Question 1

(A) Problem 9.3 in Lee

(a) We consider the vector field $V = y\frac{\partial}{\partial x} + \frac{\partial}{\partial y}$ on $\mathbb{R}^2$. Suppose $p = (p_1, p_2) \in \mathbb{R}^2$, and say we wish to find the flow $\varphi_t(p) = \gamma_p(t)$. We find the maximal integral curve in the standard way; suppose $\gamma_p(t) = (x(t), y(t))$ and $\gamma_p(0) = p$, then

$$\gamma_p'(t) = \gamma_* \left(\frac{d}{dt}\bigg|_p \right) = \frac{dx}{dt}\frac{\partial}{\partial x} + \frac{dy}{dt}\frac{\partial}{\partial y} = y\frac{\partial}{\partial x} + \frac{\partial}{\partial y} = V \implies \frac{dx}{dt} = y \quad \& \quad \frac{dy}{dt} = 1$$

$$y(t) = t + c \qquad x(t) = \frac{1}{2}t^2 + ct + d$$

but we have our initial condition, which gives

$$\gamma_p(0) = (d, c) = (p_1, p_2) \implies \varphi_t((p_1, p_2)) = (\frac{1}{2}t^2 + p_2 t + p_1, t + p_2)$$

where $t \in \mathbb{R}$, and we see this is our flow.

(b) Let $W = x\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial y}$, and we can let $\varphi_t(p) = \gamma_p(t)$, where $p = (p_1, p_2) \in \mathbb{R}^2$. Then, if $\gamma_p(0) = p$ and $\gamma_p(t) = (x(t), y(t))$, we get

$$\gamma_p'(t) = \gamma_* \left(\frac{d}{dt}\bigg|_p \right) = \frac{dx}{dt}\frac{\partial}{\partial x} + \frac{dy}{dt}\frac{\partial}{\partial y} = x\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial y} = W \implies \frac{dx}{dt} = x \quad \& \quad \frac{dy}{dt} = 2y$$

$$x(t) = Ae^t \qquad y(t) = Be^{2t}.$$

Applying our initial condition,

$$\gamma_p(0) = (A, B) = (p_1, p_2) \implies A = p_1 \quad \& \quad B = p_2$$

which then gives us that

$$\varphi_t((p_1, p_2)) = (p_1 e^t, p_2 e^{2t})$$

where $t \in \mathbb{R}$, and hence we have our flow.

(c) We see that $X = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}$, and we let $\varphi_t(p) = \gamma_p(t)$ for $p = (p_1, p_2) \in \mathbb{R}^2$, $\gamma_p(t) = (x(t), y(t))$ and $\gamma_p(0) = p$. Then,

$$\gamma_p'(t) = \gamma_* \left(\frac{d}{dt}\bigg|_p \right) = \frac{dx}{dt}\frac{\partial}{\partial x} + \frac{dy}{dt}\frac{\partial}{\partial y} = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y} = X \implies \frac{dx}{dt} = x \quad \& \quad \frac{dy}{dt} = -y$$

$$x(t) = Ae^t \qquad y(t) = Be^{-t}.$$

Applying our initial condition,

$$\gamma_p(0) = (A, B) = (p_1, p_2) \implies A = p_1 \quad \& \quad B = p_2$$

so that

$$\varphi_t((p_1, p_2)) = (p_1 e^t, p_2 e^{-t})$$

where $t \in \mathbb{R}$ and thus we have our flow.

(d) We let $Y = x\frac{\partial}{\partial y} + y\frac{\partial}{\partial x}$, and further let $\varphi_t(p) = \gamma_p(t)$, where $p = (p_1, p_2) \in \mathbb{R}^2$, $\gamma_p(t) = (x(t), y(t))$, and $\gamma_p(0) = p$, then

$$\gamma_p'(t) = \gamma_* \left(\frac{d}{dt}\bigg|_p \right) = \frac{dx}{dt}\frac{\partial}{\partial x} + \frac{dy}{dt}\frac{\partial}{\partial y} = x\frac{\partial}{\partial y} + +y\frac{\partial}{\partial x} = Y \implies \frac{dx}{dt} = y \quad \& \quad \frac{dy}{dt} = x$$

which is a pair of coupled ODEs. Suppose the following set-up

$$\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix} \quad A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \implies \frac{d}{dt}\vec{x} = A\vec{x}$$

so that our system of ODE's becomes a vector ODE. Suppose a solution of the form $e^{\lambda t}\vec{v}$ where $\vec{v} \in \mathbb{R}^2$, then

$$\frac{d}{dt}\vec{x} = A\vec{x} \implies \lambda e^{\lambda t}\vec{v} = Ae^{\lambda t}\vec{v} \implies \lambda\vec{v} = A\vec{v}$$

hence, we need to find the eigenvectors and eigen values for our solution. Thus,

$$\det(A - \lambda\mathcal{I}) = \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - 1 \implies \lambda = \pm 1$$

Furthermore, plugging in the eigenvalues, we can find that the corresponding eigenvectors to be

$$\vec{v_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \& \qquad \vec{v_2} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

With this, we see that our solution becomes

$$\gamma_p(t) = (Ae^t - Be^{-t}, Ae^t + Be^{-t}) \implies \gamma_p(0) = (p_1, p_2) = (A - B, A + B)$$

$$\implies A = \frac{p_1 + p_2}{2} \qquad \& \qquad B = \frac{p_2 - p_1}{2}$$

and hence the flow is

$$\varphi_t(p) = \left(\frac{p_1 + p_2}{2}e^t - \frac{p_2 - p_1}{2}e^{-t}, \frac{p_1 + p_2}{2}e^t + \frac{p_2 - p_1}{2}e^{-t} \right) \quad \forall t \in \mathbb{R}$$

as required.

(B) Problem 9.18 from Lee

We compute the flows of the two vector fields using the same methodology we used in the previous question. For X we have that

$$\gamma_p'(t) = \gamma_*\left(\frac{d}{dt}\bigg|_p\right) = \frac{dx}{dt}\frac{\partial}{\partial x} + \frac{dy}{dt}\frac{\partial}{\partial y} = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y} = X \implies \frac{dx}{dt} = x \quad \& \quad \frac{dy}{dt} = -y$$

which we know will give us

$$x = Ae^t \qquad \& \qquad y = Be^{-t}.$$

If we suppose $p = (p_1, p_2) \in M$, we get,

$$\gamma_p(0) = (p_1, p_2) = (A, B) \implies A = p_1 \qquad B = p_2$$

hence we have that

$$\theta_t((p_1, p_2)) = (p_1 e^t, p_2 e^{-t})$$

as our flow for X. Next, we consider the flow for Y. Again, we have that

$$\gamma_p'(t) = \gamma_*\left(\frac{d}{dt}\bigg|_p\right) = \frac{dx}{dt}\frac{\partial}{\partial x} + \frac{dy}{dt}\frac{\partial}{\partial y} = x\frac{\partial}{\partial y} + y\frac{\partial}{\partial x} = Y \implies \frac{dy}{dt} = x \quad \& \quad \frac{dx}{dt} = y.$$

We recall from question **(A)** that the solution to this will give us a flow of

$$\psi_t((p_1, p_2)) = \left(\frac{p_1 + p_2}{2}e^t - \frac{p_2 - p_1}{2}e^{-t}, \frac{p_1 + p_2}{2}e^t + \frac{p_2 - p_1}{2}e^{-t}\right) \quad \forall t \in \mathbb{R}$$

for Y.

We first compute the composition of the maps. Thus, if $J, K \subset \mathbb{R}$ are open with $0 \in J$ and $0 \in K$, not defined explicitly just yet, we can let $(s, t) \in J \times K$. Then

$$\begin{aligned}
\theta_s \circ \psi_t((p_1, p_2)) &= \theta_s\left(\frac{p_1 + p_2}{2}e^t - \frac{p_2 - p_1}{2}e^{-t}, \frac{p_1 + p_2}{2}e^t + \frac{p_2 - p_1}{2}e^{-t}\right) \\
&= \left(\frac{p_1 + p_2}{2}e^{t+s} - \frac{p_2 - p_1}{2}e^{s-t}, \frac{p_1 + p_2}{2}e^{t-s} + \frac{p_2 - p_1}{2}e^{-s-t}\right) \\
&= \psi_t \circ \theta_s((p_1, p_2)) = \psi_t(p_1 e^s, p_2 e^{-s}) \\
&= \left(\frac{p_1 e^s + p_2 e^{-s}}{2}e^t - \frac{p_2 e^{-s} - p_1 e^s}{2}e^{-t}, \frac{p_1 e^s + p_2 e^{-s}}{2}e^t + \frac{p_2 e^{-s} - p_1 e^s}{2}e^{-t}\right)
\end{aligned}$$

(C) We have that $Z = \frac{\partial}{\partial x} \in \mathcal{X}(\mathbb{R}^2)$ and $V = y\frac{\partial}{\partial x} + \frac{\partial}{\partial y}$. Then, by definition of the Lie bracket,

$$[V, Z] = (V(Z^x) - Z(V^x))\frac{d}{dx} + (V(Z^y) - Z(V^y))\frac{d}{dy} = (V(1) - Z(y))\frac{d}{dx} + (V(0) - Z(1))\frac{d}{dy}$$

$$[V, Z] = (0 - 0)\frac{d}{dx} + (0 - 0)\frac{d}{dy} = 0$$

as expected.

Question 2: Problem 9.5 from Lee

By supposition, we know that M has a nowhere vanishing vector field. Call this vector field $X \in \mathcal{X}(M)$. Furthermore, since M is compact, we know that M is complete and that the maximal integral curves admitted by this vector field will be over all of $\mathbb{R}$. Thus, we get a global flow for our manifold M. I propose that the homotopic map will be naturally induced by the flow of this manifold.

To see this, consider $p \in M$, and suppose $\gamma_p(t) : I \subset \mathbb{R}^n \to M$ an integral curve , with $\dim(M) = n$, $0 \in I$, $\gamma_p(0) = p \in M$ and $X_{\gamma_p(t)} = \gamma_p'(t)$. However, this integral curve clearly gives us our flow aswell; we simply associate $\varphi_t(p) = \gamma_p(t)$. Furthermore, since M is compact, $I = \mathbb{R}$, and our flow is global. The homotopy comes from this flow. To see this, we fix a map using the flow, in particular, let $F_1 : M \to M$ such that $F_1(p) = \varphi_1(p)$, and we see that neccessarily $id : M \to M$ is just $id(p) = p = \varphi_0(p)$. So, since $\varphi_t(p)$ is a *smooth*, global flow, we have that it is also continuous, and we get

$$\varphi_0(p) = id(p) \qquad \& \qquad \varphi_1(p) = F_1(p)$$

where we know $\varphi : \mathbb{R} \times M \to M$. Thus, we have that the identity is homotopic with F_1, where F_1 is smooth since the flow is smooth.

To finish off the proof, we need that the homotopy yields no stationary points. This follows from the vector field that originally admitted the global flow; the vector feild is non-vanishing. Suppose we did have a stationary point, say $q \in M$ under the homotopy, then we see

$$\varphi_0(q) = \varphi_1(q) \implies \gamma_q(0) = \gamma_q(1) \implies \gamma_q'(t) = 0 \implies X_{\gamma_q(t)} = 0$$

which is a contradiction, since we assumed that X is non-vanishing. Therefore we have no stationary points.

Problem 3

(a) We get the smoothness of D_XY from the components. In particular, notice that since Y^i is smooth $\forall i = 1, \dots, n$, and $X \in \mathcal{X}(M)$, we know compositions of smooth maps will be smooth as well, and hence $X(Y^i) \in \mathbb{C}^\infty(M)$. Then, by lemma, since the component functions are smooth, D_XY is smooth.

First, we need that the map is $\mathbb{R}$-bilinear. Suppose $X, Y, Z \in \mathcal{X}(M)$ and $a, b \in \mathbb{R}$, then

$$D(aX, bY) = D_{aX}bY = \sum_{i=1}^n aX(bY^i)\frac{\partial}{\partial x_i} = ab\sum_{i=1}^n X(Y^i)\frac{\partial}{\partial x_i} = abD_XY = abD(X, Y)$$

$$\begin{aligned}
D(X, Y+Z) = D_X(Y+Z) &= \sum_{i=1}^n X((Y+Z)^i)\frac{\partial}{\partial x_i} = \sum_{i=1}^n X((Y+Z)(x_i))\frac{\partial}{\partial x_i} \\
&= \sum_{i=1}^n X(Y(x_i) + Z(x_i))\frac{\partial}{\partial x_i} = \sum_{i=1}^n (X(Y^i) + X(Z^i))\frac{\partial}{\partial x_i} \\
&= \sum_{i=1}^n X(Y^i)\frac{\partial}{\partial x_i} + \sum_{i=1}^n X(Z^i)\frac{\partial}{\partial x_i} = D_XY + D_XZ = D(X, Y) + D(X, Z) \\
D(X+Y, Z) = D_{X+Y}Z &= \sum_{i=1}^n (X+Z)(Y^i)\frac{\partial}{\partial x_i} = \sum_{i=1}^n (X(Y^i)\frac{\partial}{\partial x_i} + Z(Y^i)\frac{\partial}{\partial x_i}) \\
&= D_XY + D_ZY = D(X, Y) + D(Z, Y)
\end{aligned}$$

So we see that we have $\mathbb{R}$-bilinearity in this map. Now we need that it is $C^\infty(M)$-linear in X. Let $f \in \mathbb{C}^\infty(M)$, then,

$$D(fX, Y) = D_{fX}Y = \sum_{i=1}^n (fX)(Y^i)\frac{\partial}{\partial x_i}$$

however, we note that since f is just a smooth function on M, and $\mathcal{X}(M)$ is a vector field closed under multiplication by $C^\infty(M)$ fuctions. To see this explicitly, we need this to act on a smooth function at a point; let $g \in C^\infty(M)$ and $p \in M$, then

$$\begin{aligned}
D(fX, Y)(f(p)) &= \sum_{i=1}^n (fX)(Y^i)(p)\frac{\partial g(p)}{\partial x_i} = \sum_{i=1}^n (fX)_p(Y^i)\frac{\partial g(p)}{\partial x_i} \\
&= \sum_{i=1}^n (f(p)X(p))(Y_i)\frac{\partial g(i)}{\partial x_i} = f(p)\sum_{i=1}^n X_p(Y^i)\frac{\partial g(p)}{\partial x_i} \\
&= f(p)(D_XY)(p) = (fD(X, Y))(p)
\end{aligned}$$

as required!

Now we check the Leibniz rule. We have,

$$D(X, fY)(g(p)) = D_X(fY)(g(p)) = \sum_{i=1}^{n} X(fY)^i(p)\frac{\partial g(p)}{\partial x_i}$$

$$= \sum_{i=1}^{n} X_p(fY)(x_i)\frac{\partial g(p)}{\partial x_i} = \sum_{i=1}^{n} X_p(fY^i)\frac{\partial g(p)}{\partial x_i}$$

$$= \sum_{i=1}^{n}(X_p(f)Y_p^i + f(p)X_p(Y^i))\frac{\partial g(p)}{\partial x_i} = \sum_{i=1}^{n} X_p(f)Y_p^i\frac{\partial g(p)}{\partial x_i} + \sum_{i=1}^{n} f(p)X_p(Y^i)\frac{\partial g(p)}{\partial x_i}$$

$$= X_p(f)Y_p(g) + f(p)D_X(Y)(g(p))$$

$$\implies D(X, fY) = X(f)Y + fD(X, Y)$$

as required. Thus, we have that the directional derivative is indeed an affine connection.

Now we look at $D + \Gamma$. Since Γ is a $C^\infty(\mathbb{R}^n)$-bilinear map it is also $\mathbb{R}$-bilinear, and hence we have that the sum of two $\mathbb{R}$-bilinear maps better be $\mathbb{R}$-bilinear. Furthermore, since Γ is $C^\infty(\mathbb{R}^n$-bilinear, it automatically satisfies the being $C^\infty(\mathbb{R}^n)$-linear in X. Thus, we get that the sum will also be $C^\infty(\mathbb{R}^n)$-linear. Then, all we need is that it also satisfies the Leibniz rule. We see

$$(D + \Gamma)(X, fY) = D(X, fY) + \Gamma(X, fY) = X(f)Y + fD(X, Y) + f\Gamma(X, Y) = X(f)Y + f(D + \Gamma)(X, Y)$$

as required. Thus, we see that this sum is indeed an affine connection.

To see that every affine connection is just the sum of the directional derivative, D and some $C^\infty(\mathbb{R}^n)$-bilinear map Γ, we note that we need to find Γ such that this holds, since we already have D and ∇, our affine connection. Then, notice

$$\nabla = D + \Gamma \implies \Gamma = \nabla - D.$$

So, all we need is that $\nabla - D$ satisfies the conditions for it being like Γ. In particular, we need to check that it is $C^\infty(\mathbb{R}^n)$-bilinear. We know that the sum will be $C^\infty(\mathbb{R}^n)$-linear in the first component since both Γ and D are already affine connections. All we need is to check the second component; suppose $f \in C^\infty(\mathbb{R}^n)$, and $X, Y \in \mathcal{X}(\mathbb{R}^n)$, then

$$(\nabla - D)(X, fY) = \nabla_X(fY) - D_X(fY) = X(f)Y + f\nabla_X Y - (X(f)Y + fD_X Y)$$

$$= f(\nabla_X Y - D_X Y) = f(\nabla - D)(X, Y)$$

and hence we have that this is indeed $C^\infty(\mathbb{R}^n)$-bilinear. Thus, we have that $\Gamma = \nabla - D$ is well defined and exists!

(b) To show that the Lie Derivative satisfies the Leibniz rule, we let $f, g \in C^\infty(M)$, and $X, Y \in \mathcal{X}(M)$. Further, we recall that the Lie derivative is exactly the Lie bracket, which will be much easier to compute in this case, hence

$$\mathcal{L}(X, fY)(g) = \mathcal{L}_X(fY)(g) = [X, fY](g) = X((fY)(g)) - fY(X(g)) = X(fY(g)) - fY(X(g))$$

$$= X(f)Y(g) + fX(Y(g)) - fY(X(g)) = X(f)Y(g) + f(XY - YX)(g) = X(f)Y(g) + f[X, Y]$$

$$\implies \mathcal{L}(X, fY) = X(f)Y + f\mathcal{L}(X, Y)$$

and thus we have that the Leibniz rule is satisfied. On the other hand,

$$\mathcal{L}(fX, Y)(g) = \mathcal{L}_{fX}(Y)(g) = [fX, Y](g) = fX((Y)(g)) - Y(fX(g)) = fX(Y(g)) - (Y(f)X(g) + fY(X(g)))$$

$$= fX(Y(g)) - fY(X(g)) - Y(f)X(g) = f[X, Y](g) - Y(f)X(g) \neq f[X, Y](g)$$

and hence we do not have that the Lie derivative is $C^\infty(M)$-linear in the first component.

Notice, if $M = \mathbb{R}^n$, and $p \in \mathbb{R}^n$, then

$$\mathcal{L}(X, Y)(g(p)) = [X, Y](g(p)) = (XY - YX)(g(p)) = X(Y(g(p))) - Y(X(g(p))).$$

Since this is in $\mathbb{R}^n$, we get the natural local coordinates $\{x_1, \dots, x_n\}$ and we can conclude that

$$X = \sum_{i=1}^n X^i \frac{\partial}{\partial x_i} \qquad \& \qquad Y = \sum_{i=1}^n Y^i \frac{\partial}{\partial x_i}.$$

Then, since we have the local representations, we can sub those into our expression,

$$\mathcal{L}(X, Y)(g(p)) = \sum_{i=1}^n X^i(p) \frac{\partial Y(g)}{\partial x_i}\bigg|_p - \sum_{i=1}^n Y^i(p) \frac{\partial X(g)}{\partial x_i}\bigg|_p$$

This will get ugly, so we look at just one of the expressions, as the other will follow symmetrically:

$$\sum_{i=1}^n X^i(p) \frac{\partial Y(g)}{\partial x_i}\bigg|_p = \sum_{i=1}^n X^i(p) \frac{\partial}{\partial x_i} \left(\sum_{j=1}^n Y^j(p) \frac{\partial}{\partial x_j} g\bigg|_p \right)$$

$$= \sum_{i=1, j=1}^n X^i(p) \left(\frac{\partial}{\partial x_i} Y^j(p) \frac{\partial}{\partial x_j} g + Y^j(p) \frac{\partial^2}{\partial x_i \partial x_j} g \right)$$

$$= \sum_{i=1}^n X(Y^i)(p) \frac{\partial}{\partial x_i} g\bigg|_p + \sum_{i=1, j=1}^n X^i Y^j(p) \frac{\partial^2}{\partial x_i \partial x_j} g\bigg|_p.$$

By symmetry, we get that

$$\mathcal{L}(X,Y)(g(p)) = \sum_{i=1}^{n} X(Y^i)(p)\frac{\partial}{\partial x_i}g\bigg|_p + \sum_{i=1,j=1}^{n} X^iY^j(p)\frac{\partial^2}{\partial x_i \partial x_j}g\bigg|_p$$

$$-\sum_{i=1}^{n} Y(X^i)(p)\frac{\partial}{\partial x_i}g\bigg|_p - \sum_{i=1,j=1}^{n} Y^iX^j(p)\frac{\partial^2}{\partial x_i \partial x_j}g\bigg|_p$$

$$\mathcal{L}(X,Y)(g(p)) = \sum_{i=1}^{n} X(Y^i)(p)\frac{\partial}{\partial x_i}g\bigg|_p - \sum_{i=1}^{n} Y(X^i)(p)\frac{\partial}{\partial x_i}g\bigg|_p = D_XY(g(p)) - D_YX(g(p))$$

$$\implies \mathcal{L}(X,Y) = D_XY - D_YX$$

as required.

(c) We see that, if ∇ and ∇' are affine connections, then the sum, $\nabla + \nabla'$ will not be, and this is due to the Leibniz rule not being satisfied. In particular, let $X, Y \in \mathcal{X}(M)$ and $f \in C^\infty(M)$, then

$$(\nabla + \nabla')(X, fY) = \nabla_X(fY) + \nabla'_X(fY) = X(f)Y + f\nabla_X(Y) + X(f)Y + f\nabla'_X(Y)$$

$$= 2X(f)Y + f(\nabla + \nabla')(X,Y) \neq X(f)Y + f(\nabla + \nabla')$$

and thus we have that the Leibniz rule is not satisfied.

Now we consider $t\nabla + (1-t)\nabla$, and check it is an affine connection. First, we note this is necessarily $\mathbb{R}$-bilinear and we have C^∞-linearity in the first component since scalars won't change the linearity. Thus, all we need to check is the condition that failed earlier, the Leibniz rule. Suppose $X, Y \in \mathcal{X}(M)$ and $f \in C^\infty(M)$, then

$$(t\nabla + (1-t)\nabla')(X, fY) = t\nabla_X(fY) + (1-t)\nabla'_X(fY) = t(X(f)Y + f\nabla_X(Y)) + (1-t)(X(f)Y + f\nabla'_X(Y))$$

$$= f(t\nabla_X(Y) + (1-t)\nabla'_X(Y)) + (t + 1 - t)X(f)Y = X(f)Y + f(\nabla + \nabla')(X,Y).$$

We see that the Leibniz rule is indeed satisfied, and moreover, this holds $\forall t \in \mathbb{R}$.

Now we consider an arbitrary, but finite, number of such affine connections. Suppose $\nabla_1, \dots, \nabla_l$ are affine connections on M, and $a_1, \dots, a_l \in \mathbb{R}$ such that $a_1 + \cdots + a_l = 1$. Further, let $X, Y \in \mathcal{X}(M)$ and $f \in C^\infty(M)$, then

$$(a_1\nabla_1 + \cdots + a_l\nabla_l)(X, fY) = a_1\nabla_1(X, fY) + \cdots + a_l\nabla_l(X, fY)$$

$$= a_1(X(f)Y + \nabla_1(X,Y)) + \cdots + a_l(X(f)Y + \nabla_l(X,Y)) = (a_1 + \dots a_l)X(f)Y + a_1\nabla_1(X,Y) + \cdots + a_l\nabla_l(X,Y)$$

$$= X(f)Y + (a_1\nabla_1 + \cdots + a_l\nabla_l)(X,Y)$$

and hence we again have that an arbitrary convex sum of affine connections is an affine connection.

(d) We need to show that T, the torsion of ∇, an affine connection, is $C^\infty(M)$-bilinear. Suppose $f, g \in C^\infty(M)$, and $X, Y \in \mathcal{X}(M)$, then

$$T(fX, gY) = \nabla_{fX}(gY) - \nabla_{gY}(fX) - [fX, gY]$$

$$= f(X(g)Y + g\nabla_X Y) - g(Y(f)X + f\nabla_Y X) - (fX(gY) - gY(fX))$$

We recall that the Lie Bracket only makes sense when acting on a smooth function on M, so let $h \in C^\infty(M)$, then we see that

$$[fX, gY](h) = fX(gY(h)) - gY(fX(h)) = f(X(g)Y(h) + g(X(Y(h)))) - g(Y(f)X(h) + fY(X(g)))$$

$$= fg[X, Y](h) + fX(g)Y(h) - gY(f)X(h)$$

$$\implies [fX, gY] = fg[X, Y] + fX(g)Y - gY(f)X.$$

Taking this result back into our original expression,

$$T(fX, gY) = f(X(g)Y + g\nabla_X Y) - g(Y(f)X + f\nabla_Y X) - (fg[X, Y] + fX(g)Y - gY(f)X)$$

$$= fg(\nabla_X Y - \nabla_Y X - [X, Y])$$

as required. The remaining requirements for linearity in smooth functions follow quite easily, as we already have $\mathbb{R}$-bilinearity, and that makes addition of more vector fields linear aswell. We therefore have our C^∞-bilinearity!

Question 4

(a)