PMATH 465/665: Suggested problems for the final

A. Topological spaces.

1. Give the definition of a topology and of a basis of a topology. What does it mean for a topology to be Hausdorff, second countable? What does it mean for a topological space to be compact, connected?

2. Given an example of a topology that is not Hausdorff, and of a topology that is not second countable. Moreover, give an example of a topology that is both Hausdorff and second countable.

3. Let X be a set.

   (a) Let $\tau = \{\emptyset, X\}$. Prove that τ is a topology on X, called the *trivial topology*, that is second countable but not Hausdorff.

   (b) Let $\tau = \{\text{all subsets of } X\}$. Show that τ is a topology on X called the *discrete topology.* Note that (X, τ) is called a *discrete space.* Moreover, prove that τ is Hausdorff but not necessarily second countable.

   (c) Let τ be a topology on X. Prove that (X, τ) is a 0-manifold if and only if it is a countable discrete space.

4. Let (X, τ) be a topological space and $Y \subset X$ be a subset. Give a definition of the relative topology τ_Y on Y. Verify that τ_Y is indeed a topology on Y.

5. Let (X, τ) be a Hausdorff (respectively, second countable) topological space, and let $Y \subset X$ be a subset. Show that the relative topology τ_Y on Y is also Hausdorff (respectively, second countable).

6. Let (X, τ) be a topological space and suppose there is an equivalence relation $\sim$ on X. Consider the quotient space $Y = X/\sim$. Give a definition of the quotient topology $\tau_\sim$ on Y. Verify that $\tau_\sim$ is indeed a topology on Y.

7. Let (X_1, τ_1) and (X_2, τ_2) be topological spaces and $f : X_1 \to X_2$ be a map. Define what it means for f to be continuous or a homeomorphism. Moreover, if (X_2, τ_2) is $\mathbb{R}^k$ endowed with the standard topology, define what the support of f is.

8. Prove that homeomorphisms map open sets to open sets and closed sets to closed sets. Moreover, prove that homeomorphisms map compact sets to compact sets and connected sets to connected sets (where the sets are endowed with the relative topology). Finally, prove that a homeomorphism maps a set with d connected components to a set with d connected components (with respect to the relative topologies on the sets).

   Note: You may assume the fact that continuous maps map compact sets to compact sets and connected sets to connected sets.

9. Let (X, τ) be a topological space and suppose there is an equivalence relation $\sim$ on X. Consider the quotient space $Y = X/\sim$ with the quotient topology $\tau_\sim$ on Y. Verify that the quotient map $\pi : X \to Y, p \mapsto [p]$, where $[p]$ is the equivalence class of p in Y, is always continuous. Moreover, prove that if X is compact (respectively, connected), then so is Y.

10. Let (M_1, τ_1) and (M_2, τ_2) be two topological spaces. Consider the (Cartesian) product

$$M_1 \times M_2 := \{(x_1, x_2) : \; x_1 \in M_1 \text{ and } x_2 \in M_2\}.$$

    Define what the product topology $\tau_1 \times \tau_2$ on $M_1 \times M_2$ is.

B. Topological manifolds.

1. Give the definition of a topological n-manifold, of a coordinate chart, of an atlas and of transition functions.

2. Prove that $\mathbb{R}^n$ with the standard topology is a topological n-manifold.

3. Let $U \subset \mathbb{R}^n$ be open and $f : U \to \mathbb{R}^k$ be a continuous function. Let

$$M := \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^k = \mathbb{R}^{n+k} : x \in U, y = f(x)\} \subset \mathbb{R}^{n+k}$$

be the graph of f and endow it with the relative standard topology on $\mathbb{R}^{n+k}$. Show that M is a topological n-manifold. Moreover, give an atlas for M.

4. Let $S^n = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : x_1^2 + \cdots + x_n^2 = 1\}$ be the unit sphere in $\mathbb{R}^{n+1}$ and endow it with the relative standard topology on $\mathbb{R}^{n+1}$. Show that S^n is a topological n-manifold. Moreover, give an atlas for S^n. Is it possible to find an atlas of S^n that consists of only one chart? Justify your answer.

5. Prove that any open subset of a topological n-manifold is a topological n-manifold.

6. Determine whether the following sets are topological manifolds. Justify your answers! Note that each set is a subset of $\mathbb{R}^n$ and is thus endowed with the relative standard topology.

   (a) The boundary of the square in $\mathbb{R}^2$ with vertices $(0, 0)$, $(1, 0)$, $(1, 1)$, and $(0, 1)$.

   (b) The α-curve $\{(x, y) \in \mathbb{R}^2 : y^2 = x^2(x + 1)\} \subset \mathbb{R}^2$. For a picture, see (second picture).

   (c) The cylinder $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 1\} \subset \mathbb{R}^3$.

   (d) The cone $\left\{(x, y, z) \in \mathbb{R}^3 : z = \sqrt{x^2 + y^2}\right\} \subset \mathbb{R}^3$.

   (e) The twisted cubic $\{(x, y, z) \in \mathbb{R}^3 : y = x^2, z = x^3\} \subset \mathbb{R}^3$.

   (f) The two-sided paraboloid $\{(x, y, z) \in \mathbb{R}^3 : y^2 = (x^2 + z^2)^2\} \subset \mathbb{R}^3$.

   (g) The two-sheeted paraboloid $\{(x, y, z) \in \mathbb{R}^3 : y^2 = (x^2 + z^2 + 1)^2\} \subset \mathbb{R}^3$.

   (h) $\{(x, y, z) \in \mathbb{R}^3 : z = 0\} \cup \{(x, y, z) \in \mathbb{R}^3 : x = y = 0\} \subset \mathbb{R}^3$.

   (i) $\{(x, y, z) \in \mathbb{R}^3 : y = -1\} \cup \{(x, y, z) \in \mathbb{R}^3 : x = y = 0\} \subset \mathbb{R}^3$.

7. Define real projective n-space $\mathbb{P}^n$, provide an atlas for it and compute its transition functions. Let S^n be the unit sphere in $\mathbb{R}^{n+1}$ and consider the quotient space $S^n/\sim$, where $x \sim y$ if and only if $x = \pm y$ for $x, y \in S^n$. Show that $S^n/\sim$ is homeomorphic to $\mathbb{P}^n$ and use this fact to prove that $\mathbb{P}^n$ is compact and connected.

8. Suppose that (M_1, τ_1) is a topological n_1-manifold and (M_2, τ_2) is a topological n_2-manifold. Show that $(M_1 \times M_2, \tau_1 \times \tau_2)$ is a topological $(n_1 + n_2)$-manifold.

   Note: You may assume that, if one endows $\mathbb{R}^k$ and $\mathbb{R}^l$ with the standard topology and identifies $\mathbb{R}^k \times \mathbb{R}^l$ with $\mathbb{R}^{k+l}$, then the product topology on $\mathbb{R}^k \times \mathbb{R}^l$ coincides with the standard topology on $\mathbb{R}^{k+l}$.

9. A *real n-torus* $S^1 \times \cdots \times S^1$ is defined as the product of n copies of S^1. Prove that n-tori are topological n-manifolds.

C. Smooth manifolds.

1. Define what a smooth atlas and what a smooth structure on a topological n-manifold is.

2. Show that if a topological n-manifold admits an atlas $\mathcal{A}$ consisting of a single chart, then $\mathcal{A}$ induces a smooth structure on M. Describe this smooth structure.

3. Prove that topological manifolds in questions B.2, B.3, B.4, B.5,and B.7 all admit smooth structures.

4. Problem 1.7 on p. 30 of Lee's book.

5. Show that any finite set admits the structure of a smooth 0-manifold.

6. Prove that the set $M_{m\times n}(\mathbb{R})$ of all $m \times n$ matrices with entries in $\mathbb{R}$ is a smooth mn-dimensional manifold. Moreover, prove that the general linear group

$$GL(n,\mathbb{R}) = \{A \in M_{n\times n}(\mathbb{R}) : \det A \neq 0\}$$

is a smooth n^2-dimensional manifold.

7. Prove that the Cartesian product $M_1 \times M_2$ of two smooth manifolds M_1 and M_2 is a smooth manifold. Use this to prove that n-tori admit smooth structures.

8. Let M be a smooth n-manifold and $f : M \to \mathbb{R}^k$ be a function. What does it mean for f to be smooth at a point $p \in M$? on a subset $A \subset M$? Suppose that f is smooth on $A \subset M$. Prove that f is smooth on an open set U containing A. Moreover, let $\mathcal{A}$ be a smooth atlas contained in the maximal atlas of M. Prove that if $f \circ \varphi^{-1} : \varphi(U) \subset \mathbb{R}^n \to \mathbb{R}^k$ is smooth for any chart $(U, \varphi) \in \mathcal{A}$, then f is amooth on M (in other words, it is enough to check smoothness only for the charts in a smooth atlas contained in the maximal atlas, as opposed to all the charts in the maximal atlas).

9. Let M and M' be smooth manifolds of dimensions n and n', respectively, and let $f : M \to M'$ be a map. What does it mean for F to be smooth? a diffeomorphism? If $A \subset M$ and F is smooth on A. Does it mean that F is smooth on an open subset of M containing A? Justify your answer. Moreover, let $\mathcal{A}$ be a smooth atlas contained in the maximal atlas of M and $\mathcal{A}'$ be a smooth atlas contained in the maximal atlas of M'. Prove that if

$$\varphi' \circ f \circ \varphi^{-1} : \varphi(U \cap f^{-1}(U')) \subset \mathbb{R}^n \to \mathbb{R}^{n'}$$

is smooth for any charts $(U, \varphi) \in \mathcal{A}$ and $(U', \varphi') \in \mathcal{A}'$), then F is smooth on M (again, this means that it is enough to check smoothness only for the charts in a smooth atlas contained in the maximal atlas, as opposed to all the charts in the maximal atlas).

10. Problem 2.3 p. 48 from Lee's book.

11. Problem 2.6 p. 48 from Lee's book.

12. Let M be a smooth manifold and $\{U_\alpha\}_{\alpha\in A}$ be an open cover of M. Define what a partition of unity subordinate to $\{U_\alpha\}_{\alpha\in A}$ is. Do the open sets of the open cover $\{U_\alpha\}_{\alpha\in A}$ have to be the domains of charts on M? Explain.

13. Let M be a smooth manifold and $A \subset M$ be a closed subset. Using a smooth partition of unity subordinate to an open cover of M, show that a function $f : A \to \mathbb{R}^k$ is smooth if and only if there exist an open set $U \subset M$ containing A and $\tilde{f} \in C^\infty(M)$ such that $f = \tilde{f}|_A$ and $\operatorname{supp} \tilde{f} \subset U$.

14. Let A and B be disjoint closed subsets of a smooth manifold M. Show that there exists $f \in C^\infty(M)$ such that $0 \leq f(x) \leq 1$ for all $x \in M$, f is identically 0 on A and f is identically 1 on B.

15. Let A be a closed subset and U be an open subset of a smooth manifold M with $A \subset U$. Show that there exists $f \in C^\infty(M)$ such that f is identically 1 on A and $\operatorname{supp} f \subset U$.

D. Tangent spaces.

1. Let M be a smooth manifold and $p \in M$. What is a derivation of M at p and the tangent space to M at p? Give some examples and show that the space of all derivations T_pM of M at p is an $\mathbb{R}$-vector space. Moreover, verify that if $X \in T_pM$ and $c \in \mathbb{R}$, then $X(c) = 0$. Finally, prove that if $f, g \in C^\infty(M)$ agree on a neighbourhood of p, then $X(f) = X(g)$ (which means that the derivation of a function at a point only captures local behaviour).

2. Let M and N be smooth manifolds, and $F : M \to N$ be a smooth map. Let $p \in M$. Define the pushforward map of F at p and verify that it is indeed a well-defined map (that is, $F_{*,p}(X) \in T_{F(p)}N$ for all $X \in T_pM$).

3. *Properties of the pushforward.* Let M, N and P be smooth manifolds. Moreover, let $F : M \to N$ and $G : N \to P$ be smooth maps and $p \in M$. Prove the following:

   (a) $F_* : T_pM \to T_{F(p)}N$ is linear.

   (b) $(G \circ F)_* = G_* \circ F_* : T_pM \to T_{G\circ F(p)}P$.

   (c) $(\mathrm{Id}_M)_* = \mathrm{Id}_{T_pM} : T_pM \to T_pM$.

   (d) If F is a diffeomorphism, then $F_* : T_pM \to T_{F(p)}N$ is an isomorphism.

4. Let M be a smooth n-manifold and $(U, \varphi = (x_1, \dots, x_n))$ be a local chart. Define $\partial/\partial x_i|_p$ for all $i = 1, \dots, n$ and $p \in U$, and show that $\{\partial/\partial x_1|_p, \dots, \partial/\partial x_n|_p\}$ is a basis of T_pM. Moreover, if $(V, \psi = (y_1, \dots, y_n))$ is another local chart with $U \cap V \neq \emptyset$, verify that

$$\left.\frac{\partial}{\partial y_j}\right|_p = \sum_{i=1}^n \frac{\partial x_i}{\partial y_j}(p) \left.\frac{\partial}{\partial x_i}\right|_p$$

   for all $p \in U \cap V$.

   Proof. In terms of the basis $\{\partial/\partial y_1|_p, \dots, \partial/\partial y_n|_p\}$ of T_pM, we have

$$\left.\frac{\partial}{\partial y_j}\right|_p = \sum_{i=1}^n a^i \left.\frac{\partial}{\partial x_i}\right|_p$$

   where

$$a^i = \left.\frac{\partial}{\partial y_j}\right|_p (x_i) = \frac{\partial x_i}{\partial y_j}(p),$$

   giving the formula. □

5. If a nonempty smooth n-manifold is diffeomorphic to an m-manifold, prove that $n = m$.

6. Let M^m and N^n be smooth manifolds and $F : M \to N$ be a smooth map. Let $p \in M$. If $(U, \varphi = (x_1, \dots, x_m))$ and $(V, \psi = (y_1, \dots, y_n))$ are charts of M and N containing p and $F(p)$, respectively, and $F_i := y_i \circ F$ is the i-th component function of F, for $i = 1, \dots, n$, show that

$$F_*\left(\left.\frac{\partial}{\partial x_j}\right|_p\right) = \sum_{i=1}^n \frac{\partial F_i(p)}{\partial x_i} \left.\frac{\partial}{\partial y_i}\right|_{F(p)}.$$

   Proof. In terms of the basis $\{\partial/\partial y_1|_{F(p)}, \dots, \partial/\partial y_n|_{F(p)}\}$ of $T_{F(p)}N$, we have

$$F_*\left(\left.\frac{\partial}{\partial x_j}\right|_p\right) = \sum_{i=1}^n a^i \left.\frac{\partial}{\partial y_i}\right|_p$$

where

$$a^i = F_*\left(\frac{\partial}{\partial x_j}\bigg|_p\right)(y_i) = \frac{\partial}{\partial x_j}\bigg|_p (y_i \circ F) = \frac{\partial F_i}{\partial x_j}(p),$$

giving the formula. $\square$

7. Suppose that M and N are smooth manifolds with M connected, and $F : M \to N$ is a smooth map. Prove that $F_* : T_pM \to T_{F(p)}N$ is the zero map for each $p \in M$ if and only if F is a constant map.

8. Problems 8.1 to 8.6 from Chapter 3 in Tu's book.

9. Let M be a smooth manifold. Define what a smooth curve $\gamma : I \subset \mathbb{R} \to M$ on M is and its tangent vector $\gamma'(t_0)$ at $\gamma(t_0)$ is.

10. Let M be a smooth manifold and $p \in M$. Moreover, let $p \in M$. Show that there exists a smooth curve $\gamma : I \subset \mathbb{R} \to M$ on M such that $\gamma'(t_0) = X$. This means, in particular, that one can interpret T_pM as the set of all tangent vectors to smooth curves on M passing through p.

11. Let M and N be smooth manifolds and $F : M \to N$ be a smooth map. What does it mean for F to be an immersion, an embedding, or a submersion. Given an example of each of those maps.

12. Give a definition of a immersed/embedded submanifold of a smooth manifold and some examples.

13. Prove that $O(n) := \{A \in M_{n\times n}(\mathbb{R}) : AA^T = Id_{M_{n\times n}(\mathbb{R})}\}$ is a smooth embedded submanifold of $M_{n\times n}(\mathbb{R})$. What is its dimension?

D. Smooth vector fields and flows.

1. Let M be a smooth n-manifold. Give the definition of the tangent bundle of M and of a (smooth) vector field on M, and describe what vector fields look like in terms of local coordinates $(U, \varphi = (x_1, \dots, x_n))$.

2. Let M be a smooth manifold and Y, Z be two vector fields on M. Prove that $Y = Z$ if and only if $Y(f) = Z(f)$ for all $f \in C^\infty(M)$.

3. Let M be a smooth manifold and X be a vector field on M. Prove that the following are equivalent:
   (a) X is smooth;
   (b) the component functions X^i of X are smooth in any local chart (U, φ) of M;
   (c) $X(f) \in C^\infty(M)$ for any $f \in C^\infty(M)$.

4. Problem 8.11 p. 201 in Lee. Also explain why all the vector fields are smooth.

5. Let M be a smooth manifold and $A \subset M$ be a closed subset. Suppose Y is a smooth vector field on A. Prove that given any open subset U containing A, there exists a smooth global vector field $\tilde{Y}$ on M such that $\tilde{Y}|_A = Y$ and $\operatorname{supp}\tilde{Y} \subset U$. Conclude that for any $p \in M$ and $X \in T_pM$, there exists a smooth vector field Y on M with $Y_p = X$.

   Note: If Y is a vector field on M, then $\operatorname{supp} Y := \{p \in M : Y_p \neq 0\}$ is the *support of* Y.

6. Let M and N be smooth manifolds and $F : M \to N$ be a smooth map. Moreover, let $Y \in \mathcal{X}(M)$ and $Z \in \mathcal{X}(N)$. Define what it means for Y and Z to be F-related. Suppose that X and Y are F-related. Verify that

$$Z(f) \circ F = Y(f \circ F)$$

   for all $f \in C^\infty(M)$; in particular, if F is a diffeomorphism, then $Z(f) = Y(f \circ F) \circ F^{-1}$ for all $f \in C^\infty(M)$.

7. Let M and N be smooth manifolds and $F : M \to N$ be a diffeomorphism. Prove that for any $Y \in \mathcal{X}(M)$, the pushforward $F_*(Y)$ of Y is a well-defined smooth vector field on N such that

$$(F_*(Y))_p = F_{*,p}(Y_p)$$

for all $p \in M$. Moreover, verify that $F_*(Y)(f) = Y(f \circ F) \circ F^{-1}$ for all $f \in C^\infty(M)$. What can go wrong with the above definition of $F_*(Y)$ if f is not a diffeomorphism?

8. Problem 8.10 p. 201 in Lee.

9. Problem 8.12 p. 201 in Lee.

10. Define what the Lie bracket of two vector fields is and and describe what it looks like in terms of local coordinates $(U, \varphi = (x_1, \ldots, x_n))$. Moreover, prove that the Lie bracket of two smooth vector fields is also a smooth vector field.

11. For each of the following pairs of vector fields X, Y defined on $\mathbb{R}^3$, compute the Lie bracket $[X, Y]$.

    (a) $X = y\dfrac{\partial}{\partial z} - 2xy^2\dfrac{\partial}{\partial y}$ and $Y = \dfrac{\partial}{\partial y}$.

    (b) $X = x\dfrac{\partial}{\partial y} - y\dfrac{\partial}{\partial x}$ and $Y = y\dfrac{\partial}{\partial z} - z\dfrac{\partial}{\partial y}$.

    (c) $X = x\dfrac{\partial}{\partial y} - y\dfrac{\partial}{\partial x}$ and $Y = x\dfrac{\partial}{\partial y} + y\dfrac{\partial}{\partial x}$.

12. Let M be a smooth manifold, $X, Y \in \mathcal{X}(M)$ and $f, g \in C^\infty(M)$. Prove that

$$[fX, gY] = fg[X, Y] + fX(g))Y - gY(f))X.$$

13. Let M and N be smooth manifolds and $F : M \to N$ be a diffeomorphism map. Prove that

$$F_*([X, Y]) = [F_*(X), F_*(Y)]$$

for all $X, Y \in \mathcal{X}(M)$.

14. Let M be a smooth manifold. What is a flow on M? What is the infinitessimal generator of a flow? Moreover, given $X \in \mathcal{X}(M)$, what is the integral curve of X?

    (a) Problem 9.3 from Lee's book.

    (b) Problem 9.18 from Lee's book.

    (c) Let $Z = \partial/\partial x \in \mathcal{X}(\mathbb{R}^2)$ and V be the vector field from problem 9.3 (a) in Lee's book. Verify that $[V, Z] = 0$ so that the flows of the vector fields commute. Illustrate that the flows commute by drawing a diagram.

    (d) Problems 14.3, 14.4, 14.5, 14.7, 14.8, 14.10 and 14.11 from Chapter 3 in Tu's book.

E. Distributions and the Frobenius Theorem.

1. Let M be a smooth manifold. What is a distribution? Let D be a fixed distribution. What is an integral submanifold of D? What does it mean for D to be involutive? integrable? completely integrable.

2. Let M be a smooth n-manifold and D be a completely integrable k-dimensional smooth distribution on M so that about every point $p \in M$, there exist local coordinates $(u_1, \ldots, u_n)$such that D is spanned by $\partial/\partial u_1, \ldots, \partial/\partial u_k$ on U. Prove that if $p = (c_1, \ldots, c_n)$ in those local coordinates, then $N := \{p \in U : u_{k+1} = c_{k+1}, \ldots, u_n = c_n\}$ is an integral manifold of D passing though p.

3. State the Frobenius Theorem.

4. (a) Let $U \subset \mathbb{R}^3$ be the subset where all three coordinates are positive and let D be the distribution on U spanned by the vector fields

$$X = y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}, \quad Y = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}.$$

   i. Verify that D is involutive.
   ii. Find the integral submanifolds of D.

   (b) Let D be a distribution on $\mathbb{R}^3$ spanned by

$$X = \frac{\partial}{\partial x} + yz\frac{\partial}{\partial z}, \quad Y = \frac{\partial}{\partial y}.$$

   i. Find an integral submanifold of D passing through the origin.
   ii. Is D involutive? Explain you answer in light of part (i).

F. Differential 1-forms.

1. Let M be a smooth manifold and $f \in C^\infty(M)$. What is the differential df of f? Moreover, prove that for every $p \in M$, $d_pf : T_pM \to \mathbb{R}$ is an $\mathbb{R}$-linear map, implying that $d_pf \in (T_pM)^* =: T_p^*M$.

2. Let M be a smooth n-manifold and $(U, \varphi = (x_1, \dots, x_n))$ be a local chart. Show that $\{d_px_1, \dots, d_px_n\}$ is a basis of $T_p^*M := (T_pM)^*$ for any $p \in U$ and that, in terms of this basis,

$$d_pf = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(p) d_px_i$$

for any $f \in C^\infty(M)$. Moreover, prove that this local description is independent of the local coordinates chosen. In other words, if $(V, \psi = (y_1, \dots, y_n))$ is another local chart with $U \cap V \neq \emptyset$, verify that

$$\sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(p) d_px_i = \sum_{ij=1}^{n} \frac{\partial f}{\partial y_j}(p) d_py_j$$

for all $p \in U \cap V$. Furthermore, give the definition of the cotangent bundle of M and of a (smooth) differential 1-form on M, and describe what 1-forms looks like in terms of the local coordinates $(x_1, \dots, x_n)$.

3. Let $f : \mathbb{R}^3 \to \mathbb{R}, (x, y, z) \mapsto y^2xz\sin(e^z - 1)$. Compute df.

4. Let $S^2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1\}$ be the unit sphere in $\mathbb{R}^3$ with smooth structure given by the smooth atlas

$$U_i^+ := \{(x_1, x_2, x_3) \in S^2 : x_i > 0\}, U_i^- := \{(x_1, x_2, x_3) \in S^2 : x_i < 0\}, i = 1, 2, 3,$$

and

$$\varphi_i^\pm : U_i^\pm \subset S^2 \to B_1(0) \subset \mathbb{R}^2, (x_1, x_2, x_3) \mapsto, (x_1, \dots, \hat{x}_i, \dots, x_3), i = 1, 2, 3,$$

where by $(x_1, \dots, \hat{x}_i, \dots, x_3)$ we mean (x_1, x_2, x_3) with the i-th coordinate removed. See Example 1.4 on p. 5 of Lee's book for details. Consider the scalar function $f : S^2 \to \mathbb{R}, (x, y, z) \mapsto yz$. Prove that f is smooth and compute df.

5. Suppose that M is a connected smooth manifold and that $f \in C^\infty(M)$. Prove that $df = 0$ if and only if f is constant.

6. Let M be a smooth manifold and $\omega \in \Lambda^1(M)$. Prove that the following are equivalent:

(a) ω is smooth;

(b) the component functions ω^i of ω are smooth in any local chart (U, φ) of M;

(c) $\omega(Y) \in C^\infty(M)$ for any $Y \in \mathcal{X}(M)$.

7. *Properties of the pullback.* Let M, N be a smooth manifolds and $F : M \to N$ be a smooth map. Prove the following.

   (a) $F^* : \Lambda^1(N) \to \Lambda^1(M)$ is $\mathbb{R}$-linear.

   (b) $F^*(df) = d(f \circ F)$ for all $f \in C^\infty(N)$.

   (c) $F^*(f\omega) = (f \circ F)F^*\omega$ for all $f \in C^\infty(N)$ and $\omega \in \Lambda^1(N)$.

   (d) In local coordinates $(U, \varphi = (y_1, \dots, y_n))$ on N, if $\omega = \sum_{i=1}^n \omega^i dy_i$ and $F = (F_1, \dots, F_n)$, then

   $$F^*(\omega) = \sum_{i=1}^n (\omega^i \circ F) dF_i.$$

   (e) If $\omega \in \Omega^1(N) \subset \Lambda^1(N)$, then $F^*(\omega) \in \Omega^1(M)$ so that $F^*|_{\Omega^1(N)} : \Omega^1(N) \to \Omega^1(M)$.

8. In each of the following cases, M and N are smooth manifolds, $F : M \to N$ is a map, and ω is a differential 1-form on N. In each case, explain why F and ω are smooth, and compute $F^*(\omega)$.

   (a) $M = N = \mathbb{R}^2$, $F(s,t) = (st, e^t)$, $\omega = x dy$.

   (b) $M = \mathbb{R}^2$ and $M = \mathbb{R}^3$, $F(\varphi, \theta) = ((\cos\varphi + 2)\cos\theta, (\cos\varphi + 2)\sin\theta, \sin\varphi)$, $\omega = (3y - z)dx - x^2 dz$.

   (c) $M = \{(u,v) \in \mathbb{R}^2 : u^2 + v^2 < 1\}$ and $N = \mathbb{R}^3 \backslash \{0\}$, $F((u,v) = (u, v, \sqrt{1 - u^2 - v^2})$,

   $$\omega = \frac{x}{(x^2 + y^2 + z^2)^{3/2}} dy.$$

9. Problems 17.1, 17.2 and 17.3 from Chapter 5 in Tu's book.