilities of "0" and "1" measured by Bob will be