Midterm

Due on Thrusday, March 14th before the end of the day

1 Realistic Twin paradox (Marked by Richard)

A ship takes off from Earth on May 15th 2075, and travels for five years (as clocks measure in the spaceship) with a constant acceleration $g = 9.8\ \mathrm{ms}^{-2}$ (also as measured with the instruments on board). Then it slows down for another five years at the same rate, and returns in the same way (for a total of 20 years of travel). What is the distance the spaceship has travelled? How long did it take for the whole trip as measured on Earth?

For this question you can take the speed of light to be $c = 3 \cdot 10^8$ m/s.

(BONUS QUESTION (for NO MARKS AT ALL)): if you were the ship captain, would you believe it if, on arrival to Earth, someone told you that while you were absent Earth was scheduled to be demolished to make way for a hyperspace bypass? Why?)

2 Reverse Compton effect (Marked by Richard)

When a photon collides with a charged particle which is moving with a speed very close to the speed of light in the lab frame, the photon is said to have undergone a *reverse Compton scattering*.

(a) Consider a reverse Compton scattering in which a charged particle of mass m and total energy E_1 (as seen in the lab frame), collides with a photon of total energy E_2. **Prove** that the energy (E_2') of the photon after the collision is

$$E_2' = \frac{E_2(E_1 - |\boldsymbol{p}_1|C_{12})}{(1 - C_{22'})E_2 + E_1 - |\boldsymbol{p}_1|C_{12'}}$$

where E_2' is a function of the modulus of the incoming 3-momentum of the charged particle $|\boldsymbol{p}_1|$, its incoming energy E_1, the incoming energy of the photon E_2 and the cosine of only the following three scattering angles $C_{12} \coloneqq \cos\theta_{12}$, $C_{22'} \coloneqq \cos\theta_{22'}$ and $C_{12'} \coloneqq \cos\theta_{12'}$.

Notation: The scattering angles are

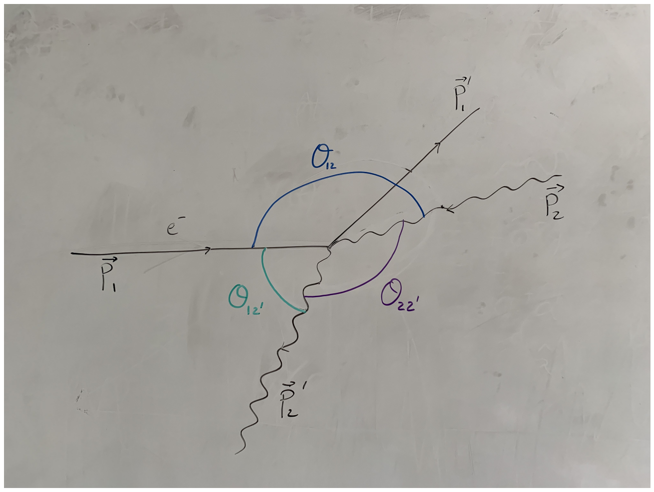

Figure 1: Schematic representation of a reverse Compton scattering.

- $\theta_{12} := \widehat{\boldsymbol{p}_1\boldsymbol{p}_2}$ is the angle between the incoming 3-momenta of the particle and the photon before the collision.
- $\theta_{12'} := \widehat{\boldsymbol{p}_1\boldsymbol{p}_2'}$ is the angle between the incoming 3-momentum of the massive particle and the outgoing 3-momentum of the photon.
- $\theta_{22'} := \widehat{\boldsymbol{p}_2\boldsymbol{p}_2'}$ is the angle between the incoming 3-momentum of the photon and the outgoing 3-momentum of the photon.
- And so on... In this question we will use the subindex 1 for magnitudes associated with the charged particle and the subindex 2 for magnitudes associated with the photon, and the primes denote magnitudes after scattering

(b) Particularize the result of the previous part to a head-on collision (i.e. $C_{12} = -1$, $C_{22'} = -1$, $C_{12'} = 1$) and an ultra-relativistic charged particle $E_1 \approx |\boldsymbol{p}_1|$. Based on your answer, how much of the initial energy of the ultra-relativistic particle is taken by the photon? What is the after-scattering energy of the particle in that limit?

(c) If space is filled with black-body radiation of temperature 3K (low-energy far-infrared photons), explain whether you should expect an abundance of ultra-relativistic charged particles in the Universe and why.

3 Is it closed? Not exactly! (Marked by Pipo)

In this exercise we introduce the definition of exterior derivative and consider whether a differential form can be expressed as the derivative of a function or not. The concept of exterior derivative is useful as it is the principle behind all the integral theorems, such as the Gauss law, in differential manifolds. Although the theory is more general, we will restrict our study to the case of one-forms and functions.

Given a function in a differential manifold, we define its differential as:

$$\mathrm{d}f = \partial_\mu f \mathrm{d}x^\mu,$$

with $\{\mathrm{d}x^\mu\}$ a coordinate basis of one-forms. On the other hand, given a 1-form $\omega = \omega_\mu \mathrm{d}x^\mu$, we define its **exterior derivative** as[1]:

$$\mathrm{d}\omega = \partial_{[\mu}\omega_{\nu]}\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu.$$

A one-form is **exact** if it is the differential of a function from the manifold to $\mathbb{R}$, that is:

$$\omega = \mathrm{d}f.$$

A one-form is **closed** if its exterior derivative is the zero two-form:

$$\mathrm{d}\omega = \mathbf{0}.$$

(a) Show that the exterior derivative of the differential of a function is the zero two-form, that is, that every exact form is closed.

(b) The opposite to the statement in part (a) is not always true: given a closed one-form it is not always possible to find a function whose differential is the one-form. However, it is true if the manifold is contractible (which, for example, would be the case if we can fully map it to $\mathbb{R}^n$ with a single chart). This result is known as the Poincaré lemma.

Let us study a simple, but subtle example. Consider the following one-form in $\mathbb{R}^2 - \{0\}$:

$$\omega = \frac{x\mathrm{d}y - y\mathrm{d}x}{x^2 + y^2}.$$

Show that $\mathrm{d}\omega = 0$.

(c) Find a function defined in a set $A \subset \mathbb{R}^2 - \{0\}$ whose differential coincides with ω.

(d) Is ω exact? If not, why?

[1] Recall the defitinion of anti-symmetrization and tensor product from the notes.

4 A plane is simply flat (Marked by Erickson)

Consider a two-dimensional plane $\mathbb{R}^2$. We can make this plane into a differentiable manifold by providing it with an atlas $\mathcal{A}$ consisting of a collection of charts[2] $\{(U_\alpha, \phi_\alpha)\}$ where α labels each chart. The simplest atlas for a plane is given by *one single chart* (U, ϕ) which covers the *whole plane*, i.e. $U = \mathbb{R}^2$ and $\phi(p) = (x^1, x^2)_p$, where ϕ is a diffeomorphism. Usually we call $x^1 \equiv x$ and $x^2 \equiv y$. With this, every point p is labelled by two numbers (x, y) and no two points share the same (x, y), otherwise ϕ^{-1} does not exist and hence not a diffeomorphism.

We know this chart very well. This chart gives us a coordinate system called **Cartesian coordinate system**. Thanks to the chart and relabelling x^1, x^2, we can now call this manifold $\mathbb{R}^2$ an *xy-plane*. We will study this very familiar manifold in great detail.

(a) Let us now make this manifold into a *Riemannian manifold*, by providing a metric compatible with the Levi-Civita connection. In this coordinate system, the components of the metric tensor can be read from the line element

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = \mathrm{d}x^2 + \mathrm{d}y^2\,. \tag{6}$$

Show explicitly that the components of Levi-Civita connection (Christoffel symbols) in Cartesian coordinates are identically zero, i.e. $\Gamma^\rho{}_{\mu\nu} = 0$ for all $\rho, \mu, \nu = 1, 2$. Hence, argue using the Riemann tensor $R_{\mu\nu\rho\sigma}$ that indeed $\mathbb{R}^2$ as Riemannian manifold is flat.

(b) A coordinate system labelling points with x^μ naturally induces a set of basis vectors in tangent spaces called *coordinate basis vectors*[3], formally written as $\Upsilon_\mu \equiv \partial_\mu$.

Here we study the interpretation of the covariant derivative and the Christoffel symbols. The **covariant derivative** of a vector $\mathsf{V} \in T_pM$ along the direction of some vector U, denoted $\nabla_{\mathsf{U}}\mathsf{V}$, generalizes the directional derivative in vector calculus. The map ∇ is called **affine connection** (assumed torsion-free) and has the property that

(1) $\nabla_{f\mathsf{U}}\mathsf{V} = f\nabla_{\mathsf{U}}\mathsf{V}$ where f is a smooth function,

(2) Leibniz rule: $\nabla_{\mathsf{U}}(f\mathsf{V}) = f\nabla_{\mathsf{U}}\mathsf{V} + (\nabla_{\mathsf{U}}f)\mathsf{V}$.

Using the fact that we can write any vector as linear combination of basis vectors $\mathsf{V} = V^\mu\Upsilon_\mu$, and noting that $\Upsilon_\mu \in T_pM$, **show that**

$$\nabla_{\mathsf{U}}\mathsf{V} = \left(\frac{\partial V^\mu}{\partial x^\nu}U^\nu + \Gamma^\mu{}_{\nu\rho}U^\nu V^\rho\right)\Upsilon_\mu \equiv (\nabla\mathsf{V})^\mu{}_\nu U^\nu \Upsilon_\mu\,. \tag{7}$$

[2] You may have known that the word "atlas" is exactly analogous to the actual atlas for the world map. In the earlier days before smartphones become popular, an atlas is a book where each page corresponds to a small piece of the world map (or even city map) with grids/coordinates for navigation, which is aptly also called a chart. Nowadays for navigation most people uses Google map.

[3] In physics notation (vector calculus), the basis vectors in Cartesian coordinates for the xy-plane are usually denoted by $\Upsilon_1 = \mathbf{i}\,, \Upsilon_2 = \mathbf{j}$.

What is the interpretation of the Christoffel symbols in terms of what they tell us about the basis vectors Υ_μ?

Hint: it helps to convince yourself that while $\nabla\mathsf{V}$ is a rank-$(1,1)$ tensor, $\nabla_{\mathsf{U}}\mathsf{V}$ is a rank-$(1,0)$ tensor, i.e. a vector! This should be expected since we know from vector calculus that directional derivative of a vector should still be a vector. This is also *essential* for deriving the result.

(c) Using part (b), show that the covariant derivative along the coordinate basis vectors Υ_μ (generalizing "partial derivative" along each coordinate), denoted $\nabla_\nu\mathsf{V} := \nabla_{\Upsilon_\nu}\mathsf{V}$, is given by

$$\nabla_\nu\mathsf{V} = \left(\frac{\partial V^\mu}{\partial x^\nu} + \Gamma^\mu{}_{\nu\rho}V^\rho\right)\Upsilon_\mu \equiv (\nabla_\nu V^\mu)\Upsilon_\mu\,. \tag{8}$$

and hence show that the basis vectors in Cartesian coordinates are the same at every point (i.e. unchanged by parallel transport), as we should expect.

Remark: this shows that in Cartesian coordinates, the covariant derivative along x^μ, the directional derivative along x^μ and the partial derivative along x^μ are all the same thing. The coefficients $\nabla_\nu V^\mu$ is sometimes written as $V^\mu{}_{;\nu}$.

(d) Now consider another chart (U',ψ), which gives what is known as **polar coordinates** (r,θ). Polar coordinates are related to Cartesian coordinates by the following

$$x = r\cos\theta\,,\qquad y = r\sin\theta\,,\qquad \tan\theta = \frac{y}{x}\,. \tag{9}$$

By comparing with Cartesian coordinates, explain why a chart (U',ψ) *cannot* cover the entire plane (hence a additional chart(s) will be needed to account for all points).

Hint: find a point q in terms of Cartesian coordinates (i.e., find $\phi(q)$) where the transition function $\psi\circ\phi^{-1}$ (which performs change of coordinates) is not a diffeomorphism. Physically, it means that the coordinates are "bad" at q as it does not label q uniquely.

(e) By performing a coordinate transformation, write down explicitly the form of the metric $g_{\mu'\nu'}$ in polar coordinates as 2×2 matrix and hence the line element $\mathrm{d}s^2$ also in polar coordinates (we use the primed coordinates $\mu',\nu' = r,\theta$ to distinguish the indices between the two coordinate systems).

(f) Work out explicitly the values of $\Gamma^{\mu'}{}_{\rho'\nu'}$ for polar coordinates. Using Eq. (8) in part (c), show whether the polar basis vectors vary[4] from point to point on the plane.

[4]You may have known the answer to this from past experience with polar coordinates or spherical coordinates in e.g. multivariable calculus or electromagnetism classes.

(g) Finally, show explicitly that the Riemann tensor $R_{\mu'\nu'\rho'\sigma'}$ also vanishes in polar coordinates, hence our plane is indeed flat[5], as we have seen in part (a).

Hint: how many *independent* components are there in $R_{\mu\nu\rho\sigma}$? This is the number of components you need to fully specify the Riemann tensor, and hence all you need to compute (instead of computing all $2^4 = 16$ components in two dimensions).

5 Plain and conformal flatness (Marked by Pipo)

Given a Lorentzian manifold $\mathcal{M}$ with a metric tensor g, another metric tensor g' is said to be *conformally related* to g if

$$\mathsf{g}' = \Omega \mathsf{g},$$

where Ω is a function[6]. More specifically, when the original metric is flat ($\mathsf{g} = \eta$), we say that the metric is *conformally flat*.

(a) Consider a conformally flat metric, whose components can be written as

$$g_{\mu\nu} = \Omega(x^\lambda)\eta_{\mu\nu},$$

where $\eta_{\mu\nu}$ is the Minkowski metric and $\Omega(x^\lambda)$ is a scalar function of the coordinates. Calculate the Christoffel symbols ${\Gamma^\rho}_{\mu\nu}$ of the connection compatible with the metric $g_{\mu\nu}$. **Note**: you do not need to compute each component one by one. You just need to write the general expression for ${\Gamma^\rho}_{\mu\nu}$ in terms of Ω, derivatives of Ω and $\eta_{\mu\nu}$.

(b) Show that for the conformally flat metric, the geodesic equation can be written as

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\Omega\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\right) + \eta^{\mu\lambda}\partial_\lambda\left(\log\sqrt{\Omega}\right) = 0.$$

(Hint: you should already know that the 4-velocity here in natural units is normalized to -1.)

[5] In particular, this shows that curvature of a manifold does not depend on the coordinate systems: a plane should be flat no matter what grid you use to label its points.

[6] In other words, g' is a product of a function $\Omega(x^\mu)$ and the original metric.