Question 1

This question motivates the use of partitions of unity. So, we first build an open cover. We note that since A is closed, we have for free that $M \setminus A$ is open. Now we need an open set around A. Since Y is a smooth vector field on A, then it is smooth $\forall p \in A$, and in particular, must be smooth for some open neighbourhood W_p with $p \in W_p$. This follows from the local coordinate definition. Say (U, ψ) is a patch with $p \in U$ and coordinates $(x_1, \dots, x_n)$, then we know

$$Y_p = Y(p) = \sum_{n=1}^{\infty} Y^i(p) \frac{\partial}{\partial x_i}|_p$$

where Y^i is smooth. Hence, we see that Y will be smooth over some open neighbourhood, which in this case is U, and hence we can let $W_p = U$. We let the new open set be the union of all such open neighbourhoods, $\cup_{p \in A} W_p$, and can suppose $\cup_{p \in A} W_p \subseteq U$, for arbitrary open set $U \supset A$, since if it is not, we can take the intersection.

Now, our open cover is $\{M \setminus A, \cup_{p \in A} W_p\}$, and hence $\exists$ a partition of unity subordinate to this cover, say $\{\varphi\} \cup \{\varphi_p\}_{p \in A}$ respectively. Further, define the extension of Y to W_p by $\hat{Y}$. Then, we let

$$\tilde{Y} = \sum_{p \in A} \varphi_p \hat{Y}$$

where the operation between the maps is the standard product, since both $\hat{Y}$ and φ_2 take points $p \in M$. Now we need to show that this is a smooth vector field. Let $q \in A$, then clearly

$$\tilde{Y}_q = \tilde{Y}(q) = \underbrace{\sum_{p \in A} \varphi_p(q)}_{1} \hat{Y}(q) = \hat{Y}(p) = Y_p$$

since $\text{supp}(\varphi) \subset M \setminus A$, so $\varphi(q) = 0 \implies \sum_{p \in A} \varphi_p = 1$. So, we have that $\tilde{Y}|_A = Y$ and is a smooth vector field since Y is a smooth vector field.

Next, we suppose $q \in M \setminus \cup_{p \in A} W_p$. Then we have that $\sum_{p \in A} \varphi_p = 0$, since $\text{supp}(\varphi_p) \subset W_p$. So,

$$\tilde{Y}_q = \tilde{Y}(q) = \sum_{p \in A} \varphi_p(q) \hat{Y}(q) = 0$$

which is to say the vector field vanishes outside of U. This is trivially a smooth vector field.

Finally, we wish to have $\tilde{Y}$ be smooth over all of M, so we need to consider the missing patch between the two closed sets we have already shown, namely we need to consider the open set $U \setminus A$. To see how this will be smooth, we need to rewrite our $\tilde{Y}$ in a slightly different, but equivalent way. Suppose $q \in U \setminus A$, then,

$$\tilde{Y}_q = \tilde{Y}(q) = \sum_{p \in A} \varphi_p(q) \hat{Y}(q) = \sum_{p \in A} \varphi_p(q) \hat{Y}(q) + \varphi(q) \hat{Y}(q) - \varphi(q) \hat{Y}(q)$$

$$= \left(\sum_{p \in A} \varphi_p(q) + \varphi(q) \right) \hat{Y}(q) - \varphi(q)\hat{Y}(q) = (1 - \varphi(q))\hat{Y}(q)$$

Further, since $\text{supp}(\varphi) \cap (\cup_{p \in A} \text{supp}(\varphi_p)) \neq \emptyset$, since otherwise the partition of unity would vanish at some point and contradict the definition, and the partition of unity uses smooth functions, we must have that $1 - \varphi(q)$ must go smoothly between 0 and 1 while switching between $\text{supp}(\varphi)$ and $\cup_{p \in A} \text{supp}(\varphi_p)$.

This actually makes the check much easier, since q must be in the support of either φ or $\cup_{p \in A} \varphi_p$. If it is in one exclusively, then we get the above cases. The only remaining case is that it is in both, since it can't be in neither. Then, we see that $(1 - \varphi(q)) \in (0, 1) \subset \mathbb{R}$ and hence

$$\tilde{Y}_q = (1 - \varphi(q))\hat{Y}_q$$

which is smooth since $\hat{Y}_q$ is smooth at this q, which is in some W_p.

Hence, we have shown that for some smooth vector field Y defined over the closed set $A \subset M$, for any open $U \subset M$ such that $A \subset U$, $\exists \tilde{Y}$ such that $\tilde{Y}$ is a smooth vector field over M and $\tilde{Y}|_A = Y$, as required.

Further, since a singleton point is also a closed set, we can extend this finding to individual points. In particular, $\forall p \in M$, and $X \in T_pM$, since X is a smooth vector field at p, a closed set, by the above proof, we can conclude that indeed $\exists$ a smooth vector field, Y over M, such that $Y|_p = X$.

Question 2

First, suppose that F is a constant map. Let $f \in C^\infty(N)$, $p \in M$, and $X \in T_pM$. Then, by definition we have that

$$F_* \circ X \circ f(p) = X(f \circ F)(p) = X(\underbrace{f(F(p))}_{\text{constant}}) = 0$$

since the derivation of a constant is zero. This must hold $\forall p \in M$, hence we have that the pushforward of a constant map is everywhere vanishing.

Now we consider the opposite implication. Suppose that F_* is the trivial push forward and everything vanishes. We pick a coordinate representation; let (U, φ) be a local patch with coordinates $\varphi = (x_1, \ldots, x_n)$ and $p \in U$. Then, we have by definition of the pushforward,

$$F_* \circ X \circ f(p) = X(f \circ F(p)) = \sum_{i=1}^{n} X^i(p) \frac{\partial f}{\partial x_i}(F(p))$$

but we assumed that this is the everywhere vanishing pushforward, then we see

$$\sum_{i=1}^{n} X^i(p) \frac{\partial f}{\partial x_i}(F(p)) = 0$$

where we know that $X^i(p)$ is smooth. Thus, we see that this is saying that the partials are everywhere vanishing at $F(p)$, which implies that

$$F(p) = C$$

for some constant C. This is true for the entire subset U, and to see that this is the same constant everywhere, we know that we can repeat the above steps with another patch, say (V, ψ), and reach a similar conclusion with another constant, C'. But, since M is connected, we know that there exists a set of overlapping patches that will connect the two open sets U and V, which would force $C = C'$. Thus, we can conclude that

$$F(p) = C \quad \forall p \in M$$

as required.

Question 3

(a) To show that this pushforward map is indeed an isomorphism, we will need to show that it is a bijection and preserves the vector space properties of the tangent space. From the linearity property of the pushforward, we get scalar multiplication and addition of vectors preserved for free, which is all we need.

Now we just need that this map is a bijection. To see injection, we just need to work through the definition. Suppose $X, Y \in T_{(p,q)}(M \times N)$, for $(p, q) \in M \times N$, and $f \in C^\infty(M)$, $g \in C^\infty(N)$, such that

$$(\pi_{1*}, \pi_{2*}) \circ X \circ (f, g)(p, q) = (\pi_{1*}, \pi_{2*}) \circ Y \circ (f, g)(p, q)$$

Then, by definition,

$$(\pi_{1*}, \pi_{2*}) \circ X \circ (f, g)(p, q) = (\pi_{1*}X, \pi_{2*}X) \circ (f, g)(p, q) = (\pi_{1*}X(f), \pi_{2*}X(g))(p, q)$$

$$(\pi_{1*}, \pi_{2*}) \circ X \circ (f, g)(p, q) = (X(f \circ \pi_1(p, q)), X(g \circ \pi_2(p, q))) = (X(f(p)), X(g(q)))$$

Through a similar argument, we get

$$(\pi_{1*}, \pi_{2*}) \circ Y \circ (f, g)(p, q) = (Y(f(p)), Y(g(q)))$$

and then by assumption we get,

$$Y(f(p)) = X(f(p)) \qquad Y(g(q)) = X(g(q))$$

which must hold $\forall p \in M, q \in N, f \in C^\infty(M)$ and $g \in C^\infty(N)$. In particular, if we let (U, φ) and (V, ψ) be local patches of M and N respectively, with $p \in U$, $q \in V$ and coordinates $\varphi = (x_1, \ldots, x_m)$, $\psi = (y_1, \ldots, y_n)$, then this result says that

$$\sum_{i=1}^{m} Y^i(p) \frac{\partial f}{\partial x_i}(p) = \sum_{j=1}^{m} X^j(p) \frac{\partial f}{\partial x_j}(p) \quad \Longrightarrow \quad Y^i(p) = X^i(p)$$

and a similar argument in V says that $Y^i(q) = X^i(q)$ $\forall p \in M, n \in N$. Thus, $Y = X$ as expected.

To see surjection, we notice the fact that the tangent spaces form vector spaces over $\mathbb{R}$. Further, we know that the dimension of the two spaces is equivalent, so if we have injection, we must have surjection. Thus, we have a bijection that preserves the vector space operations, and hence we have an isomorphism.

(b) Suppose $X \in T_pM$ and $Y \in T_qN$ for $p \in M$ and $q \in N$. Then, by the surjection, $\exists Z \in T_{(p,q)}(M \times N)$ such that $(\pi_{1*}, \pi_{2*})(Z) = (X, Y) \in T_pM \times T_qN$. Further, by lemma $\exists \gamma : I \subset \mathbb{R} \to M \times N$ such that $\gamma(t_0) = (p, q)$ and $\gamma'(t_0) = Z$. Hence, suppose $f \in C^\infty(M)$ and $g \in C^\infty(N)$, then

$$(\pi_{1*}, \pi_{2*}) \circ \gamma'(t_0)(f, g) = (\pi_{1*}, \pi_{2*}) \circ Z(f, g)(p, q) = (X(f(p)), Y(g(q)))$$

$$\Longrightarrow (\pi_{1*}, \pi_{2*}) \circ \gamma'(t_0) = (X|_p, Y|_q)$$

as required.

Question 4

Question 5

(a) To cook up some smooth vector field, we first use stereographic projection to give us our local coordinates. In particular, let $\{(U^+, \varphi^+), (U^-, \varphi^+)\}$ be our smooth atlas, where $U^+ = S^2 \backslash \{(0,0,1)\}$, $U^- = S^2 \backslash \{(0,0,-1)\}$ and the charts are

$$\varphi^+(x^1, x^2, x^3) = \frac{(x^1, x^2)}{1 - x^3} = (s, t) \qquad \varphi^-(x^1, x^2, x^3) = \frac{(x^1, x^2)}{1 - x^3} = (u, v)$$

where we use s, t, u, v as local variables in $\mathbb{R}^2$. Then, we see we would like to get the change of variables between the two coordinate systems. To do this, parameterize S^2 in terms of s, t to get,

$$x^1 = \frac{2s}{s^2 + t^2 + 1} \quad x^2 = \frac{2t}{s^2 + t^2 + 1} \quad x^3 = \frac{s^2 + t^2 - 1}{s^2 + t^2 + 1}$$

and then we plug this into our definitions for u and v,

$$u = \frac{s}{s^2 + t^2} \qquad v = \frac{t}{s^2 + t^2}$$

so we now have a change of coordinates. Now, if $p \in S^2$, then,

$$\forall p \in U^+ \quad T_pS^2 = \text{span}\left\{ \left.\frac{\partial}{\partial s}\right|_p, \left.\frac{\partial}{\partial t}\right|_p \right\} \qquad \forall p \in U^- \quad T_pS^2 = \text{span}\left\{ \left.\frac{\partial}{\partial u}\right|_p, \left.\frac{\partial}{\partial v}\right|_p \right\}$$

Consider the smooth vector field $Y = \frac{\partial}{\partial s}$ on U^+. Well, we consider what this vector field looks like in the other patch by looking at the intersect, and apply our definition,

$$Y = Y(u)\frac{\partial}{\partial u} + Y(v)\frac{\partial}{\partial v} = \frac{\partial u}{\partial s}\frac{\partial}{\partial u} + \frac{\partial v}{\partial s}\frac{\partial}{\partial v} = \left(\frac{(s^2 + t^2) - 2s^2}{(s^2 + t^2)^2}\right)\frac{\partial}{\partial u} + \left(\frac{-2st}{(s^2 + t^2)^2}\right)\frac{\partial}{\partial v}$$

which from our change of coordinates we get

$$Y = (v^2 - u^2)\frac{\partial}{\partial u} - 2uv\frac{\partial}{\partial v}$$

and we see that this vanishes identically only if $(u, v) = (0, 0)$, which happens when $p = (0, 0, 1)$, which is the north pole. Thus, since we can write,

$$Y = \begin{cases} \frac{\partial}{\partial s} & p \in U^+ \\ (v^2 - u^2)\frac{\partial}{\partial u} - 2uv\frac{\partial}{\partial v} & p \in U^- \end{cases}$$

and Y is a smooth vector field over all of S^2, we have it is only zero at $p = (0, 0, 1)$, as required.

(b) We start by first writing $\gamma_z(t)$ in the appropriate coordinates, namely, we recall that if $(a+ub, c+id) \in \mathbb{C}^2$, then the associated point in $\mathbb{R}^4$ is (a, b, c, d). With this in mind, we see that if $z_1 = a + ib, z_2 = c + id$, then

$$\gamma_z(t) = \left(e^{it}z_1, e^{it}z_2\right) = (a\cos(t) - b\sin(t) + i(b\cos(t) + a\sin(t)), c\cos(t) - d\sin(t) + i(d\cos(t) + c\sin(t)))$$

$$= (a\cos(t) - b\sin(t), b\cos(t) + a\sin(t), c\cos(t) - d\sin(t), d\cos(t) + c\sin(t)) \in \mathbb{R}^4$$

We recall stereographic projection for building our local coordinates; the atlas of S^3 will be $\{(U^+, \varphi^+)\ ,\ (U^-, \varphi^-)\}$ where $U^+ = S^3 \setminus (0,0,0,1)$ and $U^- = S^3 \setminus (0,0,0,-1)$. Also,

$$\varphi^+(x_1, x_2, x_3, x_4) = \frac{(x_1, x_2, x_3)}{1 - x_4} = (s, t, u) \qquad \varphi^-(x_1, x_2, x_3, x_4) = \frac{(x_1, x_2, x_3)}{1 + x_4} = (\alpha, \beta, \gamma)$$

are the local coordinates. We have all the machinery we need now to compute $\gamma_z'(t)$, in particular, by definition, if we suppose $z \simeq (a, b, c, d) \in U^+ \subset \mathbb{R}^4$, then

$$\gamma_z'(t) = \frac{d\gamma_z^i(t)}{dt}\frac{\partial}{\partial x^i}$$

but our coordinates are s, t, u, and so we need to find $\frac{d\gamma_z^i(t)}{dt}$ in these stereographic coordinates. This gets messy, so I will be skipping some of the algebra for simplicity,

$$\frac{d\gamma_z^1(t)}{dt} = \frac{d}{dt}(s) = \frac{d}{dt}\left(\frac{a\cos(t) - b\sin(t)}{1 - d\cos(t) - c\sin(t)}\right) = \cdots = \frac{-a\sin(t) - b\cos(t) + bd + ac}{(1 - d\cos(t) - c\sin(t))^2}$$

$$\frac{d\gamma_z^2(t)}{dt} = \frac{d}{dt}(t) = \frac{d}{dt}\left(\frac{b\cos(t) + a\sin(t)}{1 - d\cos(t) - c\sin(t)}\right) = \cdots = \frac{-b\sin(t) + a\cos(t) - ad + cb}{(1 - d\cos(t) - c\sin(t))^2}$$

$$\frac{d\gamma_z^3(t)}{dt} = \frac{d}{dt}(u) = \frac{d}{dt}\left(\frac{c\cos(t) - d\sin(t)}{1 - d\cos(t) - c\sin(t)}\right) = \cdots = \frac{-c\sin(t) - d\cos(t) + d^2 + c^2}{(1 - d\cos(t) - c\sin(t))^2}$$

and we have that

$$\gamma_z'(t) = \sum_{i=1}^{3}\frac{d\gamma_z^i(t)}{dt}\frac{\partial}{\partial x^i} = \frac{d\gamma_z^1(t)}{dt}\frac{\partial}{\partial s} + \frac{d\gamma_z^2(t)}{dt}\frac{\partial}{\partial t} + \frac{d\gamma_z^3(t)}{dt}\frac{\partial}{\partial u}$$

Looking at each component, we see that this vector field is non vanishing. A similar argument for U^- can show that it is non vanishing under the overlap.

With this in hand, if we let $Y_z = \gamma_z'(0)$, from the previous calculations we see that this field is necessarily nonvanishing over all of S^3, as we would like.

Question 6

(a) Let $f \in C^\infty(M)$. Then, from the definition of F-related, we get that, if $p \in M$,

$$Z_{F(p)} = F_{*,p}(Y_p)$$

where from definition,

$$Z_{F(p)}(f) = Z(F(p))(f) = Z(f) \circ F(p) \qquad F_{*,p}(Y_p)(f) = F_*(Y(p))(f) = Y(f \circ F(p))$$
$$\implies Z(f) \circ F(p) = Y(f \circ F(p)) \qquad \forall p \in M$$

and since it holds for all p, we can conclude

$$Z(f) \circ F = Y(f \circ F)$$

as required.

(b) Suppose $\mathcal{A}_M, \mathcal{A}_N$ smooth atlases for M and N respectively. Then, we know that $\mathcal{A}_M \times \mathcal{A}_N$ will be a smooth atlas for $M \times N$. In particular, let $(U, \varphi) \in \mathcal{A}_M$, $(V, \psi) \in \mathcal{A}_N$, $p \in U$, $q \in V$ with $U = (x_1, \dots, x_m)$ and $V = (y_1, \dots, y_n)$ local representations where $m = \dim(M)$ and $n = \dim(N)$. Then, $(U \times V, \varphi \times \psi) \in \mathcal{A}_M \times \mathcal{A}_N$, $(p, q) \in (U \times V) \subset M \times N$, with local representation $\varphi \times \psi = (x_1, \dots, x_m, y_1, \dots, y_n)$.

To build a smooth vector field on $M \times N$ with Y, we use the $\pi_1$related condition, where $f \in C^\infty(M)$, then we want that

$$Y(f) \circ \pi_1 = \tilde{Y}(f \circ \pi_1).$$

Looking at each part in local coordinates,

$$Y(f) \circ \pi_1(p, q) = Y(f(p)) = \sum_{i=1}^{m} Y^i(p) \frac{\partial f(p)}{\partial x_i} \qquad \tilde{Y}(f \circ \pi_1)(p, q) = \tilde{Y}_{(p,q)}(f(p))$$
$$= \sum_{i=1}^{m} \tilde{Y}^i((p, q)) \frac{\partial f(p)}{\partial x_i} + \sum_{j=1}^{n} \tilde{Y}^{j+m}((p, q)) \frac{\partial f(p)}{\partial y_j}$$

We notice that in order for $\tilde{Y}$ to be $\pi_1$related to Y, we must have that, for $i \in [1, m+n] \subset \mathbb{Z}$

$$\tilde{Y}^i((p, q)) = Y^i(p) \quad 0 < i \leq m \qquad \& \qquad \tilde{Y}^i((p, q)) = 0 \quad i > m$$

$\forall (p, q) \in M \times N$. We know that since $Y \in \mathcal{X}(M)$, $Y^i \in C^\infty(M)$ and we can extend this to a smooth function on $M \times N$ by making the new components zero. This is exactly what $\tilde{Y}$ is by construction; the components that do not match with Y^i vanish, and the resulting vector field is still smooth $\forall (p, q) \in U \times V$.

Thus, we have constructed a smooth vector field by looking componentwise at Y, and by definition of being $\pi_1$related by construction, we have that $Y = \pi_{1*}(\tilde{Y})$, as required.

The exact same steps can be taken for Z but the first m components would vanish in $\tilde{Z}^i$ instead. So, we can conclude the same result for Z and $\tilde{Z}$.

Question 7

(a) To show that the left multiplication map is indeed a diffeomorphism, we first show that it is a bijection. Suppose $k, h \in G$, and fix $g \in G$ such that $L_g(h) = L_g(k)$. Then, by definition of the map,

$$gh = L_g(h) = L_g(k) = gk \implies gh = gk$$

and since G is a group, we have that $g^{-1} \in G$, so

$$(g^{-1})gh = (g^{-1})gk \implies h = k.$$

So we can conclude that we have injection. Now notice that by the identity (e), inverse and associative property, we can get

$$k = ek = (gg^{-1})k = g(g^{-1}k) = L_g(g^{-1}k)$$

$\forall k \in G$, so we have surjection as well. Thus L_g is a bijection.

Now we need that L_g is also a smooth map. Take coordinate patches (U, φ) and (V, ψ) for G, which exist since G is a Lie Group and hence a smooth manifold. Furthermore, assume $L_g(U) \cap V \neq \emptyset$, as otherwise we get a trivially smooth map. Then, we see that we want

$$\psi \circ L_g \circ \varphi^{-1} : \varphi(U) \subset \mathbb{R}^n \to \psi(L_g(U) \cap V) \subset \mathbb{R}^n$$

to be smooth, where G is $n$dimensional. Suppose $h \in U \subset G$, then we see that

$$\psi \circ L_g \circ \varphi^{-1}(\varphi(h)) = \psi(L_g(h)) = \psi(gh)$$

but by assumption, we must have that $gh \in V$, and further, since G is smooth, we know that ψ is smooth, and hence we have that this composition is indeed smooth as required.

Now we need to check that it has a smooth inverse. Notice,

$$\left(\psi \circ L_g \circ \varphi^{-1}\right)^{-1}(\psi(gh)) = \varphi \circ L_g^{-1} \circ \psi^{-1}(\psi(gh)) = \varphi(L_g^{-1}(gh)) = \varphi(g^{-1}gh) = \varphi(h)$$

which is smooth by assumptions on φ. Thus we have a smooth inverse.

Therefore, since we have that L_g is a bijection that is smooth and has a smooth inverse $\forall (U, \varphi), (V, \psi)$ in the smooth atlas, we can conclude that L_g is a diffeomorphism.

(b) This follows from the *naturality* property of the Lie Bracket, in particular, since Y is left invariant, it is L_G related to itself, and similarly for Z. Then, by the *naturality* property of Lie Brackets, we must have that $[Y, Z]$ is L_g related to itself, but then by definition $[Y, Z]$ is left invariant, as required.

(c) Suppose $X, Y \in Lie(G)$. Then, we see by the $\mathbb{R}$-linearity of the pushforward that majority of the vector space properties fall out, with $Lie(G)$ being a vector space over the reals. The remaining properties follow from the associativity of the real numbers, and the vector space properties for tangent spaces, since these vectors are already a part of the tangent space. All we need for this to work is that the sum of two vectors in $Lie(G)$ is still in $Lie(G)$. In particular, if $f \in C^\infty(G)$, $h \in G$,

$$L_{g*}(X+Y)_h(f) = L_{g*}(X_h)(f) + L_{g*}(Y_h)(f) = X_{gh}(f) + Y_{gh}(f) = (X+Y)_{gh}(f) \in Lie(G)$$

as required.

To see that $Lie(G)$ is isomorphic to $T_e(G)$, we need to confirm bijection and that the vector space properties are preserved. Suppose $X, Y \in Lie(G)$, such that $ev(X) = ev(Y)$, then by definition,

$$ev(X) = ev(Y) \iff X_e = Y_e$$

but, since both X, Y are in $Lie(G)$, we have that, $\forall g \in G$,

$$X_g = X_{ge} = L_{g*}X_e = L_{g*}Y_e = Y_{ge} = Y_g \implies X = Y.$$

This gives us injection. Now we need surjection, suppose $Z \in T_e(G)$. Then, by the smooth extension lemma of vector fields we proved in $Q1$, we know $\exists W \in \mathcal{X}(G)$ such that $W_e = Z$. Now we need that this smooth vector field is in $Lie(G)$, or in particular, that W is L_g-related to itself $\forall g \in G$.