Problem 1

(a) Following the method done in Assignment 3 Question 5, we see that we can multiply the equation through by $\frac{1}{x}$ to get

$$xM'' + M' + \frac{\lambda}{x}M = 0$$

But we notice that this can be reduced to,

$$-\frac{d}{dx}\left(xM'\right) = \frac{\lambda}{x}M$$

Which if we compare with the general Sturm-Liouville problem, we see that in this case we have $\rho(x) = \frac{1}{x}$ as required.

(b) We assume that $x = e^z$. Substituting this into our PDE

$$-\frac{d}{dx}(xM') = \frac{\lambda}{x}M \quad \rightarrow \quad -\frac{d}{dz}(M') = \lambda M$$

We can see this from the following

$$dx = e^z dz \quad \& \quad \frac{dM}{dx} = \frac{dM}{dz}\frac{dz}{dx}$$

and so

$$-\frac{d}{dz}\frac{dz}{dx}\left(e^z \frac{dM}{dz}\frac{dz}{dx}\right) = \frac{\lambda}{e^z}M \quad \rightarrow \quad -e^{-z}\frac{d}{dz}(e^z e^{-z}M') = e^{-z}\lambda M$$

And thus we have that

$$-M'' = \lambda M$$

We recognize this ODE and see that the solution must be

$$M(z) = A\cos(\sqrt{\lambda}z) + B\sin(\sqrt{\lambda}z)$$

(c) We have our boundary conditions, but when $x = 1$ we see that $z = 0$ and when $x = L$ we have $z = \ln(L)$. Hence,

$$M(0) = A\cos(0) + B\sin(0) = 0 \quad \Longrightarrow \quad A = 0$$

Further, we have that

$$M(\ln(L)) = B\sin(\sqrt{\lambda}\ln(L)) = 0 \quad \Longrightarrow \quad \sqrt{\lambda}\ln(L) = n\pi \quad \Longrightarrow \quad \lambda_n = \frac{n^2\pi^2}{(\ln(L))^2}$$

where $n \in \mathbb{N} \setminus \{0\}$ with associated eigenfunction $M_n = \sin(\sqrt{\lambda_n}z)$.

(d) To see that this does indeed obey the orthogonality principle, assume that $m \neq n$ where $m.n \in \mathbb{N} \setminus \{0\}$. Then,

$$(M_n, M_m) = \int_0^{\ln(L)} \sin\left(\frac{n\pi z}{\ln(L)}\right)\sin\left(\frac{m\pi z}{\ln(L)}\right)dz = 0$$

since we have two odd functions and we integrate over their period.

Problem 2

First we need that the operator $\frac{1}{r}L$ is self-adjoint. So we show this property first. Assume two functions in our function space $f(r)$ and $g(r)$. Then,

$$\left(f, \frac{1}{r}L[g]\right) = \int_V f\frac{1}{r}L[g]rdr = \int_V f\left(-\frac{d}{dr}\left(r\frac{dg}{dr}\right) + \frac{n^2}{r}g\right)dr = -\int_V f\frac{d}{dr}\left(r\frac{dg}{dr}\right)dr + \int_V \frac{n^2}{r}fgdr$$

On the first integral we can use integration by parts,

$$-\int_V f\frac{d}{dr}\left(r\frac{dg}{dr}\right)dr = -f\left(r\frac{dg}{dr}\right)\Big|_{\partial V} + \int_V r\frac{df}{dr}\frac{dg}{dr}dr$$

integrate by parts again,

$$-\int_V f\frac{d}{dr}\left(r\frac{dg}{dr}\right)dr = -f\left(r\frac{dg}{dr}\right)\Big|_{\partial V} + r\frac{df}{dr}g\Big|_{\partial V} - \int_V \frac{d}{dr}\left(r\frac{df}{dr}\right)gdr$$

where our terms on the boundary will vanish leaving us with,

$$-\int_V f\frac{d}{dr}\left(r\frac{dg}{dr}\right)dr = -\int_V g\frac{d}{dr}\left(r\frac{df}{dr}\right)dr$$

and so our inner product becomes

$$\left(f, \frac{1}{r}L[g]\right) = -\int_V g\frac{d}{dr}\left(r\frac{df}{dr}\right)dr + \int_V \frac{n^2}{r}fgdr = \int_V g\left(-\frac{d}{dr}\left(r\frac{df}{dr}\right) + \frac{n^2}{r}f\right)dr$$

which by definition of our operator is

$$\left(f, \frac{1}{r}L[g]\right) = \int_V g\frac{1}{r}L[f]rdr = \left(\frac{1}{r}L[f], g\right)$$

which gives us our self-adjoint property. Now, say that M_n and M_m are solutions to the eigenvalue problem such that

$$\frac{1}{r}LM_n = \lambda_n M_n \quad \& \quad \frac{1}{r}LM_m = \lambda_m M_m$$

where $\lambda_m \neq \lambda_n$. Further, we know that $\frac{1}{r}L$ is self-adjoint, hence

$$\left(M_n, \frac{1}{r}LM_m\right) = \left(\frac{1}{r}LM_n, M_m\right)$$

$$(M_n, \lambda_m M_m) = (\lambda_n M_n, M_m)$$

$$(M_n, \lambda_m M_m) - (\lambda_n M_n, M_m) = 0$$

$$\lambda_m(M_n, M_m) - \lambda_n(M_n, M_m) = 0$$

$$(\lambda_m - \lambda_n)(M_n, M_m) = 0$$

But by assumption $\lambda_n \neq \lambda_m$ so we see that $(M_n, M_m) = 0$ and hence the two eigenfunctions will be orthogonal.

Problem 3

(a) To see that $\mathcal{L}$ is indeed self-ajoint we apply the definition of inner product. First let f and g be functions of $\vec{x}$, then we see

$$\left(f, \frac{1}{\rho}\mathcal{L}[g]\right) = \int_V f\frac{1}{\rho}\mathcal{L}[g]\rho dV = \int_V f\left(-\vec{\nabla}\cdot(p\vec{\nabla}g) + qg\right)dV = \int_V f\left(-\vec{\nabla}\cdot(p\vec{\nabla}g)\right)dV + \int_V fqgdV$$

We know use the vector identity $\nabla\cdot(\psi\mathbf{A}) = \psi(\nabla\cdot\mathbf{A}) + \mathbf{A}\cdot(\nabla\psi)$, which gives us

$$\left(f, \frac{1}{\rho}\mathcal{L}[g]\right) = -\int_V \left(\vec{\nabla}\cdot(fp\vec{\nabla}g) - p\vec{\nabla}g\cdot\vec{\nabla}f\right)dV + \int_V fqgdV = -\int_V \vec{\nabla}\cdot(fp\vec{\nabla}g)dV + \int_V p\vec{\nabla}g\cdot\vec{\nabla}fdV + \int_V fqgdV$$

Now we apply the Divergence theorem to the first term,

$$\begin{aligned}\left(f, \frac{1}{\rho}\mathcal{L}[g]\right) &= -\oint_{\partial V} fp\vec{\nabla}g\cdot\vec{n}dS + \int_V p\vec{\nabla}g\cdot\vec{\nabla}fdV + \int_V fqgdV \\ &= -\oint_{\partial V} fp\frac{\partial g}{\partial n}dS + \int_V p\vec{\nabla}g\cdot\vec{\nabla}fdV + \int_V fqgdV\end{aligned}$$

We now apply the boundary condition to replace $\frac{\partial g}{\partial n}$ with $-\frac{\alpha}{\beta}g$, but the by commutivity and associativity, we shift the $-\frac{\alpha}{\beta}$ to the f and again apply the boundary condition to get $\frac{\partial f}{\partial n}$,

$$\left(f, \frac{1}{\rho}\mathcal{L}[g]\right) = -\oint_{\partial V} gp\vec{\nabla}f\cdot\vec{n}dS + \int_V p\vec{\nabla}g\cdot\vec{\nabla}fdV + \int_V fqgdV$$

apply Divergence theorem again

$$= -\int_V \vec{\nabla}\cdot(gp\vec{\nabla}f)dV + \int_V p\vec{\nabla}g\cdot\vec{\nabla}fdV + \int_V fqgdV$$

further, we know $\vec{\nabla}g\cdot\vec{\nabla}f = \vec{\nabla}f\cdot\vec{\nabla}g$, and hence can use the earlier identity in reverse,

$$= -\int_V \left(\vec{\nabla}\cdot(gp\vec{\nabla}f) - p\vec{\nabla}f\cdot\vec{\nabla}g\right)dV + \int_V gqfdV = -\int_V g\left(-\vec{\nabla}\cdot(p\vec{\nabla}f)\right)dV + \int_V gqfdV$$

and we apply linearity to get

$$= \int_V g\left(-\vec{\nabla}\cdot(p\vec{\nabla}f) + qg\right)dV = \int_V g\frac{1}{\rho}\mathcal{L}[f]\rho dV = \left(\frac{1}{\rho}\mathcal{L}[f], g\right)$$

as required.

(b) Now that we have that the operator is self-adjoint, we use the same technique as we did in **Problem 2**. Assume that M_n and M_m are eigenfunctions such that

$$\frac{1}{\rho}\mathcal{L}M_n = \lambda_n M_n \quad \& \quad \frac{1}{\rho}\mathcal{L}M_m = \lambda_m M_m$$

where $\lambda_m \neq \lambda_n$. Further, we know that $\frac{1}{\rho}\mathcal{L}$ is self-adjoint, hence

$$\left(M_n, \frac{1}{\rho}\mathcal{L}M_m\right) = \left(\frac{1}{\rho}\mathcal{L}M_n, M_m\right)$$

$$(M_n, \lambda_m M_m) = (\lambda_n M_n, M_m)$$

$$(M_n, \lambda_m M_m) - (\lambda_n M_n, M_m) = 0$$

$$\lambda_m(M_n, M_m) - \lambda_n(M_n, M_m) = 0$$

$$(\lambda_m - \lambda_n)(M_n, M_m) = 0$$

But by assumption $\lambda_n \neq \lambda_m$ so we see that $(M_n, M_m) = 0$ and hence the two eigenfunctions will be orthogonal

(c) We follow a similar procedure to part **(a)**. Start by multiplying the equation in question through by u and integrating over V to get

$$\int_V \lambda u^2 \rho dV = \int_V u\mathcal{L}u dV = \int_V u\left(-\vec{\nabla}\cdot(p\vec{\nabla}u) + qu\right)dV = \int_V u\left(-\vec{\nabla}\cdot(p\vec{\nabla}u)\right)dV + \int_V uqudV$$

We apply the identity from before, $\nabla\cdot(\psi\mathbf{A}) = \psi(\nabla\cdot\mathbf{A}) + \mathbf{A}\cdot(\nabla\psi)$, which gives us

$$= -\int_V \left(\vec{\nabla}\cdot(up\vec{\nabla}u) - p\vec{\nabla}u\cdot\vec{\nabla}u\right)dV + \int_V qu^2 dV = -\int_V \vec{\nabla}\cdot(up\vec{\nabla}u)dV + \int_V p\vec{\nabla}u\cdot\vec{\nabla}udV + \int_V qu^2 dV$$

Apply divergence theorem

$$= -\oint_{\partial V} up\vec{\nabla}u\cdot\vec{n}dV + \int_V p\left(\vec{\nabla}u\right)^2 dV + \int_V qu^2 dV$$

and applying boundary conditions we see

$$\lambda\int_V u^2\rho dV = \frac{\alpha}{\beta}\oint_{\partial V} pu^2 dV + \int_V p\left(\vec{\nabla}u\right)^2 dV + \int_V qu^2 dV$$

where explicitly for λ we get

$$\lambda = \frac{\frac{\alpha}{\beta}\oint_{\partial V} pu^2 dV + \int_V p\left(\vec{\nabla}u\right)^2 dV + \int_V qu^2 dV}{\int_V u^2\rho dV}$$

where clearly $\lambda > 0$ since the integrals over the squares of u and its gradient will naturally be > 0 and all of the coefficients are positive by assumption.

Problem 4

(a) Let $u(x,t) = M(x)N(t)$, then plugging this into the PDE yields

$$MN' = DM''N - V_0M'N$$

where we can divide through by MN to get

$$\frac{N'}{N} = \frac{DM'' - V_0M'}{M}$$

which we can separate using a constant $-\lambda$,

$$N' + \lambda N = 0 \qquad \& \qquad DM'' - V_0M' + \lambda M = 0$$

Since D and V_0 are constants, we can not simply write the initial sum as a derivative of the product of two functions. Thus this is not a Sturm-Liouville problem.

(b) First we solve the spatial problem. Assume that the solution is of the form e^{rx}, then

$$e^{rx}\left(Dr^2 - V_0r + \lambda\right) = 0 \quad \implies \quad r = \frac{V_0 \pm \sqrt{V_0^2 - 4D\lambda}}{2D}$$

Now consider the BCs. We see for non-trivial temporal part,

$$M(0)N(t) = 0 \quad M(L)N(t) = 0 \qquad \implies \qquad M(0) = 0 \quad M(L) = 0$$

But such BCs would require sinusoidal solutions, since the only other option is exponential which will yield the trivial solution with these conditions. Hence, assume that

$$V_0^2 < 4D\lambda$$

hence our solution will be

$$M(x) = A\cos\left(\frac{\sqrt{4D\lambda - V_0^2}}{2D}x\right) + B\sin\left(\frac{\sqrt{4D\lambda - V_0^2}}{2D}x\right)$$

But the first BC, $M(0) = 0 \implies A = 0$. The second BC gives us

$$M(L) = B\sin\left(\frac{\sqrt{4D\lambda - V_0^2}}{2D}L\right) = 0 \quad \implies \quad \frac{\sqrt{4D\lambda - V_0^2}}{2D}L = n\pi$$

$$\implies \quad \lambda_n = \frac{n^2\pi^2 D}{L^2} + \frac{V_0^2}{4D} \quad \& \quad M_n(x) = \sin\left(\frac{n\pi}{L}x\right)$$

Where $n \in \mathbb{N} \setminus \{0\}$. Now that we have λ_n we solve the temporal ODE,

$$N' + \lambda N = 0 \qquad \implies \qquad N_n(t) = Ae^{-\lambda_n t}$$

(c) Notice that the only thing that we have changed are the boundary conditions,

$$\frac{\partial u}{\partial x}(0,t) = M'(0)N(t) = 0 \quad \& \quad \frac{\partial u}{\partial x}(L,t) = M'(L)N(t) = 0 \qquad \implies \qquad M'(0) = 0 \quad M'(L) = 0$$

due to non-trivial temporal solutions. So we see the first BC will give us

$$M'(0) = -A\frac{\sqrt{4D\lambda - V_0^2}}{2D}\sin(0) + B\frac{\sqrt{4D\lambda - V_0^2}}{2D}\cos(0) = 0 \qquad \implies \qquad B = 0$$

The second BC will give us

$$M'(L) = -A\frac{\sqrt{4D\lambda - V_0^2}}{2D}\sin\left(\frac{\sqrt{4D\lambda - V_0^2}}{2D}L\right)$$

which for non-trivial λ_n we get,

$$\frac{\sqrt{4D\lambda - V_0^2}}{2D}L = n\pi \qquad \implies \qquad \lambda_n = \frac{n^2\pi^2 D}{L^2} + \frac{V_0^2}{4D}$$

where $n \in \mathbb{N} \cup \{0\}$, and the corresponding eigenfunction is

$$M_n = \cos\left(\frac{n\pi}{L}x\right)$$

The temporal ODE will give a solution of

$$N' + \lambda_n N = 0 \qquad \implies \qquad N_n(t) = Ae^{-\lambda_n t}$$

as expected.

Problem 5

Assume that the solution to this PDE is separable. Then, we can say $u(x,t) = M(x)N(t)$, and plugging this into our PDE we get

$$MN'' - \gamma^2 M''N + c^2 MN = 0$$

we divide through by MN

$$\frac{N''}{N} - \gamma^2 \frac{M''}{M} + c^2 = 0$$

and now separate with λ to get

$$\frac{N''}{N} + c^2 = -\lambda \qquad \& \qquad \gamma^2 \frac{M''}{M} = -\lambda$$

$$N'' + (c^2 + \lambda)N = 0 \qquad \& \qquad M'' + \frac{\lambda}{\gamma^2} M = 0$$

We first solve the spatial ODE, where we recognize that the solution will be

$$M(x) = A\cos\left(\frac{\sqrt{\lambda}}{\gamma}x\right) + B\sin\left(\frac{\sqrt{\lambda}}{\gamma}x\right)$$

Notice that for non-trivial temporal solutions, we have $u(0,t) = M(0)N(t) = 0 \implies M(0) = 0$ and similarly we get $M(L) = 0$. The first BC gives us that $A = 0$, and the second gives

$$M(L) = B\sin\left(\frac{\sqrt{\lambda}}{\gamma}L\right) = 0 \qquad \Longrightarrow \qquad \lambda_n = \frac{n^2\pi^2\gamma^2}{L^2}$$

hence we have our eigenvalue λ_n with corresponding eigenfunction $M_n(x) = B\sin\left(\frac{n\pi x}{L}\right)$, which we can normalize

$$|M_n(x)| = \sqrt{\int_0^L B^2 \sin^2\left(\frac{n\pi x}{L}\right)dx} = B\sqrt{\int_0^L \frac{1-\cos\left(\frac{2n\pi x}{L}\right)}{2}dx} = B\sqrt{\frac{L}{2}} = 1$$

hence we see that our normalized eigenfunction will be $M_n(x) = \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right)$. Now we solve the temporal ODE. Again, we recognize the solution as

$$N_n(t) = A_n\cos\left(\sqrt{c^2+\lambda}t\right) + B_n\sin\left(\sqrt{c^2+\lambda}t\right)$$

So the general solution will be

$$u(x,t) = \sum_{n=1}^{\infty} M_n(x)N_n(t) = \sum_{n=1}^{\infty}\sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right)\left(A_n\cos\left(\sqrt{c^2+\lambda}t\right) + B_n\sin\left(\sqrt{c^2+\lambda}t\right)\right)$$

Applying the first IC we get

$$u(x,0) = \sum_{n=1}^{\infty} M_n(x)(A_n) \qquad \Longrightarrow \qquad f(x) = \sum_{n=1}^{\infty} M_n(x)(A_n)$$

To solve for the fourier coeffecient A_n we take the innerproduct of both sides with the n^{th} term $M_n(x)$ to get

$$(M_n(x), f(x)) = A_n$$

which can be solved with a given explicit $f(x)$. Next we solve for B_n by using the other IC,

$$u_t(x,0) = \sum_{n=1}^{\infty} M_n(x)\left((\sqrt{c^2+\lambda_n})B\_n\right) \qquad \Longrightarrow \qquad g(x) = \sum_{n=1}^{\infty} M_n(x)\left((\sqrt{c^2+\lambda_n})B_n\right)$$

again we take the inner product with the n^{th} term $M_n(x)$ to get

$$\frac{(M_n(x), g(x))}{\sqrt{c^2+\lambda_n}} = B_n$$

which can again be solved for explicitly if we have $g(x)$. We now have everything to solve the PDE,

$$u(x,t) = \sum_{n=1}^{\infty} \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right)\left((M_n(x), f(x))\cos\left(\sqrt{c^2+\lambda t}\right) + \frac{(M_n(x), g(x))}{\sqrt{c^2+\lambda_n}} \sin\left(\sqrt{c^2+\lambda t}\right)\right)$$

Problem 6

We follow the same procedure as before. Assume a solution of the form $u(x,y) = M(x)N(y)$, which gives us

$$M''N + MN'' = 0$$

which is easily separated into

$$M'' + \lambda M = 0 \qquad \& \qquad N'' - \lambda N = 0$$

We solve the x ODE first. We notice that the solution must be

$$M(x) = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x)$$

The first BC, which for non-trivial $N(y)$ is $M(0) = 0$, tells us that $A = 0$. The second BC, through a similar argument, will tell us

$$M(L_x) = B\sin(\sqrt{\lambda}L_x) = 0 \qquad \Longrightarrow \qquad \lambda_n = \frac{n^2\pi^2}{L_x^2}$$

Now that we have our eigenvalue λ_n we can normalize the corresponding eigenfunction to get

$$M_n(x) = \sqrt{\frac{2}{L_x}}\sin\left(\frac{n\pi x}{L_x}\right)$$

We now solve the y component. Notice that the general solution will be

$$N_n(y) = A_n e^{\sqrt{\lambda_n}y} + B_n e^{-\sqrt{\lambda_n}y}$$

So our combined solution is

$$u(x,y) = f(x) = \sum_{n=1}^{\infty} M_n(x)N_n(y)$$

we can now apply the ICs to solve for the Fourier coefficients A_n and B_n. The first IC gives

$$u_y(x,0) = \sum_{n=1}^{\infty} M_n(x)\left(\sqrt{\lambda_n}(A_n - B_n)\right)$$

Similar to the previous question, we can solve for B_n by taking the inner product with M_n,

$$\frac{(M_n, f(x))}{\sqrt{\lambda_n}} = A_n - B_n$$

$$\frac{(M_n, f(x))}{\sqrt{\lambda_n}} + B_n = A_n$$

The second IC tells us

$$u_y(x,L_y) = g(x) = \sum_{n=1}^{\infty} M_n(x)\left(\sqrt{\lambda_n}A_n e^{\sqrt{\lambda_n}L_y} - \sqrt{\lambda_n}B_n e^{-\sqrt{\lambda_n}L_y}\right)$$

Where we again can take the inner product to get

$$(M_n(x), g(x)) = (M_n(x), f(x)) \left(\sqrt{\lambda_n} A_n e^{\sqrt{\lambda_n} L_y} - \sqrt{\lambda_n} B_n e^{-\sqrt{\lambda_n} L_y} \right)$$

$$(M_n(x), g(x)) = \left(\sqrt{\lambda_n} \left(\frac{(M_n, f(x))}{\sqrt{\lambda_n}} + B_n \right) e^{\sqrt{\lambda_n} L_y} - \sqrt{\lambda_n} B_n e^{-\sqrt{\lambda_n} L_y} \right)$$

we now solve for B_n,

$$B_n = \frac{(M_n, g(x)) - (M_n, f(x)) e^{\sqrt{\lambda_n} L_y}}{\sqrt{\lambda_n} \left(e^{\sqrt{\lambda_n} L_y} - e^{-\sqrt{\lambda_n} L_y} \right)}$$

and hence we have the final solution is just applying these coefficients.