Problem 1

(a) Assume that $u(x, y, t) = T(t)S(x, y)$, then our PDE becomes

$$u_t = k(u_{xx} + u_{yy}) \implies T'S = k(TS_{xx} + TS_{yy})$$
$$\frac{T'}{T} = \frac{k(S_{xx} + S_{yy})}{S}$$

where we let λ be our separation constant, and hence

$$T' + \lambda T = 0 \qquad \& \qquad S_{xx} + S_{yy} = -\frac{\lambda}{k}S$$

Now let $S(x, y) = X(x)Y(y)$, so we get

$$S_{xx} + S_{yy} = -\frac{\lambda}{k}S \implies X''Y + XY'' = -\frac{\lambda}{k}XY$$
$$\frac{X''}{X} + \frac{Y''}{Y} = -\frac{\lambda}{k}$$
$$\frac{X''}{X} = -\frac{\lambda}{k} - \frac{Y''}{Y}$$

where we let μ^2 be the separation constant here, and we get

$$X'' = -\mu^2 X \qquad \& \qquad Y'' = -\left(\frac{\lambda}{k} - \mu_2\right)Y$$

We look at the boundary and initial conditions, and we see that for non-trivial spatial solutions we get

$$u(x, y, 0) = X(x)Y(y)T(0) = f(x, y) \implies T(0) = f(x, y)$$
$$u(0, y, t) = u(l, y, t) = 0 = X(l)Y(y)T(t) = X(0)Y(y)T(t) \implies 0 = X(l) = X(0)$$
$$u_y(x, 0, t) = u_y(x, l, t) = 0 = X(x)Y(l)T(t) = X(x)Y(0)T(t) \implies 0 = Y'(l) = Y'(0)$$

(b) We solve the X ODE first. By inspection we know that

$$X(x) = A\cos(\mu x) + B\sin(\mu x)$$

and applying the first BC gives

$$X(0) = A = 0$$

The second BC gives

$$X(l) = B\sin(\mu l) = 0 \implies \mu_n = \frac{n\pi}{l}$$

So our first spatial solution is

$$X_n(x) = \sin\left(\frac{n\pi x}{l}\right)$$

Next, we solve the Y ODE, where again by inspection

$$Y(y) = A\cos\left(\sqrt{\frac{\lambda}{k} - \mu^2} y\right) + B\sin\left(\sqrt{\frac{\lambda}{k} - \mu^2} y\right)$$

and the BC's give

$$Y'(0) = B\sqrt{\frac{\lambda}{k} - \mu^2} = 0 \quad \implies \quad B = 0$$

$$Y'(l) = -A\sqrt{\frac{\lambda}{k} - \mu^2}\sin\left(\sqrt{\frac{\lambda}{k} - \mu^2} l\right) = 0$$

We see that then that

$$\sqrt{\frac{\lambda}{k} - \mu^2} l = m\pi \qquad m \in \mathbb{N} \cup \{0\}$$

$$\lambda_{m,n} = \frac{m^2\pi^2 k}{l^2} - \mu_n^2 k$$

Plugging this into our eigenfunction we get

$$Y_m(y) = \cos\left(\frac{m\pi y}{l}\right)$$

and hence we have the eignefunctions for our spatial solutions.

(c) By inspection, we see that the solution is

$$T_{m,n}(t) = Ae^{-\lambda_{m,n} t}$$

(d) We know that the final solution should be the product of these individual solutions, in particular

$$u(x, y, t) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} T_{n,m}(t) Y_m(y) X_n(x) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} Ae^{-\lambda_{m,n} t}\left(\sin\left(\frac{n\pi x}{l}\right)\right)\left(\cos\left(\frac{m\pi y}{l}\right)\right)$$

(e) We use the temporal initial condition to solve for the Fourier coefficient. In particular

$$u(x, y, 0) = f(x, y) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} A\left(\sin\left(\frac{n\pi x}{l}\right)\right)\left(\cos\left(\frac{m\pi y}{l}\right)\right)$$

Since we have a double sum, we will need to project the sum over two bases. We see

$$A_{m,n} = \frac{\left(\cos\left(\frac{m\pi y}{l}\right), \left(\sin\left(\frac{n\pi x}{l}\right), f(x, y)\right)\right)}{\left(Y_m(y), Y_m(y)\right)\left(X_n(x), X_n(x)\right)}$$

Problem 2

(a) Assume the solution is of the form $u(x,y,t) = T(t)M(x,y)$, then our PDE becomes

$$T''M = Tc^2(M_{xx} + M_{yy})$$

Apply the separation constant λ and we get

$$T'' + \lambda T = 0 \quad \& \quad M_{xx} + M_{yy} = -\frac{\lambda}{c^2}M$$

Let $M(x,y) = X(x)Y(y)$, then

$$\frac{X''}{X} + \frac{Y''}{Y} = -\frac{\lambda}{c^2}$$

where we use another separation constant μ^2 to get

$$X'' + \mu^2 X = 0 \qquad \& \qquad Y'' + \left(\frac{\lambda}{c^2} - \mu^2\right) Y = 0$$

We first solve the spatial problem. Solving first for $X(x)$, we see that

$$X(x) = A\cos(\mu x) + B\sin(\mu x)$$

but our BC says that $X(0) = X(L) = 0$, so

$$X(0) = A = 0 \qquad \& \qquad X(L) = B\sin(\mu L) = 0$$

$$\implies \mu_n = \frac{n\pi}{L} \qquad n \in \mathbb{N} \setminus \{0\}$$

Now for $Y(y)$,

$$Y(y) = A\cos\left(\sqrt{\frac{\lambda}{c^2} - \mu^2 y}\right) + B\sin\left(\sqrt{\frac{\lambda}{c^2} - \mu^2 y}\right)$$

and the BC tells us $Y(0) = Y(H) = 0$ which gives us

$$Y(0) = A = 0 \qquad \& \qquad Y(H) = B\sin\left(\sqrt{\frac{\lambda}{c^2} - \mu^2 H}\right)$$

$$\implies \sqrt{\frac{\lambda}{c^2} - \mu^2 H} = m\pi \qquad m \in \mathbb{N} \setminus \{0\}$$

$$\lambda_{m,n} = \frac{m^2\pi^2c^2}{H^2} + \mu^2c^2$$

Finally, our temporal solution will be

$$T(t) = A\cos(\sqrt{\lambda_{m,n}}t) + B\sin(\sqrt{\lambda_{m,n}}t)$$

and so our solution that satisfies the BCs will be

$$u(x,y,t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}T_{m,n}(t)Y_m(y)X_n(x)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(A\cos(\sqrt{\lambda_{m,n}}t)+B\sin(\sqrt{\lambda_{m,n}}t)\right)\left(\sin\left(\frac{m\pi y}{H}\right)\right)\left(\sin\left(\frac{n\pi x}{L}\right)\right)$$

which with our first initial condition gives

$$u(x,y,0)=f(x,y)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}AY_m(y)X_n(x)$$

$$\implies A_{m,n}=\frac{(X_n(x),(Y_m(y),f(x,y)))}{(Y_m(y),Y_m(y))(X_n(x),X_n(x))}$$

Similarly, the second initial condition gives us

$$u_t(x,y,0)=g(x,y)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}B\sqrt{\lambda_{m,n}}Y_m(y)X_n(x)$$

$$\implies B_{m,n}=\frac{(X_n(x),(Y_m(y),g(x,y)))}{\lambda_{m,n}(Y_m(y),Y_m(y))(X_n(x),X_n(x))}$$

and so we have completely solved the PDE.

(b) Refer to attached plots.

(c) We have that

$$\sin\left(\frac{\pi y}{H}\right)\sin\left(\frac{2\pi x}{L}\right)+\sin\left(\frac{2\pi y}{H}\right)\sin\left(\frac{\pi x}{L}\right)=2\sin\left(\frac{\pi y}{H}\right)\sin\left(\frac{\pi x}{L}\right)\cos\left(\frac{\pi x}{L}\right)+2\sin\left(\frac{\pi x}{L}\right)\sin\left(\frac{\pi y}{H}\right)\cos\left(\frac{\pi y}{H}\right)$$

$$=2\sin\left(\frac{\pi y}{H}\right)\sin\left(\frac{\pi x}{L}\right)\left(\cos\left(\frac{\pi x}{L}\right)+\cos\left(\frac{\pi y}{H}\right)\right)$$

The nodal lines will be the values of (x,y) over which these functions will vanish

$$\sin\left(\frac{\pi y}{H}\right)\sin\left(\frac{\pi x}{L}\right)\left(\cos\left(\frac{\pi x}{L}\right)+\cos\left(\frac{\pi y}{H}\right)\right)=0$$

$$\sin\left(\frac{\pi y}{H}\right)\sin\left(\frac{\pi x}{L}\right)=0 \qquad \& \qquad \cos\left(\frac{\pi x}{L}\right)+\cos\left(\frac{\pi y}{H}\right)=0$$

The sine terms give us $(x,y)=(0,0),(L,0),(0,H)$ and (L,H), all of which lie on the corners. The cosines give us that

$$\cos\left(\frac{\pi x}{L}\right)=-\cos\left(\frac{\pi y}{H}\right)$$

we notice that since both of the cosines only have half a period over the domain, we expect to be able to find a continuous set of solutions to this transcendental equation. Since the entire function itself is smooth (a product of smooth functions) we expect that the roots form a continuous set. Then, since we require that the boundary points be the corners, we end up with lines between the corners.

Problem 3

We first recall what the laplacian in polar coordinates is and see that

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

We approach this as we normally would, so let $u(r,\theta) = R(r)\Theta(\theta)$, and get

$$\frac{\Theta}{r}\left(rR'' + R'\right) + \frac{R}{r^2}\Theta'' = 0 \qquad \Longrightarrow \qquad \frac{r}{R}(rR'' + R') + \frac{\Theta''}{\Theta} = 0$$

$$\frac{r}{R}(rR'' + R') = -\frac{\Theta''}{\Theta}$$

Let λ^2 be the separation constant, which gives us

$$\frac{r}{R}(rR'' + R') = \lambda^2 \qquad \& \qquad -\frac{\Theta''}{\Theta} = \lambda^2$$

The angular ODE gives us that

$$\Theta(\theta) = A\cos(\lambda\theta) + B\sin(\lambda\theta)$$

It is safe to assume continuity, so $\Theta(0) = \Theta(2\pi)$ and $\Theta'(0) = \Theta'(2\pi)$. Then

$$A = A\cos(2\pi\lambda) + B\sin(2\pi\lambda) \qquad \& \qquad \lambda B = -A\lambda\sin(\lambda\theta) + B\lambda\cos(\lambda\theta)$$

which solving results in the usual result

$$\Theta(\theta) = A_n\cos(n\theta) + B_n\sin(n\theta) \qquad n \in \mathbb{N}\cup\{0\}$$

We look to the radial ODE and notice that the solution must be of the form

$$R(r) = Dr^n + Er^{-n}$$

but we require that $u(r,\theta)$ be bounded as $r \to \infty$, so we can assume that $D = 0$, and hence

$$R_n(r) = E_n r^{-n}$$

and thus our solution becomes

$$u(r,\theta) = \sum_{n=0}^{\infty}\left(A_n\cos(n\theta) + B_n\sin(n\theta)\right)r^{-n}$$

but we have that

$$u(a,\theta) = f(\theta) = \sum_{n=0}^{\infty}\left(A_n\cos(n\theta) + B_n\sin(n\theta)\right)a^{-n}$$

$$A_n = a^n\frac{(\cos(n\theta), f(\theta))}{(\cos(n\theta), \cos(n\theta))} \qquad \& \qquad B_n = a^n\frac{(\sin(n\theta), f(\theta))}{(\sin(n\theta), \sin(n\theta))}$$

and so our problem is solved.

Problem 4

(a) First we let $u(r, \theta, t) = T(t)M(r, \theta)$. Then,

$$MT' = DT\nabla^2 M - kTM$$
$$\frac{T'}{T} = \frac{D\nabla^2 M}{M} - k$$
$$T' + \lambda T = 0 \qquad \& \qquad \nabla^2 M = \frac{k-\lambda}{D}M$$

where λ is our separating constant. Now we let $M(r, \theta) = R(r)\Theta(\theta)$, and hence

$$\frac{\Theta}{r}\left(rR'' + R'\right) + \frac{R}{r^2}\Theta'' = \frac{k-\lambda}{D}R\Theta$$
$$\frac{r}{R}\left(rR'' + R'\right) + \frac{\Theta''}{\Theta} = \frac{k-\lambda}{D}r^2$$
$$\frac{r}{R}\left(rR'' + R'\right) - \frac{k-\lambda}{D}r^2 = \mu^2 \qquad \& \qquad -\frac{\Theta''}{\Theta} = \mu^2$$

We recognize the solution to the angular ODE, and further, we recognize this to be exactly the same as in **Problem 3**, and hence we will find that

$$\Theta_n(\theta) = A_n\cos(n\theta) + B_n\sin(n\theta) \qquad \mu_n = n \in \mathbb{N} \cup \{0\}$$

and for the radial equation we see that

$$r^2R'' + rR' + \left(\frac{\lambda-k}{D}r^2 - n^2\right) = 0$$

which we recognize to be the Bessel DE, and hence the solution will be of the form

$$R(r) = D_nJ_n\left(\frac{\lambda-k}{D}r\right) + E_nY_n\left(\frac{\lambda-k}{D}r\right)$$

which are the Bessel Functions of the first and second kind. However, we need that $R(l) = 0$, so we can get conditions on λ since if the solution is bounded at the centre, $r = 0$ we require that $E_n = 0$, so

$$R(l) = D_nJ_n\left(\frac{\lambda-k}{D}l\right) = 0$$

which will give us $\lambda_{n,m}$. Once we have that, we can find the temporal solution

$$T_{n,m}(t) = Ce^{-\lambda_{n,m}t}$$

and the final solution will the product of all three over the double sum

$$u(r, \theta, t) = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} T_{n,m}(t)R_{n,m}(r)\Theta_n(\theta)$$

(b) If we let $k = 0$ and $D \neq 0$, we can see that our angular solution remains unchanged, and so does our temporal solution (excluding that λ is changing). The real change occurs with the Bessel Functions in the Radial equation. We notice that for $k = 0$ we get exactly what we would get for the 2D case in a circularly symmetric system under polar coordinates. This naturally will be similar to the 1D case since we have a symmetry in the other degree of freedom.

Problem 5

(a) Assume that $u = M(x,y)N(z)$, then we have that

$$M_{xx}N + M_{yy}N + MN'' = 0$$
$$\frac{M_{xx} + M_{yy}}{M} + \frac{N''}{N} = 0$$
$$M_{xx} + M_{yy} + \lambda^2 M = 0 \qquad \& \qquad N'' = \lambda^2 N$$

which are the corresponding ODEs for M and N for the separation constant λ^2.

(b) Let $M(x,y) = X(x)Y(y)$, then we have that

$$X''Y + XY'' + \lambda^2 XY = 0$$
$$\frac{X''}{X} + \frac{Y''}{Y} + \lambda^2 = 0$$
$$X'' + (\lambda^2 - \mu^2)X = 0 \qquad \& \qquad Y'' + \mu^2 Y = 0$$

where μ^2 is another separation constant. We see that for Y we get

$$Y(y) = A\cos(\mu y) + B\sin(\mu y)$$

and the BC's give us

$$Y(0) = A = 0 \qquad \to \qquad Y(H) = B\sin(\mu H) = 0$$
$$\implies \mu_n = \frac{n\pi}{H} \qquad n \in \mathbb{N}\setminus\{0\}$$

For X we get

$$X(x) = D\cos\left(\sqrt{\lambda^2 - \mu^2}x\right) + E\sin\left(\sqrt{\lambda^2 - \mu^2}x\right)$$

which in the BC's gives

$$X(0) = D = 0 \qquad \to \qquad X(L) = E\sin\left(\sqrt{\lambda^2 - \mu^2}L\right) = 0$$
$$\implies \sqrt{\lambda^2 - \mu^2}L = m\pi$$
$$\lambda_{n,m} = \sqrt{\frac{m^2\pi^2}{L^2} + \mu^2}$$

Finally we can solve the N component

$$N_{n,m}(z) = A_{n,m}e^{\lambda_{n,m}z} + B_{n,m}e^{-\lambda_{n,m}z}$$

and the solution will simply be plugging all of this into our original assumption

$$u(x,y,z) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} N_{n,m}(z)X_m(x)Y_n(y)$$

$$u(x,y,z) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left(A_{n,m}e^{\lambda_{n,m}z} + B_{n,m}e^{-\lambda_{n,m}z}\right)\sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi y}{H}\right)$$

where we now apply the z BC's,

$$u_y(x,y,0) = 0 = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left(A_{n,m}\lambda_{n,m}e^{\lambda_{n,m}z} - B_{n,m}\lambda_{n,m}e^{-\lambda_{n,m}z}\right)\sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi y}{H}\right)$$

But this is the zero function in an inner product space, so we can assume $A_{n,m} = B_{n,m}$. The second z condition will give us

$$u(x,y,H) = f(x,y) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{n,m}\left(e^{\lambda_{n,m}H} + e^{-\lambda_{n,m}H}\right)\sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi y}{H}\right)$$

Taking inner products with the orthogonal basis in both m and n we see

$$A_{n,m} = \frac{\left(\sin\left(\frac{n\pi y}{H}\right), \left(\sin\left(\frac{m\pi y}{L}\right), f(x,y)\right)\right)}{\left(\sin\left(\frac{m\pi y}{L}\right), \sin\left(\frac{m\pi y}{L}\right)\right)\left(\sin\left(\frac{n\pi y}{L}\right), \sin\left(\frac{n\pi y}{L}\right)\right)\left(e^{\lambda_{n,m}H} + e^{-\lambda_{n,m}H}\right)}$$

as required.