Question 1

Question 2

This follows from explicitly spelling out what equivalent representations are. In particular, suppose G is our group with $g \in G$, and $D : G \to GL(V)$, where V is a vector space, will be our representation. Then, $D' : G \to GL(W)$, where W is another vector space. Then, we say D and D' are equivalent representations if V and W are isomorphic, that is $\exists$ an isomorphism $\alpha : V \to W$, and

$$\alpha \circ D(g) \circ \alpha^{-1} = D'(g) \quad \forall g \in G\,.$$

However, since we need that V and W are over the same field for our representations, we can see that α is just a change of basis, and can be thought of as an invertable matrix, say P, such that

$$PD(g)P^{1} = D'(g)\,.$$

From here, we know that Trace is preserved under basis transformations, since similar matrices share their Trace, so

$$Tr(D(g)) = Tr(D'(g))\,.$$

So, if two representations do not preserve trace, we can immediatly conclude they are not equivalent from the contrapositive of the above.

Question 3

(a) To simplify our table, we notice that rotation can be written as $e^{\frac{in\pi}{2}}$, so we use this to represent our rotations. The operation is multiplication, and thus which shows all the properties we need for this to be

$\times$	$\mathbf{1}$	$\boldsymbol{e^{\frac{i\pi}{2}}}$	$\boldsymbol{e^{i\pi}}$	$\boldsymbol{e^{\frac{i3\pi}{2}}}$
$\mathbf{1}$	1	$e^{\frac{i\pi}{2}}$	$e^{i\pi}$	$e^{\frac{i3\pi}{2}}$
$\boldsymbol{e^{\frac{i\pi}{2}}}$	$e^{\frac{i\pi}{2}}$	$e^{i\pi}$	$e^{\frac{i3\pi}{2}}$	1
$\boldsymbol{e^{i\pi}}$	$e^{i\pi}$	$e^{\frac{i3\pi}{2}}$	1	$e^{\frac{i\pi}{2}}$
$\boldsymbol{e^{\frac{i3\pi}{2}}}$	$e^{\frac{i3\pi}{2}}$	1	$e^{\frac{i\pi}{2}}$	$e^{i\pi}$

a group. The only missing one explicitly is associativity, but that follows from exponential multiplication being associative.

(b) Looking at the table, we see that this is indeed abelian, which we know is true since the multiplication table is symmetric!

(c) The subgroup is the π rotation. Looking at the multiplication table, we can notice that $\{1, e^{i\pi}\}$ is a closed set under the group operation of multiplication, and since it is a subset of a group it is naturally a subgroup.

(d) We recognize that the non-trivial eigenvectors will be $|0,y\rangle$ and $|x,0\rangle$. In particular, we see

$$M_x\,|0,y\rangle = |0,y\rangle \quad \& \quad M_x\,|x,0\rangle = |-x,0\rangle = -\,|x,0\rangle$$

which tells us the eigenvalues are 1 and -1 respectively.

(e) For the rotation operator, we look at each value of n individually to get the eigenvectors and hence eigenvalues. We skip the identity, since it gives a trivial eigenvalue of 1 with the entire space, except the $\vec{0}$ being eigenvectors. So, first for $n=1$ we get the rotation $D_z\left(\frac{\pi}{2}\right)$, which we know will swap components and make one negative, so this will only have the trivial eigenvector, and similar for $n=3$ for the same reason. This leaves the $n=2$ case, which is a π rotation, $D_z(\pi)$, and hence only flips the signs of our states and will give an eigenvalue of -1.

(f) It suffices to show that the rotation and reflection operator do not commute in a particular case to conclude they do not commute in general. Consider $D_z(\pi/2)$, then

$$D_z(\pi/2)M_x\,|x,y\rangle = D_z(\pi/2)\,|-x,y\rangle = |-y,-x\rangle \quad \& \quad M_xD_z(\pi/2)\,|x,y\rangle = M_x\,|-y,x\rangle = |y,x\rangle$$

$$\implies D_z(\pi/2)M_x \neq M_xD_z(\pi/2) \implies D_z(n\pi/2)M_x \neq M_xD_z(n\pi/2)\,.$$

(g) First, we see that for a state $|x,y\rangle$, $D_z(\pi)\,|x,y\rangle = -\,|x,y\rangle$. So, we can conclude that all states are eigenstates of $D_z(\pi)$. On the other hand, we recall that the eigenstates of M_x are $|x,0\rangle$ and $|0,y\rangle$. So, we want to choose eigenstates of the hamiltonian H to be simultaneous eigenstates of these two operators, which means they must take the form either $|x,0\rangle$ or $|0,y\rangle$.