Question 1

(a) We prove the claim directly. Suppose that $G{=}Z(G)$ is indeed cyclic, then $9{=}; a\ 2\ G$ such that $aZ(G)$ generates the entire quotient group. We know that cosets partition the group, and in particular if $g; h\ 2\ G$, then $9 g_Z; h_Z\ 2\ Z(G)$ and $n; m\ 2\ \mathbb{Z}^+$ such that

$$g = a^n g_Z \qquad \& \qquad h = a^m h_Z :$$

Then, using the fact that g_Z and h_Z are in the centre,

$$gh = (a^n g_Z)(a^m h_Z) = a^n a^m g_Z h_Z = a^m a^n h_Z g_Z = (a^m h_Z)(a^n g_Z) = hg$$

and so we see that G is abelian.

(b) We rst use Lagrange's theorem, since we know $Z(G)$ is a normal subgroup of G, then we know that $jZ(G)$ divides jGj. So, we get that $jZ(G)j = 1; p$ or p_2. Well, if $jZ(G)j = p^2$, then $G = Z(G)$ and G is abelian.

Further, we recall that the centre of p-groups is non-trivial, so we know that $jZ(G)j \neq 1$, and hence we assume $jZ(G)j = p$. Then, $jGj{=}jZ(G)j = jG : Z(G)j = p$ and we recall that groups of prime order are cyclic, and hence from the above proof we know that G is abelian.

Thus, we have shown what is required.