Question 1

(a) Since $\alpha \in K$ is algebraic over F, and $F \subset E$ as a subfield, then α is algebraic over E. By definition, $\exists\, f(x) \in F[x]$ minimial polynomial of α over F. There are two cases to consider for this polynomial:

(i) Suppose $f(x)$ is irreducable in $E[x]$ where the coeffecients of $f(x)$ are sent under the canonical inclusion map. Then, since $f(\alpha) = 0$ and it is monic by construction, $f(x)$ is the minimal polynomial over E of α. With this determined

$$[E(\alpha) : E] = \deg_E(f(x)) = \deg_F(f(x)) = [F(\alpha) : F]\,.$$

(ii) Suppose $f(x)$ is reducable in $E[x]$ where the coeffecients are mapped as before. Then, $\exists\, h(x), g(x) \in E[x]$ such that $\deg_E(h(x)) \geq 1$, $g(x)$ is the minimal polynomial of α over E and $f(x) = h(x)g(x)$. Therefore,

$$[E(\alpha) : E] = \deg_E(g(x)) \leq \deg_E(h(x)g(x)) = \deg_F(f(x)) = [F(\alpha) : F]\,.$$

(b) We first note that $E(\alpha)/E$, $E(\alpha)/F$, $F(\alpha)/F$ and $E(\alpha)/F(\alpha)$ are all finite extensions. So, we are motivated to apply the Tower Theorem:

$$[E(\alpha) : F] = [E(\alpha) : F(\alpha)][F(\alpha) : F] \geq [E(\alpha) : F(\alpha)][F(\alpha) : E]$$

where the inequality comes from **(a)**. Then, with another application of the Tower theorem

$$[E(\alpha) : F(\alpha)] \leq \frac{[E(\alpha) : F]}{[E(\alpha) : E]} = \frac{[E(\alpha) : E][E : F]}{[E(\alpha) : E]} = [E : F]$$

as required.

Question 2

Since $F(\alpha,\beta)/F(\alpha), F(\alpha,\beta)/F(\beta), F(\alpha)/F, F(\beta)/F$, and $F(\alpha,\beta)/F$ are all finite extensions, we can apply the Tower Theorem. Let $\deg_F(\alpha) = m$, and $\deg_F(\beta) = n$, then

$$[F(\alpha,\beta) : F] = [F(\alpha,\beta) : F(\alpha)][F(\alpha) : F] = [F(\alpha,\beta) : F(\alpha)]m$$

$$[F(\alpha,\beta) : F] = [F(\alpha,\beta) : F(\beta)][F(\beta) : F] = [F(\alpha,\beta) : F(\beta)]n\,.$$

So we clearly have that $n \,|\, [F(\alpha,\beta) : F]$ and $m \,|\, [F(\alpha,\beta) : F]$, and since these are coprime, $nm \,|\, [F(\alpha,\beta) : F]$.

Now we use the previous question to see that

$$[F(\alpha,\beta) : F] = [F(\alpha,\beta) : F(\alpha)][F(\alpha) : F] \leq [F(\beta) : F][F(\alpha) : F] = mn$$

and since $mn \,|\, [F(\alpha,\beta) : F]$, we have that

$$[F(\alpha,\beta) : F] = mn = [F(\beta) : F][F(\alpha) : F]\,.$$

Question 3

(a) From the previous assignment, we recall that $\deg_{\mathbb{Q}}(i+\sqrt{2}) = 4$ and $\deg_{\mathbb{Q}}(\cos(\frac{\pi}{9})) = 3$, which are coprime. So, from the previous lemma

$$[\mathbb{Q}(\alpha, \beta) : \mathbb{Q}] = [\mathbb{Q}(\alpha) : \mathbb{Q}][\mathbb{Q}(\beta) : \mathbb{Q}] = (3)(4) = 12\,.$$

(b) By the Tower theorem,

$$[\mathbb{Q}(\sqrt{p}, \sqrt{q}) : \mathbb{Q}] = [\mathbb{Q}(\sqrt{p}, \sqrt{q}) : \mathbb{Q}(\sqrt{p}][\mathbb{Q}(\sqrt{p}) : \mathbb{Q}]\,.$$

We recognize that each square root of a prime will have a minimal polynomial of order 2, namely $x^2 - p$ and $x^2 - q$ over $\mathbb{Q}$. So, all we need to show is that this minimal polynomial is the same in the adjoined field over the opposite extension. In particular, in this case we consider $x^2 - q$ over $\mathbb{Q}(\sqrt{p})$. Clearly this is still irreducible, as otherwise we would have a power of $\sqrt{p}$ that is q, but they are distinct primes, and so that can not be the case. Thus,

$$[\mathbb{Q}(\sqrt{p}, \sqrt{q}) : \mathbb{Q}] = [\mathbb{Q}(\sqrt{p}, \sqrt{q}) : \mathbb{Q}(\sqrt{p}][\mathbb{Q}(\sqrt{p}) : \mathbb{Q}] = 2 \cdot 2 = 4\,.$$

(c) We notice that this polynomial is reducible, in particular,

$$x^3 + x + 1 = (x + 1)(x^2 - x + 2)\ \in \mathbb{Z}_3[x]\,.$$

Here, we see that $x^2 - x + 2$ is irreducible and so is $x + 1$, so we just need to consider α in these two cases.

Suppose $\alpha = 2$, then the minimal polynomial is $x + 1$, so $\deg_{\mathbb{Z}_3}(\alpha) = 1$, and thus

$$[\mathbb{Z}_3(\alpha) : \mathbb{Z}_3] = 1\,.$$

Suppose α is a root of the irreducible and monic $x^2 - x + 2$, then we have that

$$[\mathbb{Z}_3(\alpha) : \mathbb{Z}_3] = 2\,.$$

(d) Since $\mathbb{R}(t) = \mathbb{R}(t, t^2)$, we can think of $\mathbb{R}(t^2) = F$ as a new field, and then we see that the question becomes to find $[F(t) : F]$, which is to say we want a minimal polynomial in F of t. So, what polynomial in $\mathbb{R}(t^2)[x]$ has a root of t. Well, we see that the natural choice would be $x^2 - t^2 \in \mathbb{R}(t^2)[x]$. So, the minimal polynomial, call it $p(x)$, must divide this polynomial, and hence $\deg_{\mathbb{R}(t^2)}(t) \in \{1, 2\}$. But, if the degree were 1, then that would imply that $t \in \mathbb{R}(t^2)$, which is false, so we must have that the degree of t is 2. Then, $[\mathbb{R}(t) : \mathbb{R}(t^2)] = 2$, as required.

Question 4

We build the natural homomorphism and show that it is an isomorphism. That is, for any rational polynomial $\frac{f(\alpha)}{g(\alpha)} \in F(\alpha), g(\alpha) \neq 0$, which is by definition of adjoining, we define the map $\varphi : F(\alpha) \to F(x)$ by

$$\varphi\left(\frac{f(\alpha)}{g(\alpha)}\right) = \frac{f(x)}{g(x)} \in F(x)\,.$$

This is the natural homomorphism, and since all we do is relabel, this better be a homomorphism. Notice here, however, that since α is trancendental over F, $\ker(\varphi) = \{0\}$. That is, the kernel is trivial, and so we have injection. Moreover, notice that the map has a natural inverse that is also injective, thus we have surjection, and

$$F(\alpha) \cong F(x)$$

as needed. Since β is trancendental like α, the above holds for it aswell and we get $F(\beta) \cong F(x)$, and by chaining isomorphisms, $F(\beta) \cong F(\alpha)$.

Question 5

We use the Tower theorem and what we did in **Question 1** to prove this statement. First, since $f(x) \in \mathbb{Q}[x]$ is irreducible, then by Kronecker's theorem, $\exists\, \alpha$ in some field extension of $\mathbb{Q}$ such that $f(\alpha) = 0$. Thus, we see that $f(x)$ is the minimal polynomial of α over $\mathbb{Q}$, up to some unit multiple to make it monic. This gives us the following extensions we can use

$$E(\alpha)/\mathbb{Q} \quad E(\alpha)/\mathbb{Q}(\alpha) \quad \mathbb{Q}(\alpha)/\mathbb{Q} \quad E(\alpha)/E\,.$$

Since these are all finite (due to $E/\mathbb{Q}$ being finite), we apply the Tower Theorem along with inequalities from **Question 1** to get

$$[E(\alpha) : Q] = [E(\alpha) : E][E : \mathbb{Q}] \leq [\mathbb{Q}(\alpha) : \mathbb{Q}] \cdot 2 = 4n$$

$$[E(\alpha) : Q] = [E(\alpha) : \mathbb{Q}(\alpha)][\mathbb{Q}(\alpha) : \mathbb{Q}] \leq [E : \mathbb{Q}] \cdot 2n = 4n$$

$$\implies 2n \,|\, [E(\alpha) : Q] \quad [E(\alpha) : Q] \leq 4n$$

$$\implies [E(\alpha) : Q] \in \{2n, 4n\}\,.$$

So, we need only consider these two cases of $[E(\alpha) : Q]$.

Suppose $[E(\alpha) : Q] = 4n$, then by the Tower Theorem, $[E(\alpha) : E] = 2n$, and this says that the minimal polynomial of α over E is of order $2n$, but we already know that $f(x) \in E[x]$ is the minimal polynomial over $\mathbb{Q}$, and thus can not have reduced in degree in the extension, and hence $\deg_E f(x) = 2n$ and it is irreducible.

Suppose $[E(\alpha) : Q] = 2n$, then by the Tower Theorem, $[E(\alpha) : E] = n$, that is the minimal polynomial of α over E has degree n. Since $f(x) \in \mathbb{Q}[x]$ was the minimal polynomial (up to unit multiple), we expect its extension to contain the minimal polynomial over E, but the degree must be n. Suppose the minimal polynomial is $p(x) \in E[x]$, then we must have that $f(x) = p(x)h(x)$, with $h(x) \in E[x]$ some other degree n polynomial. Clearly $p(x)$ is irreducible, so we need to consider the reducibility of $h(x)$. However, since α was chosen arbitrarily, any root of $h(x)$ will follow a similar path and hence will require it also being of degree n. Thus, $h(x)$ better be irreducible.