Assignment 3

Due Friday April 4th before the end of the day

1 Perfect fluids and energy conditions (Marked by Pipo)

(a) Find the constraints on density and pressure imposed by the different energy conditions on a perfect fluid of equation of state $p = w\rho$ where w is a constant.

(b) What is the speed of sound of the fluid with equation of state $p = w\rho$? **Hint:** the speed of sound can be computed from the equation of state through dimensional analysis (don't use natural units for that analysis) ρ is an energy density and p is a pressure.

(c) From the previous parts, you already have the possible values of the speed of sound allowed by the energy conditions. Which ones are compatible with the speed of sound in the fluid being subluminal?

2 Let's make a star (Marked by Richard)

It is time to at least once, solve Einstein's equations. You have learned in class (through Birkhoff) that Schwarzschild metric gives the metric outside a static (time-independent), spherically symmetric mass such as a black hole or a spherical star. But what happens inside the star? We will complete our analysis of a static spherical star here, which will be a good approximation for e.g. the Sun.[1] This will also demonstrate how calculations in general relativity are typically 'difficult' to do by hand, even if the problem has a lot of symmetries (here we have spherically symmetric which is a *great* simplification).

Let us assume that the interior of the star is made of a *perfect fluid*[2]. The perfect fluid stress-energy tensor is given by

$$T^{\mu\nu} = (\rho + p)u^\mu u^\nu + pg^{\mu\nu} \,,$$

[1] Of course, the Sun rotates a little and hence not exactly spherically symmetric but the approximation is very good because the Sun's gravitational field is "weak" by relativistic standards.

[2] Recall that perfect fluid has isotropic pressure, no heat conduction and zero viscosity. Inside this fluid you will only feel pressure and no heat flow or 'friction' associated to moving inside a fluid.

where $p = p(r)$ is the pressure inside the fluid and $\rho = \rho(r)$ is the mass density of the fluid as measured in the comoving frame of the fluid, u^μ is the 4-velocity of the fluid and $g_{\mu\nu}$ is the metric of the star. The metric of a static spherically symmetric spacetime is given by

$$\mathrm{d}s^2 = -B(r)\mathrm{d}t^2 + A(r)\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2\,,$$

where $A(r), B(r)$ are unknown functions of radial coordinate r. To know the gravitational field (or curvature) generated by the star, it all boils down to solving Einstein's field equations

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G T_{\mu\nu}\,.$$

(a) First, show that the Einstein field equations can be written in another form which makes this particular problem much simpler, namely

$$R_{\mu\nu} = 8\pi G\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right)$$

where $T = T^\mu{}_\mu$, the trace of the stress-energy tensor.

(b) Using the comoving frame (where the fluid is at rest), compute all the nonzero components of $T_{\mu\nu}$ in terms of $A(r), B(r), \rho, p$.

(c) For the rest of this question it may be useful to use the following notation to remove clutter: $B' = \mathrm{d}B/\mathrm{d}r$. Compute the following nonzero Christoffel symbols (the others are zero):

$$\begin{array}{llll}
\Gamma^t{}_{rt}\,, & \Gamma^t{}_{tr}\,, & & \\
\Gamma^r{}_{tt}\,, & \Gamma^r{}_{rr}\,, & \Gamma^r{}_{\theta\theta}\,, & \Gamma^r{}_{\varphi\varphi}\,, \\
\Gamma^\theta{}_{r\theta}\,, & \Gamma^\theta{}_{\theta r}\,, & \Gamma^\theta{}_{\varphi\varphi}\,, & \\
\Gamma^\varphi{}_{r\varphi}\,, & \Gamma^\varphi{}_{\varphi r}\,, & \Gamma^\varphi{}_{\theta\varphi}\,, & \Gamma^\varphi{}_{\varphi\theta}\,.
\end{array}$$

(d) The components of Ricci tensor can be obtained from the Riemann tensor by contraction, i.e. $R_{\mu\nu} = R_{\mu\rho\nu}{}^\rho$. More explicitly, we get

$$R_{\mu\nu} = R_{\mu\rho\nu}{}^\rho = \partial_\sigma\Gamma^\sigma{}_{\mu\nu} - \partial_\mu\Gamma^\sigma{}_{\sigma\nu} + \Gamma^\rho{}_{\mu\nu}\Gamma^\sigma{}_{\rho\sigma} - \Gamma^\rho{}_{\sigma\nu}\Gamma^\sigma{}_{\rho\mu}\,.$$

Compute R_{tt} and show that Einstein field equation gives

$$\frac{B''}{2A} - \frac{B'}{4A}\left(\frac{A'}{A} + \frac{B'}{B}\right) + \frac{B'}{Ar} = 8\pi G\left(\frac{B(\rho + 3p)}{2}\right)\,. \tag{5}$$

(e) R_{tt} should already give a flavour how 'painful' computation of Ricci tensor components can be[3]. If you were to repeat the (masochistic) calculation for the other nonzero components, you would get[4]

$$R_{rr} = -\frac{B''}{2B} + \frac{B'}{4B}\left(\frac{A'}{A} + \frac{B'}{B}\right) + \frac{A'}{Ar} = 8\pi G\left(\frac{A(\rho - p)}{2}\right), \tag{6}$$

$$R_{\theta\theta} = 1 + \frac{r}{2A}\left(\frac{A'}{A} - \frac{B'}{B}\right) - \frac{1}{A} = 8\pi G\left(\frac{r^2(\rho - p)}{2}\right). \tag{7}$$

Now, by considering

$$\frac{R_{tt}}{2B} + \frac{R_{rr}}{2A} + \frac{R_{\theta\theta}}{r^2},$$

prove that

$$A(r) = \left(1 - \frac{2GM(r)}{r}\right)^{-1}, \quad \text{where } M(r) = \int_0^r 4\pi(r')^2\rho(r')\,\mathrm{d}r'. \tag{8}$$

Hint: you may find $\frac{\mathrm{d}}{\mathrm{d}r}(r/A)$ useful.

(f) If we define $B(r) := e^{2\phi(r)}$, then local divergence-free property of $T_{\mu\nu}$ i.e. $\nabla_\nu T^{\mu\nu} = 0$ gives[5]

$$\frac{\mathrm{d}p(r)}{\mathrm{d}r} = -\frac{\rho + p}{2}\frac{B'}{B} = -\frac{\rho + p}{2}\frac{\mathrm{d}}{\mathrm{d}r}\log|g_{tt}| = -(\rho + p)\frac{\mathrm{d}\phi(r)}{\mathrm{d}r}.$$

By using this and the expression $A(r)$ found previously on $R_{\theta\theta}$, show that[6]

$$\frac{\mathrm{d}\phi(r)}{\mathrm{d}r} = \frac{GM(r)}{r^2}\frac{1 + 4\pi p(r)r^3/M(r)}{1 - 2GM(r)/r}. \tag{9}$$

In the limit where $2GM/r \ll 1$ outside the star, what does this expression become?

Remark: To further solve this equation one needs a relationship between ρ (or equivalently M) and p, i.e., an equation of state. In that sense, the Sun will use a plasma equation of state, for a neutron star you would use something that would look like an atomic nucleus kind of equation of state, etc... The equation of state tells you about the nature of the matter that forms the star, and indeed it affects the motion of the test particles inside the star, but not outside.

[3] These were originally computed at the time when computers practically did not exist.

[4] The $R_{\varphi\varphi}$ does not add anything new since $R_{\varphi\varphi} = R_{\theta\theta}\sin^2\theta$ and $T_{\varphi\varphi} = T_{\theta\theta}\sin^2\theta$.

[5] Feel free to try it if you are bored.

[6] You need to demand hydrostatic equillibrium, which means that the pressure at the surface of the planet/star is zero. If it were more the star would keep expanding. If it were less, the star will collapse.

3 Tidal forces in GR (Marked by Erickson)

Consider two neighbouring test particles free-falling under the gravitational field created by a source. According to general relativity, these two particles follow timelike geodesics $\gamma_A(\tau), \gamma_B(\tau)$, where τ is the proper time of one of the particles (say, particle A). Let V be the 4-velocity vector field for these geodesics. If ξ is a separation vector between the two geodesics at proper time τ, i.e. $\xi^\mu(\tau) = x^\mu(\gamma_B(\tau)) - x^\mu(\gamma_A(\tau))$, then the geodesic deviation equation says that the second directional derivative along the geodesics is controlled the Riemann tensor (i.e. curvature):

$$\frac{D^2\xi^a}{d\tau^2} := \nabla_{\mathsf{V}}\nabla_{\mathsf{V}}\xi^a = -R_{cbd}{}^a\xi^b V^c V^d \tag{18}$$

This expression can be interpreted as the *tidal force per unit mass* of an extended object whose spatial extent is given by the spatial vector ξ^a (since the left hand side has units of acceleration, i.e. force per unit mass).

To a good approximation, we can approximate the line element of spacetime outside Earth using a spherically symmetric coordinate system $\{x^\mu\} = \{t, r, \theta, \varphi\}$, which reads

$$\mathrm{d}s^2 = -(1+2\Phi)\,\mathrm{d}t^2 + (1-2\Phi)\left(\mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\,\mathrm{d}\varphi^2\right), \qquad \Phi(r) = -\frac{GM}{r}, \tag{19}$$

where r is the radial coordinate with origin at the centre of the Earth, and Φ is the Newtonian gravitational potential. This is a “weak-field” approximation since relativistic corrections occur at order $O(\Phi^2)$, i.e. where $|\Phi| \ll 1$ (which holds even for the Sun: $|\Phi| \sim 10^{-6}$).

(a) If the spacetime is flat, find the general solution for $\xi^\mu(\tau)$. How does geodesic deviation equation account for initially non-parallel geodesics in flat spacetime?

Hint: think about the growth of ξ^a with τ.

(b) Consider the coordinate basis vectors $\{\partial_\mu\}$ associated with the coordinates $\{t, r, \theta, \varphi\}$ for the metric outside the Earth and define the separation vector ξ to be purely spatial (i.e. $\xi^0 = 0$ at $\tau = 0$). Let us assume that both particles are initially at rest in this coordinate system at $\tau = 0$. Show that in this basis, the geodesic deviation equation becomes

$$\frac{D^2\xi^a}{d\tau^2} = R^a{}_{ttb}V^tV^t\xi^b\,. \tag{20}$$

Remark: the geodesic deviation guarantees that the spatial vector ξ can remain spatial at all τ.

(c) For simplicity, let us assume that the two particles A and B are on the equatorial plane ($\theta = \pi/2$). Recall that the particles are at the same radial coordinate $r = R$ at $\tau = 0$.

- Sketch the setup.
- Show that to first-order in M, tidal forces bring the two particles (whose trajectory are initially parallel) *closer* together during free-fall (i.e. their geodesics *spatially converge*[7]). **Note:** You do **not** need to solve any differential equation to conclude that the geodesics converge.
- Using Earth's values $M_E = 6 \times 10^{24}$ kg, $R_E = 6400$ km, give an order of magnitude estimate[8] for this tidal effect (i.e. $\sim 10^x$ N for some x) for the tidal force experienced by a 1-meter rod aligned according to the setup above. If factors of c is needed, you can take the speed of light can be taken to be $c = 3 \times 10^8$ m/s.

Hint: in this setup, think about component(s) of ξ^a you need to evaluate. For example, you certainly do *not* need to compute ξ^θ since it is simply zero because the particles are both on the same equatorial plane. This will help you decide which component of Riemann tensor relevant for this problem.

4 Symmetries of Kerr geometry (AKA Killing me Hardly) (Marked by Richard)

In this exercise we will go through some preliminary steps on the relation between geodesics and constants of motion in general relativity. In classical mechanics, when the trajectory of a particle is constrained to some generalized surface due to symmetry of the problem we say that there are constants of motion. Since finding geodesics is so strongly related with solving a system of Euler-Lagrange equations, something similar happens when geodesics are analyzed. Using constants of motion to solve differential equations in classical mechanics is the figure of merit of the so-called Hamilton-Jacobi method[9]. This method proves especially useful in situations where orbital solutions for the geodesics are expected, for instance planets, stars, or black holes.

We will study in this exercise the Hamilton-Jacobi equation for a rotating black hole geometry described by the Kerr metric. In the Boyer-Lindquist coordinates the line

[7] This should not be surprising, since in Newtonian physics, gravity is already attractive. In fact, by choosing the appropriate set of geodesics, you can show that geodesic deviation underlies the phenomenon of *tidal forces* on Earth that causes the tides due to the Moon and the Sun.

[8] Though not required for this assignment, check online about what this order of magnitude of tidal stress is equivalent to and convince yourself that this amount makes sense. For example, the weight of an apple is about 1 N.

[9] A method, that can be argued is an important part of the toolbox of classical mechanics.

element reads[10]:

$$\mathrm{d}s^2 = -\frac{\Delta}{\Sigma}\left(\mathrm{d}t - a\sin^2\theta\mathrm{d}\varphi\right)^2 + \frac{\Sigma}{\Delta}\mathrm{d}r^2 + \Sigma\mathrm{d}\theta^2 + \frac{\sin^2\theta}{\Sigma}\left(\left(r^2+a^2\right)\mathrm{d}\varphi - a\mathrm{d}t\right)^2, \quad (27)$$

where

$$\Sigma = r^2 + a^2\cos^2\theta, \quad \Delta = r^2 - 2Mr + a^2, \quad a = J/M, \quad (28)$$

and where we work in Planck units (that for our purposes means $c = 1$, $G_N = 1$). We first want to find all the Killing vector fields of this geometry. As we know, Killing fields are associated with conserved quantities along geodesics (which are like constants of motion). These allow us (as you already made use of in classical mechanics) to integrate (or solve) the geodesic equations in this geometry.

(a) Find the two obvious Killing fields[11]. Is this a stationary geometry? What are the corresponding conserved quantities along geodesics?

Remark[12]: In fact, as it turns out, these are all the Killing vectors fields of this spacetime. In addition, we have the constraint that the normalization $\kappa = u^2$ of the tangent vector $u^\mu = \mathrm{d}x^\mu/\mathrm{d}\lambda$ is conserved for any geodesic motion. However, for integrability in four dimensions we need another constant of motion (given that we do not just consider motion in the equatorial plane with $\theta = \pi/2$). It turns out, though, that there exists a hidden conserved quantity that we could not find just by identifying the Killing vectors. This hidden symmetry is can be understood in terms of a tensor Killing field (of rank larger than 1). Instead of symmetries of a configuration space (with relation to isometries and hence with a intuitive geometrical interpretation), the higher-rank tensor symmetries rather represent the symmetries of phase space. Since their presence is not directly encoded in the geometry, but can be recognized by motion of particles and fields in the geometry, these symmetries are often called hidden symmetries. In our case this constant of motion is known as the Carter's constant and its role is similar to that of the Runge-Lenz vector for the Kepler problem.

(b) The **Hamilton-Jacobi equation** is equivalent to the Euler-Lagrange equation and Hamilton's equation of motion in mechanics: for a system with N generalized coordinates, the Hamilton-Jacobi equation takes the form of a **single** *first-order partial differential equation of N generalized coordinates* (instead of systems of N second-order ODEs for Euler-Lagrange equation and $2N$ systems of first-order PDEs for Hamilton's equations of motion).

[10] And therefore, you have the metric tensor coefficients $g_{\mu\nu}$

[11] Any cyclic coordinates?

[12] not a question, not a hint! just some material for your bedtime stories.

In curved spacetimes, Hamilton-Jacobi equation takes the form

$$\frac{\partial S}{\partial \lambda}+\frac{1}{2}g^{\mu\nu}\frac{\partial S}{\partial x^\mu}\frac{\partial S}{\partial x^\nu}=0, \tag{29}$$

where S is called Hamilton's principal function. **Prove** Eq. (29). You can use that:

- The principal function S is (up to a constant equal to the action)[13],

$$S(x^\mu,\lambda)=\int_{\lambda_0}^{\lambda}\mathrm{d}\lambda L(x^\mu,\dot{x}^\mu,\lambda)=\int_{\lambda_0}^{\lambda}\mathrm{d}\lambda\frac{1}{2}g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu\,. \tag{30}$$

- The principal function as a general line integral from (x_0,λ_0) to (x,λ) can be written as $S=\int_{x_0}^{x}\frac{\partial S}{\partial x^\mu}\mathrm{d}x^\mu+\int_{\lambda_0}^{\lambda}\frac{\partial S}{\partial\lambda}\mathrm{d}\lambda$.
- One can take variations of the principal function in the previous step (keeping the extremes constant), and write them in terms of the Hamiltonian that is defined via Legendre transformation $H=\frac{\partial L}{\partial\dot{x}^\mu}\dot{x}^\mu-L$,

(c) To show complete integrability of geodesic motion it is sufficient to prove that the Hamilton-Jacobi equation *additively separates*. By this we mean that the solution to the Hamilton-Jacobi equation can be written as the sum of functions of each variable, i.e.

$$S=-\frac{1}{2}\kappa\lambda+\mathcal{E}t-h\varphi-R(r)-\Lambda(\theta), \tag{31}$$

where[14] $\mathcal{E}=u_t$ and $h=-u_\varphi$. In order to write down the Hamilton-Jacobi equation explicitly, the only work we have to do is to invert the metric. Let's do this using the orthonormal frame formalism[15]. Since the metric tensor can be written as $\mathsf{g}=-\left(\Upsilon^1\right)^2+\left(\Upsilon^2\right)^2+\left(\Upsilon^3\right)^2+\left(\Upsilon^4\right)^2$, where[16] Υ^μ are the basis 1-forms, **find explicitly**[17] the dual basis vectors Υ_μ by imposing

$$\langle\Upsilon^\mu,\Upsilon_\nu\rangle=\Upsilon^\mu\left(\Upsilon_\nu\right)=\delta^\nu_\mu. \tag{32}$$

Hint: Remember that $\langle\mathrm{d}t,\partial_t\rangle=1,\langle\mathrm{d}\varphi,\partial_t\rangle=0,\langle\mathrm{d}\varphi,\partial_\varphi\rangle=1$.
Write down the inverse metric $\mathsf{g}^{-1}=-\left(\Upsilon_1\right)^2+\left(\Upsilon_2\right)^2+\left(\Upsilon_3\right)^2+\left(\Upsilon_4\right)^2$.

[13] Note that this definition of the action differs from the one given in the lectures. However, it can be shown that for a path parametrised by an affine parameter, the Euler-Lagrange equations for both Lagrangian definitions are equivalent. If the parametrisation is not affine, then the Euler-Lagrange equations for the definition here will satisfy the ones coming from the lecture definition, but the converse is not true.

[14] Do you recognize the first three terms? they look like something you may have seen before.

[15] This is your first contact with something called tetrads (or Vielbein), which is how grown-ups do local stuff in general relativity, like femionic fields, etc...

[16] the squares on the one-forms in the line element, mean tensor product, not multiplication of Υ_μ with itself, and definitely not a contraction.

[17] as a function of $\partial_r,\partial_\theta,\partial_\phi,\partial_t$

(d) Insert the ansatz (31) into the Hamilton-Jacobi equation. **Show** that this equation allows for separation of variables[18]; denote the corresponding separation constant[19] C.

Remark: If you feel a bit underwhelmed that you worked with a black hole metric that seems cool but you haven't really done any physics with yet, check the next question...

5 Symmetries of Kerr geometry II (Marked by Richard)

Now that we have identified another constant for our geodesics, and therefore we can fully integrate the problem, it's time to do some physics with it. In the following we will solve the geodesic equations (aka, the equations of motion).

(a) Let us remark that in the Hamilton-Jacobi theory one can identify $\partial_\mu S$ with the tangent vector u to the geodesic as follows: $\partial_\mu S = u_\mu$. This justifies the ansatz (31) for the Killing terms $\mathcal{E}$, h, as well as explains the fact that the Hamilton-Jacobi equation basically expresses that $\kappa = -2\partial S/\partial\lambda = g^{\mu\nu}u_\mu u_\nu = u^2$. **Can you find** the equations for $\dot{t}$, $\dot{r}$, $\dot{\theta}$ and $\dot{\varphi}$? (Hint: you don't need to use the geodesic equations for this; since one can write these equations of motion without using the geodesic equations the system is called 'completely integrable').

(b) Using $\partial_\mu S = u_\mu$, show that C is quadratic in the tangent vector to the geodesic and can be written as

$$C = K^{\mu\nu}u_\mu u_\nu, \quad K = r^2 g^{-1} + \frac{1}{\Delta}\left(\left(r^2+a^2\right)\partial_t + a\partial_\varphi\right)^2 - \Delta\partial_r^2. \tag{41}$$

Here, $K_{\mu\nu}$ is a Killing tensor[20], obeying $\nabla_{(\rho}K_{\mu\nu)} = 0$. Thus we see that C in fact derives from an additional (hidden) symmetry of the Kerr geometry that is more general than a Killing vector.

(c) In particular, consider the motion in the equatorial plane $\theta = \pi/2$. Show that the constant C is no longer independent of other constants and can be expressed as

$$C = (a\mathcal{E} - h)^2. \tag{42}$$

[18] in the PDEs kind of sense.

[19] This is the "famous" Carter's constant. B. Carter, Phys. Rev. 174, 1559 (1968)

[20] The previously mentioned generalisation of a Killing vector which is higher order in the momentum of the particle. One can also construct so-called Killing-Yano tensors. They always gives rise to a rank-2 Killing tensor. Such Killing tensors give symmetry operators for the Klein-Gordon equation. Their 'square roots', Killing-Yano tensors, give symmetry operators for the Dirac equation.

Show that in this case the radial equation of motion can be written as

$$\dot{r}^2 - \mathcal{E}^2 + V(r) = 0. \tag{43}$$

where $V(r)$ is an effective potential. **Find** the effective potential $V(r)$.

(d) A Kerr black hole is a generalization of a Schwarzschild black hole by allowing the black hole to rotate with an angular momentum J. Hence, show that, by taking the appropriate limit, you recover the equations of motion of lightlike and timelike geodesics in the equatorial plane.