Question 1

(a) We find the splitting field of $f(x) = x^{11} - 2$ by finding the roots. In particular, notice that we have an immediate root of $\sqrt[11]{2}$, and if we suppose ξ_{11} the 11th root of unity, then we have that the roots of this polynomial will be $\sqrt[11]{2}, \xi_{11}\sqrt[11]{2}, \xi_{11}^2\sqrt[11]{2}, \ldots, \xi_{11}^{10}\sqrt[11]{2}$. So, the splitting field will be $\mathbb{Q}(\sqrt[11]{2}, \xi_{11})$. Notice, $\deg_{\mathbb{Q}}\left(\sqrt[11]{2}\right) = 11$ and $\deg_{\mathbb{Q}}(\xi_{11}) = 10$ since 11 is prime. These two are coprime, and thus by a lemma we proved in the previous assignment,

$$[\mathbb{Q}(\sqrt[11]{2}, \xi_{11}) : \mathbb{Q}] = [\mathbb{Q}(\sqrt[11]{2}) : \mathbb{Q}][\mathbb{Q}(\xi_{11}) : \mathbb{Q}] = 11 \cdot 10 = 110\,.$$

(b) We find the splitting field of $f(x) = x^4 - x^2 + 4$ by first finding the roots:

$$f(x) = (x^2+2)^2 - 5x^2 = (x^2 + 2 - \sqrt{5}x)(x^2 + 2 + \sqrt{5})$$

$$f(x) = \left(x - \frac{\sqrt{5} - i\sqrt{3}}{2}\right)\left(x - \frac{\sqrt{5} + i\sqrt{3}}{2}\right)\left(x - \frac{-\sqrt{5} - i\sqrt{3}}{2}\right)\left(x - \frac{-\sqrt{5} + i\sqrt{3}}{2}\right).$$

These aren't nice roots, but we only need to adjoin $\sqrt{5}$ and $i\sqrt{3}$ onto $\mathbb{Q}$ to have the splitting field, that is, the splitting field is $\mathbb{Q}(\sqrt{5}, i\sqrt{3})$. To compute $[\mathbb{Q}(\sqrt{5}, i\sqrt{3}) : \mathbb{Q}]$, we notice

$$\deg_{\mathbb{Q}}(\sqrt{5}) = \deg_{\mathbb{Q}}(x^2 - 5) = 2 \quad \deg_{\mathbb{Q}}(\sqrt{3}) = \deg_{\mathbb{Q}}(x^2 + 3) = 2\,.$$

Since these are not coprime, we can't apply our little trick. Instead, we use the fact that $f(x)$ is irreducible over $\mathbb{Q}$. This would imply that any field extension using this polynomial better have degree atleast 4, and since $f(x)$ will split over $\mathbb{Q}(\sqrt{5}, i\sqrt{3})$, it better be atleast degree 4. However, from last assignment, we also have an inequality that says $[\mathbb{Q}(\sqrt{5}, i\sqrt{3}) : \mathbb{Q}] \leq 4$, and thus we would require that $[\mathbb{Q}(\sqrt{5}, i\sqrt{3}) : \mathbb{Q}] = 4$.

So, to see that $f(x)$ is irreducible, we use the mod-3 irreducibility test. Then,

$$\bar{f}(x) = x^4 - x^2 + 1 \quad \bar{f}(0) = 1, \bar{f}(1) = 1, \bar{f}(2) = 1\,.$$

So, we need to check if this factors into degree 2 irreducible polynomials. The irreducible polynomials of $\mathbb{Z}_3[x]$ are

$$x^2 + 2\,,\ x^2 + 1\,,\ x^2 + x + 2\,,\ x^2 - x + 2\,,\ 2x^2 + x + 1\,,\ 2x^2 - x + 1\,.$$

Immediatly, we see that the polynomials with leading coefficient 2 can only multiply one another since the result better be monic, but the product is $x^4 - x + 1$. Similarly, the polynomials with only an x^2 term can only multiply one another since the result with another polynomial would leave them with an extra x^3 term, and they multiply together to $x^4 - 1$. Finally, the remaining two multiply to $x^4 + 4$. Thus, we have irreducibility, and

$$[\mathbb{Q}(\sqrt{5}, i\sqrt{3}) : \mathbb{Q}] = 4\,.$$

(c)