Question 1

(1.1) We know that physics is the same at any point in space, so consider our potential at the current origin to be $U(\vec{r}_1, \vec{r}_2)$. Then, if we do a change of coordinate, we can take our origin to be at the position of the second particle, then we see

$$\vec{r}_2 \mapsto \vec{0} \quad \& \quad \vec{r}_1 \mapsto \vec{r}_1 - \vec{r}_2$$

which is a translation, and physics is invariant under this transformation. Then, under this transformation, we see

$$U(\vec{r}_1, \vec{r}_2) \mapsto U(\vec{r}_1 - \vec{r}_2, 0) = \boxed{U(\vec{r}_1 - \vec{r}_2)}$$

as required.

(1.2) We see that

$$\dot{\vec{p}}_1 = -\partial_{\vec{r}_1} U(r_1 - r_2) = -U' \quad \& \quad \dot{\vec{p}}_2 = -\partial_{\vec{r}_2} U(r_1 - r_2) = U'$$

and so adding our equations

$$\dot{\vec{p}}_1 + \dot{\vec{p}}_2 = -U' + U' = 0 \implies \boxed{\frac{d}{dt}(\vec{p}_1 + \vec{p}_2) = 0}.$$

This tells us that the total momentum of our system is unchanged with time, and so momentum is conserved!

Question 2

To be latexed

Question 3

Looking at our diagrams, we first compute our matrix element for the two lowest order diagrams.

$$\begin{aligned} -i\mathcal{M}_1 &= (-ig)^2 \int \frac{d^4q}{(2\pi)^4} \frac{i}{q^2 - m^2} (2\pi)^4 \delta^4(P_1 + P_2 - q)(2\pi)^4 \delta^4(q - P_3 - P_4) \\ &= -g^2 \frac{i}{(P_1 + P_2)^2 - m^2} \end{aligned}$$

$$\implies \boxed{\mathcal{M}_1 = \frac{g^2}{(P_1 + P_2)^2 - m^2}}.$$

Similarly for the second diagram, we see

$$\begin{aligned} -i\mathcal{M}_2 &= (-ig)^2 \int \frac{d^4q}{(2\pi)^4} \frac{i}{q^2 - m^2} (2\pi)^4 \delta^4(P_1 - P_4 - q)(2\pi)^4 \delta^4(q - P_3 + P_2) \\ &= -g^2 \frac{i}{(P_3 - P_2)^2 - m^2} \end{aligned}$$

$$\implies \boxed{\mathcal{M}_2 = \frac{g^2}{(P_3 - P_2)^2 - m^2}}.$$

So, adding these together we get

$$\mathcal{M} = \mathcal{M}_1 + \mathcal{M}_2 = \frac{g^2}{(P_1 + P_2)^2 - m^2} + \frac{g^2}{(P_3 - P_2)^2 - m^2}$$

$$\boxed{\mathcal{M} = g^2 \left(\frac{1}{(P_1 + P_2)^2 - m^2} + \frac{1}{(P_3 - P_2)^2 - m^2} \right)}.$$

Now, if we assume that initially A is at rest in the lab frame, we know $\vec{p}_2 = \vec{0}$, so $E_2 = m$. Keeping this in mind, we compute our denominators,

$$\begin{aligned}
(P_1 + P_2)^2 - m^2 &= (E_1 + E_2)^2 - (\vec{p}_1 - \underbrace{\vec{p}_2}_{0})^2 - m^2 \\
&= \underbrace{E_1^2 - p_1^2}_{0} + 2E_1E_2 + \underbrace{E_2^2 - m^2}_{0} \\
&= 2E_1 m
\end{aligned}$$

where we from now on call $E_1 = E$, since it is the incident energy. Next,

$$\begin{aligned}
(P_3 - P_2)^2 - m^2 &= (E_3 - E_2)^2 - (\vec{p}_3 - \vec{p}_2)^2 - m^2 \\
&= \underbrace{E_3^2 - p_3^2}_{0} - 2E_3E_2 + \underbrace{E_2^2 - m^2}_{0} \\
&= -2E_3 m\,.
\end{aligned}$$

So, we have found our components, and hence

$$\mathcal{M} = g^2 \left(\frac{1}{2E_1 m} + \frac{1}{-2E_3 m} \right)$$

$$\boxed{\mathcal{M} = \frac{g^2}{2m} \left(\frac{1}{E_1} - \frac{1}{E_3} \right)}$$

Before we apply Fermi's Golden Rule, we need to get a little identity to get E_3 in terms of E_1 and θ. Using conservation of four-momentum,

$$\begin{aligned}
P_1 + P_2 &= P_3 + P_4 \\
P_4 &= P_1 + P_2 - P_3 \\
P_4^2 &= (P_1 + P_2 - P_3)^2 \\
m^2 &= P_1^2 + P_2^2 + P_3^2 + 2P_1P_2 - 2P_1P_3 - 2P_2P_3 \\
m^2 &= m^2 + 2mE_1 - 2E_1E_3(1 - \cos\theta) - 2mE_3 \\
2E_1 m &= 2mE_3 + 2E_1E_3(1 - \cos\theta) \\
\boxed{\frac{E_1 m}{E_3}} &\boxed{= m + E_1(1 - \cos\theta)}.
\end{aligned}$$

Then, plugging what we know into Fermi's Golden Rule for the cross-section, we get

$$\begin{aligned}\left(\frac{d\sigma}{d\Omega}\right)_{\text{Lab}} &= \left(\frac{1}{8\pi}\right)^2 \frac{\mathcal{S}|\mathcal{M}|^2 p_3^2}{mp_1(p_3(E_1+m) - p_1 E_3\cos\theta)} \\ &= \left(\frac{g^2}{16m\pi}\right)^2 \mathcal{S}\left(\frac{1}{E_1} - \frac{1}{E_3}\right)^2 \frac{E_3^2}{mE_1} \frac{1}{E_3(E_1+m) - E_1E_3\cos\theta} \\ &= \left(\frac{g^2}{16m\pi}\right)^2 \mathcal{S}\left(\frac{1}{E_1} - \frac{1}{E_3}\right)^2 \frac{E_3}{mE_1} \frac{1}{m + E_1(1-\cos\theta)}\end{aligned}$$

and knowing that $\mathcal{S} = 1$ and using our identity for $\frac{mE_1}{E_3}$, we get

$$\begin{aligned}\left(\frac{d\sigma}{d\Omega}\right)_{\text{Lab}} &= \left(\frac{g^2}{16m\pi}\right)^2 \left(\frac{1}{E_1} - \frac{m+E_1(1-\cos\theta)}{E_1 m}\right)^2 \left(\frac{1}{m+E_1(1-\cos\theta)}\right) \frac{1}{m+E_1(1-\cos\theta)} \\ &= \left(\frac{g^2}{16m\pi}\right)^2 \left(\frac{-E_1(1-\cos\theta)}{E_1 m}\right)^2 (m + E(1-\cos\theta))^{-2}\end{aligned}$$

$$\boxed{\left(\frac{d\sigma}{d\Omega}\right)_{\text{Lab}} = \left(\frac{g^2}{16m\pi}\right)^2 \left(\frac{1-\cos\theta}{m(m+E(1-\cos\theta))}\right)^2}$$

as required.

Question 4

Question 5

(5.1) We recall the Euler Lagrange Equation is

$$\partial_\mu \left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\right) = \frac{\partial\mathcal{L}}{\partial\psi} \quad \& \quad \partial_\mu \left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\bar{\psi})}\right) = \frac{\partial\mathcal{L}}{\partial\bar{\psi}}\,.$$

So, finding our pieces we see

$$\begin{aligned}\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)} &= \bar{\psi}i\gamma^\mu & \qquad \frac{\partial\mathcal{L}}{\partial\psi} &= -m\bar{\psi} \\ \frac{\partial\mathcal{L}}{\partial(\partial_\mu\bar{\psi})} &= 0 & \qquad \frac{\partial\mathcal{L}}{\partial\bar{\psi}} &= (i\gamma^\mu\partial_\mu - m)\psi\end{aligned}$$

$$\implies \boxed{i\partial_\mu\left(\bar{\psi}\gamma^\mu\right) = -m\bar{\psi}} \quad \& \quad \boxed{(i\gamma^\mu\partial_\mu - m)\psi = 0}$$

(5.2) We see that we get

$$\gamma^\nu\partial_\nu(i\gamma^\mu\partial_\mu - m)\psi = 0$$

and so since the gamma matrices are constants, we can pull through and see

$$\begin{aligned} 0 &= \gamma^\nu \partial_\nu (i\gamma^\mu \partial_\mu - m)\psi \\ &= (i\gamma^\nu \partial_\nu \gamma^\mu \partial_\mu - m\gamma^\nu \partial_\nu)\psi \\ &= (i\gamma^\nu \partial_\nu \gamma^\mu \partial_\mu - m\gamma^\mu \partial_\mu)\psi \\ 0 &= (i\gamma^\nu \partial_\nu - m)\gamma^\mu \partial_\mu \psi \end{aligned}$$

What does this tell us about the components?

(5.3) Using our previously found Equations of Motion, we see

$$\begin{aligned} \partial_\mu J^\mu &= \partial_\mu \left(-e\bar{\psi}\gamma^\mu\psi\right) \\ &= -e\left((\partial_\mu \bar{\psi}\gamma^\mu)\psi + \bar{\psi}(\partial_\mu \gamma^\mu \psi)\right) \\ &= -e\left((im\bar{\psi})\psi + \bar{\psi}(-im\psi)\right) \\ &= -e\left(im\bar{\psi}\psi - im\bar{\psi}\psi\right) \\ &= \boxed{0} \end{aligned}$$

as required.

(5.4) First we sho how the ajoint transforms. Notice that $\bar{\psi}\psi$ is a scalar, so it must transform like a scalar, that is $\left(\bar{\psi}\psi\right)' = \bar{\psi}\psi$, so

$$\begin{aligned} \bar{\psi}\psi &= \left(\bar{\psi}\psi\right)' \\ &= \bar{\psi}'\psi' \\ &= \bar{\psi}'\left(S\psi\right) \\ \bar{\psi} &= \bar{\psi}' S \\ \boxed{\bar{\psi}' = \bar{\psi}S^{-1}}&. \end{aligned}$$

Now that we know how $\bar{\psi}$ transforms, we can see how these other quantities transform. As before, we recall that $\bar{\psi}\psi$ is a scalar, so it is fixed under a Lorentz Transform. Next, we see

$$\begin{aligned} \left(\bar{\psi}\gamma^\mu\psi\right)' &= \bar{\psi}'\gamma^\mu\psi' \\ &= \left(\bar{\psi}S^{-1}\right)\gamma^\mu\left(S\psi\right) \\ &= \bar{\psi}S^{-1}\gamma^\mu S\psi \\ &= \bar{\psi}\Lambda^\mu{}_\nu \gamma^\nu \psi \\ &= \boxed{\Lambda^\mu{}_\nu \bar{\psi}\gamma^\nu\psi} \end{aligned}$$

so it transforms like a vector. Finally, for the four-current we see

$$\begin{aligned}
(J^\mu)' &= \left(-e\bar{\psi}\gamma^\mu\psi\right)' \\
&= -e\left(\bar{\psi}\gamma^\mu\psi\right)' \\
&= -e\bar{\psi}S^{-1}\gamma^\mu S\psi \\
&= -e\bar{\psi}\Lambda^\mu{}_\nu\gamma^\nu\psi \\
&= \boxed{\Lambda^\mu{}_\nu J^\nu}
\end{aligned}$$

which also transforms like a vector.