Problem 1

(a) From the homogeneous PDE we can conclude that the solution to the spatial ODE formed from separation is

$$M(x) = A\sin\left(\frac{\sqrt{\lambda}}{c}x\right) + B\cos\left(\frac{\sqrt{\lambda}}{c}x\right)$$

where the initial conditions will give us

$$M_n(x) = A\sin\left(\frac{n\pi x}{L}\right) \qquad n \in \mathbb{N}\setminus\{0\}$$

where $\lambda_n = \left(\frac{cn\pi x}{L}\right)^2$. We can now assume that the forced term must be a function such that it is in the space of functions spanned by the basis $\{M_n(x)\}$ with time-dependent coefficients. Thus,

$$F(x)\sin(\omega t) = \sum_{n=1}^{\infty} f_n(t)M_n(x).$$

Taking the inner product of both sides with $M_n(x)$ we get

$$f_n(t) = (M_n(x), F(x)\sin(\omega t)).$$

If we assume that the temporal variable under separation was $N(t)$, then from class we saw that the final solution becomes

$$N_n(t) = \int_0^t f_n(\tau)\frac{\sin\left(\sqrt{\lambda_n}(t-\tau)\right)}{\sqrt{\lambda_n}}d\tau + N_n(0)\cos\left(\sqrt{\lambda_n}t\right) + \frac{N_k'(0)}{\sqrt{\lambda_n}}\sin\left(\sqrt{\lambda_n}t\right).$$

Our initial conditions tell us that $N_n(0) = 0 = N_n'(0)$, so we expect

$$N_n(t) = \int_0^t f_n(\tau)\frac{\sin\left(\sqrt{\lambda_n}(t-\tau)\right)}{\sqrt{\lambda_n}}d\tau.$$

We know what $f_n(t)$ is, thus

$$N_n(t) = \int_0^t (M_n(x), F(x)\sin(\omega\tau))\frac{\sin\left(\sqrt{\lambda_n}(t-\tau)\right)}{\sqrt{\lambda_n}}d\tau = \frac{(M_n(x), F(x))}{\sqrt{\lambda_n}}\int_0^t \sin(\omega\tau)\sin\left(\sqrt{\lambda_n}(t-\tau)\right)d\tau.$$

Solving the integral (as was done in lecture) we get

$$N_n(t) = \frac{(M_n(x), F(x))}{\sqrt{\lambda_n}}\left(\frac{\omega\sin\left(\sqrt{\lambda_n}t\right) - \sqrt{\lambda_n}\sin(\omega t)}{\omega^2 - \lambda_n}\right).$$

(b) Taking the limit as $\omega \to \frac{\pi nc}{L} = \sqrt{\lambda_n}$ we see that from direct substitution we will get $\frac{0}{0}$. Applying L'Hôpital's rule we see that

$$\lim_{\omega\to\sqrt{\lambda_n}} N_n(t) = \frac{(M_n(x), F(x))}{\sqrt{\lambda_n}}\left(\frac{\sin\left(\sqrt{\lambda_n}t\right) - \sqrt{\lambda_n}t\cos(\omega t)}{2\lambda_n}\right).$$

Problem 2

(a) Similar to **Problem 1**, we can see that the spatial ODE will give us a sinusoidal solution, but here we have Neumann boundary conditions. So,

$$M_n(x) = \cos\left(\frac{\sqrt{\lambda_n}}{c}x\right) \qquad \lambda_n = \left(\frac{nc\pi}{L}\right)^2 \quad n \in \mathbb{N} \cup \{0\}.$$

Our forcing term here, unlike in the previous problem, is only time dependent. We still need to write the function as some linear combination of the basis formed by $\{M_n(x)\}$. However, notice that $n = 0$ is a part of the solution. This value corresponds with $M_0(x) = 1$, and hence

$$F(t) = (1)F(t) = M_0(x)F(t)$$

So our forcing term is a “scalar” multiple of the first eigenfunction/basis-function. Thus, any inner product with the other basis vectors will be 0, and so our ODE for $N_n(t)$ becomes,

$$(F(t), M_n(x)) = \frac{dN_n}{dt} + \lambda_n N_n \quad \to \quad F(t) = \frac{dN_0}{dt} + \lambda_0 N_0$$

Which can be solved with an integrating factor $e^{\lambda_0 t}$ to get

$$e^{\lambda_0 t}F(t) = e^{\lambda_0 t}\frac{dN_0}{dt} + e^{\lambda_0 t}\lambda_0 N_0$$

$$e^{\lambda_0 t}F(t) = \frac{d}{dt}\left(e^{\lambda_0 t}N_0\right)$$

$$\int_0^t e^{\lambda_0 \tau}F(\tau)d\tau = \int_0^t \frac{d}{d\tau}\left(e^{\lambda_0 \tau}N_0\right)d\tau = e^{\lambda_0 t}N_0(t) - N_0(0)$$

From the initial condition we see that $N_0(0) = (f(x), M_0(x))$, and thus

$$N_0(t) = e^{-\lambda_0 t}(f(x), M_0(x)) + e^{-\lambda_0 t}\int_0^t e^{\lambda_0 \tau}F(\tau)d\tau.$$

(b) Letting $F(t) = e^{-t}$, we see

$$N(t) = e^{-\lambda_0 t}(f(x), 1) + e^{-\lambda_0 t}\int_0^t e^{\lambda_0 \tau}e^{-\tau}d\tau$$

$$N(t) = e^{-\lambda_0 t}(f(x), 1) + e^{-\lambda_0 t}\int_0^t e^{(\lambda_0 - 1)\tau}d\tau$$

$$N(t) = e^{-\lambda_0 t}(f(x), 1) + \frac{e^{-\lambda_0 t}}{\lambda_0 - 1}\left(e^{(\lambda_0 - 1)t} - 1\right)$$

$$N(t) = e^{-\lambda_0 t}\left((f(x), 1) + \frac{e^{(\lambda_0 - 1)t} - 1}{\lambda_0 - 1}\right).$$

However, $\lambda_0 = 0$, so

$$N(t) = \left((f(x), 1) - e^{-t} + 1\right).$$

as required.

Problem 3

(a) We first take the Fourier transform of the PDE to get,

$$\frac{d\mathcal{F}[u]}{dt} - c^2(-is)^2\mathcal{F}[u] = \mathcal{F}[F(x,t)]$$

where we let $\mathcal{F}[u] = U(s,t)$ and $\mathcal{F}[F(x,t)] = H(s,t)$,

$$\frac{dU}{dt} + c^2s^2U = H(s,t)$$

This isn't immediately solvable, so we use the Laplace transform

$$\tau\mathcal{L}[U] - U(s,0) + c^2s^2\mathcal{L}[U] = \mathcal{L}[H(s,t)].$$

From initial conditions, we see that $U(s,0) = 0$, and thus we isolate $\mathcal{L}[U]$ to get

$$\mathcal{L}[U] = \frac{1}{\tau + c^2s^2}\mathcal{L}[H(s,t)]$$

$$\mathcal{F}[u] = U = \mathcal{L}^{-1}\left(\frac{1}{\tau + c^2s^2}\mathcal{L}[H(s,t)]\right)$$

We see that this is heading towards a double convolution.

$$u(x,t) = \mathcal{F}^{-1}\left(\mathcal{L}^{-1}\left(\frac{1}{\tau + c^2s^2}\right) * \mathcal{F}[F(x,t)]\right)$$

$$u(x,t) = \int_0^t \int_{-\infty}^{\infty} G(x-s, t-\tau)F(s,\tau)dsd\tau$$

where the function $G(s,\tau)$ is defined by the inverses of the $\frac{1}{\tau+c^2s^2}$, as required.

(b) If our initial condition becomes $u(x,0) = f(x)$ then we see that $\mathcal{F}[u(x,0)] = U(s,0) = \mathcal{F}[f(x)]$. Then, since the procedure is the same as **(a)** up till that initial condition,

$$\mathcal{L}[U] = \frac{1}{\tau + c^2s^2}\mathcal{L}[H(s,t)] + \frac{1}{\tau + c^2s^2}\mathcal{F}[f(x)]$$

and so if we follow the notation from the prior parts we see

$$u(x,t) = \int_0^t \int_{-\infty}^{\infty} G(x-s, t-\tau)F(s,\tau)dsd\tau + \int_{-\infty}^{\infty} f(s)\int_0^t G(x-s,\tau)dtds$$

as required.

Problem 4

We notice how x is unbounded in it's domain, so we will first apply the Fourier Transform to the PDE in the x dependence, letting $U(\lambda, t) = \mathcal{F}[u(x,t)]$,

$$\mathcal{F}[u_{tt}] - \mathcal{F}[c^2 u_{xx}] - \mathcal{F}[a^2 u] = 0$$

$$\frac{\partial^2 U}{\partial t^2} + c^2\lambda^2 U - a^2 U = 0$$

$$\frac{\partial^2 U}{\partial t^2} + \left(c^2\lambda^2 - a^2\right) U = 0$$

We recognize this as an ODE in t where the solution for U will be

$$U(\lambda, t) = A(\lambda)\cos\left(\sqrt{c^2\lambda^2 - a^2}t\right) + B(\lambda)\sin\left(\sqrt{c^2\lambda^2 - a^2}t\right).$$

Changing the initial conditions to the Fourier space,

$$\mathcal{F}[u(x,0)] = U(\lambda, 0) = \mathcal{F}[f(x)] \qquad \mathcal{F}[u_t(x,0)] = \frac{\partial U(\lambda, 0)}{\partial t} = 0$$

which tells us that $A(\lambda) = \mathcal{F}[f(x)]$ and $B(\lambda) = 0$. So we have that

$$U(\lambda, t) = \mathcal{F}[u(x,t)] = \mathcal{F}[f(x)]\cos\left(\sqrt{c^2\lambda^2 - a^2}t\right)$$

$$u(x,t) = \mathcal{F}^{-1}\left(\mathcal{F}[f(x)]\cos\left(\sqrt{c^2\lambda^2 - a^2}t\right)\right)$$

which by convolution theorem is

$$u(x,t) = \int_{-\infty}^{\infty} f(x)g(x-\lambda, t)d\lambda$$

where $g(x,t) = \mathcal{F}^{-1}[\cos\left(\sqrt{c^2\lambda^2 - a^2}t\right)]$.

Problem 5

Again, we see that x is unbounded, so we take the Fourier Transform with respect to this variable,

$$-\lambda^2\mathcal{F}[u] + \frac{\partial^2 \mathcal{F}[u]}{\partial y^2} = 0.$$

This is an ODE in $\mathcal{F}[u] = U(\lambda, y)$ where we recognize the solution to be

$$U(\lambda, y) = Ae^{\lambda y} + Be^{-\lambda y}.$$

Next, we take the Fourier Transform of the boundary conditions,

$$\mathcal{F}[u(x,0)] = U(\lambda, 0) = \mathcal{F}[e^{-|x|}] \qquad \mathcal{F}[u(x,L)] = U(\lambda, L) = 0$$

where the first boundary condition evaluates some more,

$$U(\lambda,0) = \mathcal{F}[e^{-|x|}] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i\lambda x}e^{-|x|}dx = \frac{1}{\sqrt{2\pi}}\left(\int_0^{\infty} e^{i\lambda x}e^{-x}dx + \int_{-\infty}^{0} e^{i\lambda x}e^{x}dx\right)$$

$$U(\lambda,0) = \frac{1}{\sqrt{2\pi}}\left(\frac{1}{i\lambda - 1}\left(e^{x(i\lambda-1)}\right)_0^{\infty} + \frac{1}{i\lambda+1}\left(e^{x(i\lambda+1)}\right)_{-\infty}^{0}\right)$$

$$U(\lambda,0) = \frac{1}{\sqrt{2\pi}}\left(\frac{1}{1-i\lambda} + \frac{1}{i\lambda+1}\right)$$

$$U(\lambda,0) = \frac{\sqrt{2}}{\lambda^2+1}.$$

Applying these BCs to the ODE solution, we see

$$A + B = \frac{\sqrt{2}}{\lambda^2+1} \qquad Ae^{\lambda L} + Be^{-\lambda L} = 0$$

where we throw these in a matrix to solve,

$$\begin{bmatrix} 1 & 1 \\ e^{\lambda L} & e^{-\lambda L} \end{bmatrix}\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{\lambda^2+1} \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} A \\ B \end{bmatrix} = \frac{1}{e^{-\lambda L} - e^{\lambda L}}\begin{bmatrix} e^{-\lambda L} & -1 \\ -e^{\lambda L} & 1 \end{bmatrix}\begin{bmatrix} \frac{\sqrt{2}}{\lambda^2+1} \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}e^{-\lambda L}}{(\lambda^2+1)(e^{-\lambda L}-e^{\lambda L})} \\ -\frac{\sqrt{2}e^{\lambda L}}{(\lambda^2+1)(e^{-\lambda L}-e^{\lambda L})} \end{bmatrix}$$

Now that we have A and B, we see that we can compute the final answer to be

$$u(x,y) = \mathcal{F}^{-1}\left(\frac{\sqrt{2}e^{-\lambda L}}{(\lambda^2+1)(e^{-\lambda L} - e^{\lambda L})}e^{\lambda y} - \frac{\sqrt{2}e^{\lambda L}}{(\lambda^2+1)(e^{-\lambda L} - e^{\lambda L})}e^{-\lambda y}\right)$$

as required.

Problem 6

(a) We start with taking the Fourier transform and letting $U(\lambda, t) = \mathcal{F}[u(x,t)]$,

$$\frac{\partial U}{\partial t} + aU - ib\lambda U + c^2\lambda^2 U = \mathcal{F}[F(x,t)]$$

$$\frac{\partial U}{\partial t} + \left(a - ib\lambda + c^2\lambda^2\right) U = \mathcal{F}[F(x,t)].$$

This is a first order forced ODE in U, so we multiply through by an integrating factor,

$$e^{(a-ib\lambda+c^2\lambda^2)t}\frac{\partial U}{\partial t} + \left(a - ib\lambda + c^2\lambda^2\right) e^{(a-ib\lambda+c^2\lambda^2)t} U(\lambda,t) = e^{(a-ib\lambda+c^2\lambda^2)t}\mathcal{F}[F(x,t)]$$

$$\frac{\partial}{\partial t}\left(e^{(a-ib\lambda+c^2\lambda^2)t}U(\lambda,t)\right) = e^{(a-ib\lambda+c^2\lambda^2)t}\mathcal{F}[F(x,t)]$$

where we integrate to see

$$\int_0^t \frac{\partial}{\partial \tau}\left(e^{(a-ib\lambda+c^2\lambda^2)\tau}U(\lambda,\tau)\right) d\tau = \int_0^t e^{(a-ib\lambda+c^2\lambda^2)\tau}\mathcal{F}[F(x,\tau)]d\tau$$

$$e^{(a-ib\lambda+c^2\lambda^2)t}U(\lambda,t) - U(\lambda,0) = \int_0^t e^{(a-ib\lambda+c^2\lambda^2)\tau}\mathcal{F}[F(x,\tau)]d\tau$$

$$U(\lambda,t) = e^{-(a-ib\lambda+c^2\lambda^2)t}\int_0^t e^{(a-ib\lambda+c^2\lambda^2)\tau}\mathcal{F}[F(x,\tau)]d\tau + e^{-(a-ib\lambda+c^2\lambda^2)t}U(\lambda,0)$$

where we know $U(\lambda,0) = \mathcal{F}[u(x,0)] = \mathcal{F}[g(x)]$ and thus the final result in Fourier space will be

$$U(\lambda,t) = e^{-(a-ib\lambda+c^2\lambda^2)t}\int_0^t e^{(a-ib\lambda+c^2\lambda^2)\tau}\mathcal{F}[F(x,\tau)]d\tau + e^{-(a-ib\lambda+c^2\lambda^2)t}\mathcal{F}[g(x)]$$

as required.

(b) Assuming that $F(x,t) = 0$, we see that

$$U(\lambda,t) = \mathcal{F}[u(x,t)] = e^{-(a-ib\lambda+c^2\lambda^2)t}\mathcal{F}[g(x)]$$

$$u(x,t) = \mathcal{F}^{-1}\left(e^{-(a-ib\lambda+c^2\lambda^2)t}\mathcal{F}[g(x)]\right)$$

$$u(x,t) = e^{-at}\mathcal{F}^{-1}\left(e^{-c^2\lambda^2 t}\left(e^{ib\lambda}\mathcal{F}[g(x)]\right)\right)$$

and by the convolution theorem and the identities provided

$$u(x,t) = \frac{e^{-at}}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2c^2 t}}e^{-\frac{(x-\lambda)^2}{4c^2 t}}g(\lambda - b)d\lambda.$$

The integral of this will be dependent upon $g(x)$.