

Question 1

1.1 The speed of light is exactly 299792456 m/s, which rounded to 1% give us $c \approx 3.00 \times 10^8$. We also know $\hbar \approx 1.05 \times 10^{-34} \text{ m}^2\text{kg/s} = 6.58 \times 10^{-16} \text{ eV} \cdot \text{s}$ up to 1% error.

1.2

(1.2.1) The given mass is only in units of energy, and we want a dimension of mass. So, we recall that energy is $\frac{[M][L]^2}{[T]^2}$, where $[M]$, $[L]$, $[T]$ are dimensions of mass, length and time respectively. Then, we see that we only need to get rid of the length/time dimensions twice, which is just our dimensions for c , so

$$938 \text{ MeV} \rightarrow \frac{938 \text{ MeV}}{c^2}$$

will be the true mass.

(1.2.2) We recall that a unit of energy is the eV, so to get a length from this quantity that has units of energy, we recognize \hbar has units of energy-time and we can get length from the speed of light. That is,

$$\lambda = \frac{2\pi}{E_\gamma} \rightarrow \frac{2\pi}{E_\gamma} \cdot \frac{\hbar}{c}$$

will be the true wavelength.

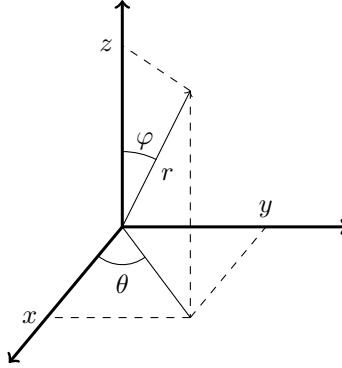
(1.2.3) We recall that the dimensions of the inverse square-root gravitational constant are $\frac{[M]^{1/2}[T]}{[L]^{3/2}}$, and we want dimensions of $[M]$. So, we can see that the corresponding factor we need to multiply by to get the units back is $\sqrt{\hbar c}$, since this will have dimensions of $\frac{[M]^{1/2}[L]^{3/2}}{[T]}$ and thus

$$m_{Pl} = \frac{1}{\sqrt{G}} \rightarrow \sqrt{\frac{\hbar c}{G}}$$

as required.

Question 2

2.1 We first draw what our spherical coordinates will look like relative to the standard euclidean $\{x, y, z\}$ coordinates:



which we will convert into the (r, θ, φ) spherical coordinates. Notice we get

$$x = r \sin(\varphi) \cos(\theta) \quad y = r \sin(\varphi) \sin(\theta) \quad z = r \cos(\varphi)$$

for our conversion. Then, we recall that the metric is $ds^2 = dx^2 + dy^2 + dz^2$, so checking each component, we get

$$dx = \sin(\varphi) \cos(\theta) dr + r \cos(\varphi) \cos(\theta) d\varphi - r \sin(\varphi) \sin(\theta) d\theta$$

$$dy = \sin(\varphi) \sin(\theta) dr + r \cos(\varphi) \sin(\theta) d\varphi + r \sin(\varphi) \cos(\theta) d\theta$$

$$dz = \cos(\varphi) dr - r \sin(\varphi) d\varphi$$

and we can compute the square to get

$$\begin{aligned} dx^2 &= \sin^2(\varphi) \cos^2(\theta) dr^2 + r \sin(\varphi) \cos(\varphi) \cos^2(\theta) dr d\varphi - r \sin^2(\varphi) \cos(\theta) \sin(\theta) dr d\theta \\ &\quad + r \cos(\varphi) \sin(\varphi) \cos^2(\theta) dr d\varphi + r^2 \cos^2(\varphi) \cos^2(\theta) d\varphi^2 - r^2 \cos(\varphi) \sin(\varphi) \cos(\theta) \sin(\theta) d\varphi d\theta \\ &\quad - r \sin^2(\varphi) \sin(\theta) \cos(\theta) dr d\theta - r^2 \sin(\varphi) \cos(\varphi) \sin(\theta) \cos(\theta) d\varphi d\theta + r^2 \sin^2(\varphi) \sin^2(\theta) d\theta^2 \\ dy^2 &= \sin^2(\varphi) \sin^2(\theta) dr^2 + r \sin(\varphi) \cos(\varphi) \sin^2(\theta) d\varphi dr + r \sin^2(\varphi) \sin(\theta) \cos(\theta) dr d\theta \\ &\quad + r \cos(\varphi) \sin(\varphi) \sin^2(\theta) dr d\varphi + r^2 \cos^2(\varphi) \sin^2(\theta) d\varphi^2 + r^2 \cos(\varphi) \sin(\varphi) \sin(\theta) \cos(\theta) d\varphi d\theta \\ &\quad + r \sin^2(\varphi) \cos(\theta) \sin(\theta) dr d\theta + r^2 \sin(\varphi) \cos(\varphi) \cos(\theta) \sin(\theta) d\varphi d\theta + r^2 \sin^2(\varphi) \cos^2(\theta) d\theta^2 \\ dz^2 &= \cos^2(\varphi) dr^2 - 2r \sin(\varphi) \cos(\varphi) dr d\varphi + r^2 \sin^2(\varphi) d\varphi^2. \end{aligned}$$

So, adding these together and simplifying

$$\begin{aligned} ds^2 &= dx^2 + dy^2 + dz^2 \\ &= dr^2 + r^2 d\varphi^2 + r^2 \sin^2(\varphi) d\theta^2 \end{aligned}$$

and the matrix will be

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2(\varphi) \end{bmatrix}.$$

2.2 To see this, we recall how the vectors will change under a lorentz transform

$$A^\mu \rightarrow \bar{A}^\mu = \Lambda^\mu{}_\nu A^\nu \quad A_\mu \rightarrow \bar{A}_\mu = g_{\mu\nu} \bar{A}^\nu = g_{\mu\nu} \Lambda^\nu{}_\eta A^\eta.$$

Then, we see that

$$A^\mu B_\mu \rightarrow \bar{A}^\mu \bar{B}_\mu = \Lambda^\mu{}_\nu A^\nu g_{\mu\sigma} \Lambda^\sigma{}_\eta B^\eta = \underbrace{\Lambda^\mu{}_\nu g_{\mu\sigma} \Lambda^\sigma{}_\eta}_{g_{\nu\eta}} A^\nu B^\eta = A^\nu B_\nu$$

and hence we have that the contraction is indeed Lorentz invariant. Moreover, we see

$$A^\mu B_\mu = A_\nu g^{\nu\mu} B_\mu = A_\nu B^\nu$$

and so the two quantities are the same, as we would expect.

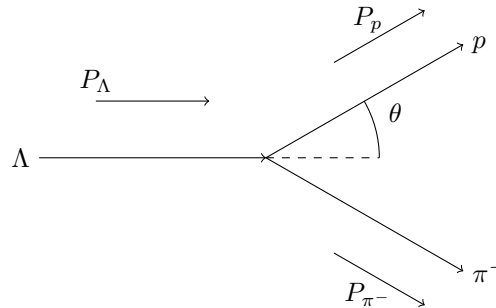
Question 3

3.1 Consider the following table, as required.

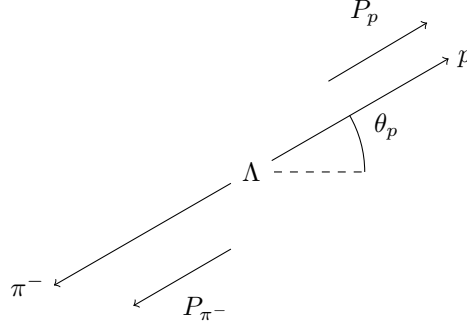
	mass MeV	Composition
p	938.27	uud
Λ	1115.68	uds
π^-	139.57	$d\bar{u}$

3.2

3.2.1 In the Lab Frame we will see



but in the COM frame we would see



3.2.2 In the COM frame, we know that $\vec{p}_p = \vec{p}$ and $\vec{p}_{\pi^-} = -\vec{p}$, and that $\vec{p}_\Lambda = \vec{0}$. Using this, we see that

$$s_i = (P_\Lambda)^\mu (P_\Lambda)_\mu = E_\Lambda^2$$

$$s_f = (P_p + P_{\pi^-})^\mu (P_p + P_{\pi^-})_\mu = (E_p + E_{\pi^-})^2 - (\vec{p} - \vec{p})^2 = (E_p + E_{\pi^-})^2.$$

Moreover, we know that

$$E_p^2 = m_p^2 + p^2 \quad \& \quad E_{\pi^-}^2 = m_{\pi^-}^2 + p^2$$

which gives

$$E_p^2 - E_{\pi^-}^2 = m_p^2 - m_{\pi^-}^2$$

$$(E_p - E_{\pi^-})(E_p + E_{\pi^-}) = m_p^2 - m_{\pi^-}^2$$

$$(E_p - m_\Lambda + E_p)m_\Lambda = m_p^2 - m_{\pi^-}^2$$

$$E_p = \frac{m_p^2 - m_{\pi^-}^2 + m_\Lambda^2}{2m_\Lambda}$$

and in a similar manner we get

$$E_{\pi^-} = \frac{-m_p^2 + m_{\pi^-}^2 + m_\Lambda^2}{2m_\Lambda}.$$

The expressions for our momentum come from the relationship used before, that is

$$p_p = \sqrt{E_p^2 - m_p^2} \quad \& \quad p_{\pi^-} = \sqrt{E_{\pi^-}^2 - m_{\pi^-}^2}.$$

Plugging in some numbers, we get

$$E_p \approx 943.645 \text{ MeV} \quad E_{\pi^-} \approx 172.035 \text{ MeV} \quad p_p \approx 100.58 \text{ MeV} \approx p_{\pi^-}.$$

3.2.3 To compute the boost we would need, we only need the velocity that the Λ Hyperon is traveling at, as that will uniquely determine the Lorentz boost transformation. First, we get a nice relation we can use:

$$\begin{aligned}
 E^2 &= p^2 c^2 + m^2 c^4 \\
 &= \gamma^2 m^2 v^2 c^2 + m^2 c^4 \\
 &= m^2 c^4 (\gamma^2 \beta^2 + 1) \\
 &= m^2 \left(\frac{\beta^2}{1 - \beta^2} + 1 \right) \quad (\text{set } c = 1) \\
 E^2 &= m^2 \gamma^2.
 \end{aligned}$$

Notice that the information for the velocity is cooked into γ , so this will make computing the boost even easier. We see

$$\gamma_\Lambda^2 = \frac{E_\Lambda^2}{m_\Lambda^2} = \frac{m_\Lambda^2 + p_\Lambda^2}{m_\Lambda^2} \approx 4.214 \quad \beta^2 = 1 - \frac{1}{\gamma^2} \approx 0.763.$$

Then plugging this into our Lorentz Transform,

$$\Lambda = \begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \sqrt{\frac{m_\Lambda^2 + p_\Lambda^2}{m_\Lambda^2}} & -\frac{p_\Lambda}{m_\Lambda} & 0 & 0 \\ -\frac{p_\Lambda}{m_\Lambda} & \sqrt{\frac{m_\Lambda^2 + p_\Lambda^2}{m_\Lambda^2}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2.053 & -1.566 & 0 & 0 \\ -1.566 & 2.053 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

where we have taken the positive root of β^2 since we suppose a boost in the positive direction.

3.2.4 We start by expressing the 4-momentum using θ_p , and what we get is

$$\begin{aligned}
 P_p^\mu &= (E_p, \vec{p}) = (E_p, p \cos(\theta_p), p \sin(\theta_p), 0) = (E_p, \sqrt{3}/2 p, p/2, 0) \\
 P_{\pi^-}^\mu &= (E_{\pi^-}, -p \cos(\theta_p), -p \sin(\theta_p), 0) = (E_{\pi^-}, -\sqrt{3}/2 p, -p/2, 0)
 \end{aligned}$$

where I have abused my notation, but it is understood that the vectors are column vectors. Then, we see that

$$\begin{aligned}
 \bar{P}_p^\mu &= (\Lambda^\nu{}_\mu)^{-1} P_p^\mu = \begin{bmatrix} \sqrt{\frac{m_\Lambda^2 + p_\Lambda^2}{m_\Lambda^2}} & \frac{p_\Lambda}{m_\Lambda} & 0 & 0 \\ \frac{p_\Lambda}{m_\Lambda} & \sqrt{\frac{m_\Lambda^2 + p_\Lambda^2}{m_\Lambda^2}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} E_p \\ \frac{\sqrt{3}}{2} p \\ p/2 \\ 0 \end{bmatrix} = \begin{bmatrix} E_p \sqrt{\frac{m_\Lambda^2 + p_\Lambda^2}{m_\Lambda^2}} + \frac{\sqrt{3}}{2} \frac{p_\Lambda}{m_\Lambda} p \\ \frac{p_\Lambda}{m_\Lambda} E_p + \sqrt{\frac{m_\Lambda^2 + p_\Lambda^2}{m_\Lambda^2}} \frac{\sqrt{3}}{2} p \\ p/2 \\ 0 \end{bmatrix} \\
 \bar{P}_{\pi^-}^\mu &= (\Lambda^\nu{}_\mu)^{-1} P_{\pi^-}^\mu = \begin{bmatrix} \sqrt{\frac{m_\Lambda^2 + p_\Lambda^2}{m_\Lambda^2}} & \frac{p_\Lambda}{m_\Lambda} & 0 & 0 \\ \frac{p_\Lambda}{m_\Lambda} & \sqrt{\frac{m_\Lambda^2 + p_\Lambda^2}{m_\Lambda^2}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} E_{\pi^-} \\ -\frac{\sqrt{3}}{2} p \\ -p/2 \\ 0 \end{bmatrix} = \begin{bmatrix} E_{\pi^-} \sqrt{\frac{m_\Lambda^2 + p_\Lambda^2}{m_\Lambda^2}} - \frac{\sqrt{3}}{2} \frac{p_\Lambda}{m_\Lambda} p \\ \frac{p_\Lambda}{m_\Lambda} E_{\pi^-} - \sqrt{\frac{m_\Lambda^2 + p_\Lambda^2}{m_\Lambda^2}} \frac{\sqrt{3}}{2} p \\ -p/2 \\ 0 \end{bmatrix}.
 \end{aligned}$$

This isn't the most intuitive form, but all the values are known, so we can compute it explicitly, which will help us find θ . Computing \bar{P}_p^μ explicitly yields

$$\bar{P}_p^\mu \approx \begin{bmatrix} 2073.71 \text{ MeV} \\ 1656.57 \text{ MeV} \\ 50.29 \text{ MeV} \\ 0 \end{bmatrix}$$

and so $\theta \approx \arctan\left(\frac{50.29}{1656.57}\right) \approx 1.74^\circ$ Degrees. For π^- we find through a similar computation $\theta_{\pi^-} \approx \arctan\left(-\frac{50.29}{95.28}\right) \approx -27.83^\circ$ Degrees.

Question 4

4.1 To show that $\mathbf{O}(n)$ is a group under multiplication, we need only show the definition of a group is satisfied. In particular, if $M, N \in \mathbf{O}(n)$, notice

$$MN(MN)^t = MN(N^t M^t) = MNN^t M^t = MM^t = I \implies MN \in \mathbf{O}(n),$$

which is closure (Notice we don't have to show $(MN)^t MN = I$ since we showed the inverse of MN is its transpose and inverses are unique from linear algebra). Next, since $II^t = II = I$, we have an identity $I \in \mathbf{O}(n)$. Matrix multiplication is associative, and since $\mathbf{O}(n) \subset M_{n \times n}(\mathbb{R})$, we have associativity for free. Finally, we show inverses are also orthogonal. We know they exist, since

$$\det(MM^t) = \det(I) \implies (\det(M))^2 = 1 \implies \det(M) = \pm 1.$$

But, since M is orthogonal, by definition $M^{-1} = M^t$, so

$$M^{-1}(M^{-1})^t = M^t(M^t)^t = M^t M = I \implies M^{-1} \in \mathbf{O}(n).$$

So, we can conclude that $\mathbf{O}(n)$ is indeed a group.

4.2 To show that $\mathbf{SO}(n)$ is a group, we need only show that it is a subgroup, so our criterion aren't as restrictive. In particular, we get associativity for free, since $\mathbf{SO}(n) \subset \mathbf{O}(n)$, and since $\det(I) = 1$, $I \in \mathbf{SO}(n)$, and so we have the identity as well. All we need is closure and inverses. Well, notice if $M, N \in \mathbf{SO}(n)$, then

$$\det(MN) = \underbrace{\det(M)}_1 \underbrace{\det(N)}_1 = 1 \implies MN \in \mathbf{SO}(n).$$

For inverses, we note

$$\det(M^{-1}) = \det(M^t) = \det(M) = 1 \implies M^{-1} \in \mathbf{SO}(n).$$

Thus, we have shown $\mathbf{SO}(n)$ is indeed a subgroup of $\mathbf{O}(n)$.