

Question 1

(a) First, notice that $f(x)$ is separable and irreducible by assumption, and so the splitting field of $f(x)$, K/\mathbb{Q} , will be Galois. Notice, that the statement follows if we can show that the expression is fixed by the elements of $\text{Gal}(K/\mathbb{Q}) = \text{Gal}(f(x))$, since K/\mathbb{Q} is Galois $\implies \text{Fix}(\text{Gal}(f(x))) = \mathbb{Q}$, then anything fixed by the Galois group must be in \mathbb{Q} . Suppose $\varphi \in \text{Gal}(f(x))$, then notice

$$g(\alpha_1, \alpha_2, \dots, \alpha_n) = \frac{3 + \alpha_1}{\alpha_1^7} + \dots + \frac{3 + \alpha_n}{\alpha_n^7} = \sum_{k=1}^n \frac{3 + \alpha_k}{\alpha_k^7}$$

, and so we have

$$\varphi(g(\alpha_1, \alpha_2, \dots, \alpha_n)) = \varphi\left(\sum_{k=1}^n \frac{3 + \alpha_k}{\alpha_k^7}\right)$$

and with homomorphism properties,

$$\varphi(g(\alpha_1, \alpha_2, \dots, \alpha_n)) = \sum_{k=1}^n \frac{3 + \varphi(\alpha_k)}{\varphi(\alpha_k)^7}.$$

However, we know that the elements of the Galois group have a unique element of S_n associated with them through an isomorphism, and so if $\pi \in S_n$ is the symmetry group element associated with φ , we see that

$$\varphi(g(\alpha_1, \alpha_2, \dots, \alpha_n)) = \sum_{k=1}^n \frac{3 + \alpha_{\pi(k)}}{\alpha_{\pi(k)}^7} = g(\alpha_1, \alpha_2, \dots, \alpha_n)$$

up to reordering. Thus, we are done.

(b) Since K/F is Galois, we know that $[K : F] = |\text{Gal}(K/F)| \implies p \mid |\text{Gal}(K/F)|$. By Cauchy's Theorem, $\exists \varphi \in \text{Gal}(K/F)$ such that $|\varphi| = p$, and moreover $\langle \varphi \rangle = H < \text{Gal}(K/F)$. The Fundamental Theorem of Galois Theory tells us that $\text{Fix}(H) \in \mathcal{E}$, and moreover that $\text{Fix}(\text{Gal}(K/\text{Fix}(H))) = \text{Fix}(H)$, and so $K/\text{Fix}(H)$ is a Galois group, and thus $[K : \text{Fix}(H)] = |\text{Gal}(K/\text{Fix}(H))| = |H| = p$, as required.

(c) We have the tower of fields $\mathbb{C}/F/\mathbb{Q}$. Proceed by contrapositive, suppose $F \not\subseteq \mathbb{R}$, then:

Case 1: Suppose $\mathbb{R} \subset F$, but \mathbb{R}/\mathbb{Q} is not separable, and so from Assignment 7 this would imply F/\mathbb{Q} is not separable, and thus not Galois, which is a contradiction.

Case 2: We now can't say anything about the two subfields of \mathbb{C} except $F \cap \mathbb{R} \supseteq \mathbb{Q}$. Notice, since F/\mathbb{Q} is Galois, it is simple, so $\exists \alpha \in F$ such that $F = \mathbb{Q}(\alpha)$. Notice, $\alpha \notin F \cap \mathbb{R}$, since if $\alpha \in \mathbb{R}$, then $\mathbb{Q}(\alpha) \subset \mathbb{R}$, and we get **Case 1**. Clearly $\alpha \notin \mathbb{Q}$. Since F/\mathbb{Q} is Galois, the minimal polynomial of α over \mathbb{Q} is separable. Call it $f(x)$, and thus $[F : \mathbb{Q}] = \deg(f(x))$. Notice, $f(\alpha) = 0 \implies$ we have a linearly dependent result $a + ib = 0$, since $\alpha = c + id \in \mathbb{C}$, but this is only possible if i can be written as a linear combination of elements in \mathbb{Q} , and hence the i must be in \mathbb{Q} , and thus $2 \mid \deg(f(x)) = [\mathbb{Q}(\alpha) : \mathbb{Q}] = [F : \mathbb{Q}]$.

Question 2

(a) Since K is a splitting field of $p_1(x)p_2(x)\dots p_m(x)$, it is normal. All we need for K to be Galois is that $p_1(x)p_2(x)\dots p_m(x)$ be separable. However, we know from hypothesis that each $p_i(x)$ is a minimal polynomial, and moreover is separable. Furthermore, each $p_i(x)$ is distinct, so we have no overlapping roots $\implies p_1(x)p_2(x)\dots p_m(x)$ is separable, and thus K is normal.

(b) Since K/F is Galois, from A7 we know that if $K/F(\alpha_1, \dots, \alpha_n)/F$ is a tower of fields then $F(\alpha_1, \dots, \alpha_n)$ is separable. However, we know that $\alpha_i \in K \ \forall i \in \{1, \dots, n\}$, and so $F \subset K$, and the result follows.

(c) By the above theory, we see that we need the distinct minimal polynomials of our roots, which we see to be

$$q_1(x) = x^3 - 2 \quad \& \quad q_2(x) = x^7 - 3$$

which are irreducible by 2 and 3 - Eisenstein respectively. These are clearly distinct, so we need to find the splitting field of $q_1(x)q_2(x) = (x^3 - 2)(x^7 - 3)$. For simplicity, let $\alpha = \sqrt[3]{2}$ and $\beta = \sqrt[7]{3}$, then notice that the roots we need in our splitting field must be $\alpha, \alpha\xi_3, \alpha\xi_3^2, \beta, \beta\xi_7, \beta\xi_7^2, \dots, \beta\xi_7^6$. We recognize that the splitting field is thus $\mathbb{Q}(\alpha, \beta, \xi_3, \xi_7)$, which is minimal by construction.

Question 3

(a) We know that K/\mathbb{Q} is Galois from **Question 2**, since $x^2 - 7$ is separable (with roots $\pm\sqrt{7}$) and irreducible (7-Eisenstein) and $x^4 + x^3 + x^2 + x + 1$ is separable and irreducible (5th cyclotomic polynomial). So, the splitting field of the product is the Galois Closure of some intermediate field that contains only one root of each polynomial.

(b) Since we have two separable minimal polynomials, we know that the elements of the Galois group must be transitive within each polynomial, and we can only send roots to roots that share the same minimal polynomial. Thus, if we call $\alpha = \sqrt{7}$, we get

	α	$-\alpha$	ξ_5	ξ_5^2	ξ_5^3	ξ_5^4	S_6
φ_1	α	$-\alpha$	ξ_5	ξ_5^2	ξ_5^3	ξ_5^4	ε
φ_2	α	$-\alpha$	ξ_5^2	ξ_5^4	ξ_5	ξ_5^3	(3465)
φ_3	α	$-\alpha$	ξ_5^3	ξ_5	ξ_5^4	ξ_5^2	(3564)
φ_4	α	$-\alpha$	ξ_5^4	ξ_5^3	ξ_5^2	ξ_5	(36)(45)
φ_5	$-\alpha$	α	ξ_5	ξ_5^2	ξ_5^3	ξ_5^4	(12)
φ_6	$-\alpha$	α	ξ_5^2	ξ_5^4	ξ_5	ξ_5^3	(12)(3465)
φ_7	$-\alpha$	α	ξ_5^3	ξ_5	ξ_5^4	ξ_5^2	(12)(3564)
φ_8	$-\alpha$	α	ξ_5^4	ξ_5^3	ξ_5^2	ξ_5	(12)(36)(45)

(c) By inspection, we see that there are two independent components to our group elements, and so we see that $\text{Gal}(K/\mathbb{Q}) \cong \mathbb{Z}_2 \times \mathbb{Z}_4$.

(d) Since we have that $\text{Gal}(K/\mathbb{Q}) \cong \mathbb{Z}_2 \times \mathbb{Z}_4$, and we know that subfields correspond with subgroups, we can get an idea of the subfield structure by considering the subgroups of $\mathbb{Z}_2 \times \mathbb{Z}_4$. We note that the subgroups must be \mathbb{Z}_2 , $\mathbb{Z}_2 \times \mathbb{Z}_2$ and \mathbb{Z}_4 . To represent these subgroups, I used the generators as elements, $(a, b) \in \mathbb{Z}_2 \times \mathbb{Z}_4$. Notice, the index of the subgroup is found by checking the number of elements that have been removed in each subgroup.

The subfield diagram is found by using the Fundamental Theorem of Galois Theory, and we just flip the diagram to obtain the appropriate positions. The subgroups of S_6 are chosen such that we get a similar group as the appropriate group from the group diagram. The fix is decided by looking at the table we have made. The only fix that is not obvious is $\text{Fix}(((36)(45)))$. Clearly α is fixed, but we see that we have swapped ξ_5 and ξ_5^4 so we better expect that $\xi_5 + \xi_5^4$ is fixed. This gives us our diagram.

Question 4

To see that $\mathbb{Q}(\sqrt[8]{2}, i)$ is Galois, we show that it is the splitting field of $f(x) = x^8 - 2$. Notice that the roots of $f(x)$ are $\sqrt[8]{2}, \xi_8 \sqrt[8]{2}, \dots, \xi_8^7 \sqrt[8]{2}$, so we only need to show that ξ_8 is in $\mathbb{Q}(\sqrt[8]{2}, i)$. Notice,

$$\xi_8 = e^{i \frac{2\pi}{8}} = e^{i \frac{\pi}{4}} = \cos\left(\frac{\pi}{4}\right) + i \sin\left(\frac{\pi}{4}\right) = \frac{1}{2} + \frac{i}{2} \in \mathbb{Q}(\sqrt[8]{2}, i).$$

Thus, clearly all of the roots must be in $\mathbb{Q}(\sqrt[8]{2}, i)$ and the separable polynomial $f(x)$ splits over this field, thus it must be normal: our Galois theory applies. Thus, to consider the number of possible fields between these two extensions, we consider the number of groups between the corresponding Galois groups. In particular, the subgroup that corresponds with the subfields $\mathbb{Q}(i)$ is just the Fix of the subgroup that fixes i . Thus, we just need the other fields that correspond with groups that fix i as well, which we can from inspection see are just $\mathbb{Q}(\sqrt{2}, i)$ and $\mathbb{Q}(\sqrt[4]{2}, i)$. Thus, we have two intermediate subfields.