Question 1

(a) We prove the claim directly. Suppose that $G/Z(G)$ is indeed cyclic, then $\exists/, a \in G$ such that $aZ(G)$ generates the entire quotient group. We know that cosets partition the group, and in particular if $g, h \in G$, then $\exists\, g_Z, h_Z \in Z(G)$ and $n, m \in \mathbb{Z}^+$ such that

$$g = a^n g_Z \qquad \& \qquad h = a^m h_Z\,.$$

Then, using the fact that g_Z and h_Z are in the centre,

$$gh = (a^n g_Z)(a^m h_Z) = a^n a^m g_Z h_Z = a^m a^n h_Z g_Z = (a^m h_Z)(a^n g_Z) = hg$$

and so we see that G is abelian.

(b) We first use Lagrange's theorem, since we know $Z(G)$ is a normal subgroup of G, then we know that $|Z(G)$ divides $|G|$. So, we get that $|Z(G)| = 1, p$ or p_2. Well, if $|Z(G)| = p^2$, then $G = Z(G)$ and G is abelian.

Next, we suppose $|Z(G)| = 1$. By the class equation, we have that

$$|G| = |Z(G)| + \sum |G : C_G(a_i)|$$

and since $p \nmid |Z(G)|$, we see that $p \nmid |G : C_G(a_i)|$. However, since the centre is trivial, we know that $a_i \notin Z(G)$ and the groups $C_G(a_i)$ are proper subgroups of G, and hence $p \,|\, |G : C_G(a_i)|$, which is a contradiction. Thus, the centre is non-trival.

Next we assume $|Z(G)| = p$. Then, $|G|/|Z(G)| = |G : Z(G)| = p$ and we recall that groups of prime order are cyclic, and hence from the above proof we know that G is abelian.

Thus, we have shown what is required.

Question 2

We first factor 132 into it's primes to get $2^2 \cdot 3 \cdot 11$. Looking at n_{11}, we see that

$$n_{11} = 1 \pmod{11} \qquad \& \qquad n_{11} | 12$$

so we require that $n_{11} = 1$ or 12. If $n_{11} = 1$ then we have a normal subgroup so we assume $n_{11} = 12$. Next, we look at n_3,

$$n_3 = 1 \pmod 3 \qquad \& \qquad n_3 | 44 = 2^2 \cdot 11\,.$$

We have way more choices than we did with our Sylow 11-group, so we reduce this by counting the number of elements that are actually left after having 12 Sylow 11-subgroups. We see that $132 - (12 \times 10) = 12$, so we must have that $n_3 = 1, 4$. However, if $n_3 = 4$, then there are no Sylow 2-subgroups, which is a contradiction with Sylows first theorem, and hence $n_3 = 1$ and we see that we have no simple groups of order 132.

Question 3

(a) We prove this by looking at the action of left multiplication to cosets of H in G. In particular, since $|G : H| = n > 1$, we know that $\{a_1H, a_2H, \ldots, a_nH\}$ forms the set of left cosets, where $a_0 = e$. Suppose $g \in G$, then the action will be $g \cdot a_iH = ga_iH = a_jH$, which we can define as a map, call it f_g. Notice that this action of left multiplication only permutes the representatives. So, define a map $f : G \to S_n$ with $f(g) = f_g$, and thus elements of G are sent to their action on the left cosets of H, which is exactly a subset of S_n. Now we need to convince ourselves this is injection and homomorphism. Well, notice that the homomorphism property holds quite naturally, as if $g, h \in G$, then

$$f(gh)(a_i) = f_{gh}(a_i) = (gh)(a_i) = g(ha_i) = g(f_h(a_i)) = f_g(f_h(a_i)) = f(g) \circ f(h)(a_i)$$

and $f(e)(a_i) = f_e(a_i) = e(a_i) = a_i$. To see injection, we need to show that distinct elements map uniquely. Suppose $f(g) = f(h)$, then

$$f(g)(a_i) = f_g(a_i) = ga_i \qquad \& \qquad f(h)(a_i) = f_h(a_i) = ha_i$$

$$f(g) = f(h) \implies ga_i = ha_i \implies g = h$$

and so we have injection.

Thus there is indeed an injective homomorphism from G to a subgroup of S_n (where we know it is a subgroup since it is a homomorphism).

(b) Since G is not simple, we can assume no proper normal subgroups and hence we have that $n_p > 1$. Furthermore, we see that $|G : N_G(P)| = n_p$, where we note that the normalizer forms a subgroup. This means we can apply the previous proof! Thus, we have an injective map $\varphi : G \to S_{n_p}$, and in particular, $\varphi(G) \leq S_{n_p}$, so $|G|$ divides $|S_{n_p}| = n_p!$ by Lagrange's Theorem.

(c) We start by factoring, in particular we see that $|G| = 48 = 2^4 \cdot 3$. We first note that

$$n_3 = 1 (\operatorname{mod} 3) \qquad \& \qquad n_3 \,|\, 16\,.$$

So, we have that $n_3 \in \{1, 4, 16\}$. We assume $n_3 \neq 1$, since then we would have that G has a normal subgroup and hence not simple. Suppose instead that $n_3 = 4$, but by the contrapositive of the previous proof, we see that then $|G| = 48$ does not divide $4!$, and since $3 \,|\, |G|$, and $|G| \neq 3$, we see that G would not be simple. Then, we only have $n_3 = 16$ left, which by counting tells us that we have $48 - (16 \cdot 2) = 16$ elements left. Well, we know that then $n_2 = 1$ since there is only enough room for one Sylow 2-subgroup, but then G has a normal subgroup and is not simple. Thus, we have no simple subgroups of order 48.

Question 4

We prove this statement by induction on the group order. First, notice that $|G| = 2$ gives us the trivial Sylow 2-subgroup of itself and the statement is trivially true. Thus, suppose $|G| = p^n m > 2$ where $n, m \in \mathbb{Z}$, and $p \nmid m$, and we assume the result for all groups of smaller order. We consider two cases:

Case 1 Suppose $p \,|\, |Z(G)|$. Then by Cauchy's Theorem we have that $\exists\, a \in Z(G)$ such that $|a| = p$, and $< a > \leq Z(G)$, and thus $< a >$ is normal in G. Then, we can consider the subgroup $G/ < a >$, which has order less than G, and hence by induction the result holds and we have a Sylow p-subgroup and subgroups of all orders of p less than the Sylow p-subgroup. We can order these subgroups by their order in p and call them

$$\{\bar{H}_1, \bar{H}_2, \ldots, \bar{H}_{n-1}\}\,.$$

We notice that if $n = 1$, then we only have $< a >$ as our subgroup and that gives us our result, so we suppose $n > 1$. Then, we see the order of our subgroups is given by $\{p, p^2, \ldots, p^{n-1}\}$ respectively. Then, by the correspondance theorem, we have that

$$\bar{H}_i = H_i / < a >\,,\ 1 \leq i \leq n-1$$

where $H \leq G$, and buy looking at orders

$$p^i = \frac{|H_i|}{p} \implies |H| = p^{i+1}\,.$$

So, we see that G has subgroups of orders p^k for $k \in \{1, \ldots, p\}$.

Case 2 Suppose $p \nmid |Z(G)|$. Then by the class equation, we see that

$$p^n m = |Z(G)| + \sum |G : C_G(a_i)|$$

and thus $\exists\, a_i$ such that $p \nmid |G : C_G(a_i)| \implies p^n \,|\, |C_G(a_i)|$. We know that $C_G(a_i) \neq G$ since $a_i \notin Z(G)$, so $|C_G(a_i)| < |G|$, and by induction $C(a_i)$ has a Sylow p-subgroup and subgroups of order p^k for $k = 1, \ldots, n$.

Thus, we have that G has subgroups of order p^k for $k = 1, \ldots, n$ given that $|G| = p^n m$.

Question 5

We prove this statement by contradiction. Suppose that $\exists Q \leq G$ a p-subgroup such that it is not contained in any of the Sylow p-subgroups. In particular, suppose P is a Sylow p-subgroup of G and that G, and thus Q, acts on P by conjugation to produce $K = \{gPg^{-1} : g \in G\}$, where we order them to get $K = \{P = P_1, P_2, \ldots, P_r\}$. So, we have assumed $Q \cap P_i = \{e\} \leq Q$, $i \in \{1, \ldots, r\}$. Then, by theorem,

$$|K| = \sum_{i=1}^{r} |Q : Q \cap P_i| = \sum_{i=1}^{r} |Q|/1 \implies p \mid |K| .$$

But, we know from the third Sylow Theorem that $n_p = |K| = 1 \pmod{p}$, and hence this is a contradiction. So, we must have that $Q \leq P_i$ for some $P_i \in K$.