Problem 1

(a) We solve this using simple algebra,

$$\begin{aligned} \frac{z}{1-z} &= 2-3i \\ z &= (1-z)(2-3i) \\ z+(z-1)(2-3i) &= 0 \\ z+2z-3iz-2+3i &= 0 \\ z(3-3i) &= 2-3i \end{aligned} \rightarrow \begin{aligned} z &= \frac{2-3i}{3-3i} \\ z &= \frac{(2-3i)(3+3i)}{18} \\ z &= \frac{6+6i-9i+9}{18} \\ z &= \frac{15-3i}{18} \end{aligned} \tag{1}$$

(b) We apply the quadratic formula

$$\begin{aligned} z &= \frac{(i-2) \pm \sqrt{(2-i)^2+4(4)(8)}}{16} \\ z &= \frac{i-2 \pm \sqrt{(4-4i-1)+128}}{16} \\ z &= \frac{i-2 \pm \sqrt{131-4i}}{16} \end{aligned}$$

Problem 2

(a) Assume $z = a+bi \in \mathbb{C}$, with $a, b \in \mathbb{R}$. $\text{Im}\{z\} = b$, and hence, $b > 0$.Furthermore,

$$\frac{1}{z} = \frac{\bar{z}}{z\bar{z}} = \frac{a-bi}{|z|^2} = \frac{a-bi}{a^2+b^2}$$

We aslo have, $\text{Im}\{\frac{1}{z}\} = -\frac{b}{a^2+b^2}$. However, $b > 0$, thus, $\text{Im}\{\frac{1}{z}\} < 0$.

(b) Assume $z \in \mathbb{C}$, $|z| = 1$ but $z \neq 1$. Take $z = a+bi$, $a, b \in \mathbb{R}$. In particular.

$$|z| = 1 \implies \sqrt{a^2+b^2} = 1, a \neq 1$$

$$\frac{1}{1-z} = \frac{1-\bar{z}}{(1-z)(1-\bar{z})} = \frac{1-\bar{z}}{1-\bar{z}-z+|z|^2} = \frac{1-(a-bi)}{1-(a-bi)-(a+bi)+a^2+b^2}$$
$$= \frac{1-a+bi}{1-2a+a^2+b^2}$$

In particular, we notice $\text{Re}\{\frac{1}{1-z}\} = \frac{1-a}{1-2a+a^2+b^2}$, yet $a \neq 1$. Yet, we note $a^2+b^2 = 1^2 = 1$, and thus

$$\frac{1-a+bi}{1-2a+a^2+b^2} = \frac{1-a}{2(1-a)} = \frac{1}{2}$$

Problem 3

(a) Notice the real and imaginary part are both 3, thus

$$\text{Arg}(3+3i) = \theta = \frac{\pi}{4}$$

(b) To compute $\arg(\frac{1+i}{2\sqrt{3}+2i})$, we first rewrite it in standard form,

$$\frac{1+i}{2\sqrt{3}+1} = \frac{(1+i)(2\sqrt{3}-2i)}{4(3)+4} = \frac{2\sqrt{3}-2i+2\sqrt{3}i+2}{16} = \frac{2(\sqrt{3}+1)-2(1-\sqrt{3})i}{16} = \frac{\sqrt{3}+1(\sqrt{3}-1)i}{8}$$

Thus, we can conclude $\arg(\frac{1+i}{2\sqrt{3}+2i}) = \arctan(\frac{\sqrt{3}-1}{\sqrt{3}+1}) + 2k\pi$.

(c) We know $-\pi \in \mathbb{R}$, in particular we know it is on the negative side of the real number line, thus, $\text{Arg}(-\pi) = -\pi$.

Problem 4

(a) We convert the right side of the equation to polar form. First find the norm,

$$|-1+\sqrt{3}i| = 2$$

We also have that $\text{Im}\{-1+\sqrt{3}i\} = \sqrt{3}$ and $\text{Re}\{-1+\sqrt{3}i\} = -1$, and thus $\theta = \frac{2\pi}{3}$. Hence,

$$z^3 = -1+\sqrt{3}i = 2e^{i\frac{2\pi}{3}} \implies z = (2)^{\frac{1}{3}}e^{i\frac{2\pi}{9}+\frac{2k\pi}{3}}$$

Thus we have $k \in \{0,1,2\}$, and in particular, $z_0 = (2)^{\frac{1}{3}}e^{i\frac{2\pi}{9}}$, $z_1 = (2)^{\frac{1}{3}}e^{\frac{2\pi}{3}(\frac{i}{3}+1)}$, $z_2 = (2)^{\frac{1}{3}}e^{\frac{2\pi}{3}(\frac{i}{3}+2)}$.

(b) We first get the standard form,

$$z^5 = \frac{-2i}{1+i} = \frac{-2i(1-i)}{(1+i)(1-i)} = \frac{-2i+2}{2} = 1-i$$

Now we rewrite it in polar from,

$$1-i \implies \theta = \frac{3\pi}{4}$$

and thus, $z^5 = \sqrt{2}e^{i\frac{3\pi}{20}} \implies z = (2)^{\frac{1}{10}}e^{\frac{\pi}{5}(i\frac{3}{4}+2k)}$ for $k \in \mathbb{N}, 0 \leq k \leq 4$.

Problem 5

(a) To compute this argument, we change the complex value into polar form,

$$|\sqrt{3}-i| = \sqrt{3+1} = 2$$

$$\mathrm{Arg}(\sqrt{3}-i) = \frac{5\pi}{6}$$

and thus,

$$z = 2^7 \left(\cos\left(\frac{35\pi}{6}\right) + i\sin\left(\frac{35\pi}{6}\right)\right)$$

(b) We solve this series by rewriting it in terms of the exponential form,

$$\sum_{n=0}^{\infty} \frac{\sin(n\pi)}{2^n} = \sum_{n=0}^{\infty} \frac{\Im\{e^{i\frac{n\pi}{40}}\}}{2^n} = \Im\left\{\sum_{n=0}^{\infty} \frac{e^{i\frac{n\pi}{40}}}{2^n}\right\} = \Im\left\{\sum_{0}^{\infty}\left(\frac{e^{i\frac{\pi}{40}}}{2}\right)^n\right\}$$

We recognize that this is exactly the geometric series, $a = 1$, $r = \frac{e^{\frac{\pi}{40}}}{2}$, and thus,

$$\Im\left\{\sum_{0}^{\infty}\left(\frac{e^{i\frac{\pi}{40}}}{2}\right)^n\right\} = \Im\left\{\frac{1-\left(\frac{e^{\frac{i\pi}{40}}}{2}\right)^{21}}{1-\frac{e^{\frac{i\pi}{40}2}}{2}}\right\} = \Im\left\{\frac{1-\frac{e^{\frac{i21\pi}{40}}}{2^{21}}}{1-\frac{e^{\frac{i\pi}{40}}}{2}}\right\} = \Im\left\{\frac{\left(1-\frac{e^{\frac{i21\pi}{40}}}{2^{21}}\right)\left(1-\frac{e^{\frac{-i\pi}{40}}}{2}\right)}{1-\frac{e^{\frac{-i\pi}{40}}}{2}-\frac{e^{\frac{i\pi}{40}}}{2}+\frac{1}{4}}\right\}$$

$$= \Im\left\{\frac{1-\frac{e^{\frac{-i\pi}{40}}}{2}-\frac{e^{\frac{i21\pi}{40}}}{2^{21}}+\frac{1}{2^{22}}e^{i\frac{20\pi}{40}}}{\frac{5}{4}-\frac{1}{2}\left(e^{\frac{i\pi}{40}}+e^{\frac{-i\pi}{40}}\right)}\right\} = \frac{1}{\frac{5}{4}-\cos\left(\frac{\pi}{40}\right)}\left(\frac{1}{2}\sin\left(\frac{\pi}{40}\right)-\frac{1}{2^{21}}\sin\left(\frac{21\pi}{40}\right)+\frac{1}{2^{22}}\sin\left(\frac{\pi}{2}\right)\right)$$

(Bonus)

Problem 6

Let $p(z)$ be a non-constant polynomial of degree $n \geq 1 \in \mathbb{N}$. Then, by the FTA, $\exists z_0 \in \mathbb{C}$ s.t. $p(z_0) = 0$. Since $z \in \mathbb{C}$ and $\mathbb{C}$ is a field, then the ring formed by the polynomials over this field is a Euclidean domain, and we can use the Euclidean algorithm

$$p(z) = (z - z_0)q_0(z) + r_0(z)$$

for $q_i(z), r_i(z) \in \mathbb{P}(\mathbb{C})$, which is the ring of polynomials over $\mathbb{C}$. In particular, notice that,

$$p(z_0) = (z_0 - z_0)q_0(z_0) + r_0(z_0)$$
$$0 = r_0(z_0)$$

Thus, we can apply the Euclidean algorithm again to r_0,

$$r_0 = (z - z_0)q_1 + r_1$$

First of all, by the algorithm, the degree of r_0 is necessarily less than the degree of q_0, which has a degree of $n - 1$ by equality. Doing this recursively, we note after n iterations, the remainder must reduce to $(z - z_0)$. By the distributive property of rings, we can rewrite this as,

$$p(z) = (z - z_0)h(z) \qquad h(z) = \sum_{i=0}^{n-1} q_i$$

we notice $h(z) \in \mathbb{P}(\mathbb{C})$, and thus by the FTA, $h(z_1) = 0$ for $z_1 \in \mathbb{C}$. We notice that $\deg(h) = n - 1$, and naturally we can recursively do this process again for $n - 2$ more times. Thus,

$$p(z) = (z - z_0)(z - z_1)\ldots(z - z_{n-1})$$

As required.

Problem 7

We approach this problem in the same way we would any problem with complex numbers and powers, we rewrite it in polar form.

$$\begin{gathered}
(z-2018)^{2n}+(z+2018)^{2n}=0\\
\left(|z-2018|e^{i\,\mathrm{Arg}(z-2018)}\right)^{2n}+\left(|z+2018|e^{i\,\mathrm{Arg}(z+2018)}\right)=0\\
|z-2018|^{2n}e^{i2n\,\mathrm{Arg}(z-2018)}+|z+2018|^{2n}e^{i2n\,\mathrm{Arg}(z+2018)}=0\\
|z-2018|^{2n}e^{i2n\,\mathrm{Arg}(z-2018)}=-|z+2018|^{2n}e^{i2n\,\mathrm{Arg}(z+2018)}\\
|z-2018|^{2n}e^{i2n\,\mathrm{Arg}(z-2018)}=|z+2018|^{2n}e^{i\pi}e^{i2n\,\mathrm{Arg}(z+2018)}
\end{gathered}$$

We take $n \geq 1$ and an integer. Furthermore, by definition of equality,

$$|z-2018|^{2n}=|z+2018|^{2n} \qquad e^{i2n\,\mathrm{Arg}(z-2018)}=e^{i\pi}e^{i2n\,\mathrm{Arg}(z+2018)}$$

we also get,

$$\begin{gathered}
i2n\,\mathrm{Arg}(z-2018)=i\pi+i2n\,\mathrm{Arg}(z+2018)\\
\mathrm{Arg}(z-2018)-\mathrm{Arg}(z+2018)=\frac{\pi}{2n}\\
\mathrm{Arg}\left((z-2018)(\bar{z}+2018)\right)=\frac{\pi}{2n}\\
\mathrm{Arg}(|z|^2+2018z-2018\bar{z}-2018^2)=\frac{\pi}{2n}
\end{gathered}$$

we let $z=a+bi$, for $a,b\in\mathbb{R}$. Then, expanding,

$$\mathrm{Arg}(a^2+b^2+2018a+2018bi-2018a+2018bi-2018^2)=\frac{\pi}{2n}$$

$$\operatorname{Arg}(a^2 + b^2 - 2018^2 + (2)2018bi) = \frac{\pi}{2n}$$
$$\arctan\left(\frac{2(2018)b}{a^2 + b^2 - 2018^2}\right) = \frac{\pi}{2n}$$
$$\frac{2(2018)b}{a^2 + b^2 - 2018^2} = \tan\left(\frac{\pi}{2n}\right)$$

We notice that this implies that a is identically zero and that z is necessarily only imaginary.