Problem 1

(a) To show this relation, we simply do a direct computation.

$$\begin{aligned}
\lim_{a\to 1}\int_0^\infty \left(-1\frac{\partial}{\partial a}\right)^N e^{-ax}dx &= \lim_{a\to 1}\int_0^\infty (-1)^N\left(\frac{\partial}{\partial a}\right)^N e^{-ax}dx \\
&= \lim_{a\to 1}\int_0^\infty (-1)^N(-1)^N x^N e^{-ax}dx \\
&= \lim_{a\to 1}\int_0^\infty x^N e^{-ax}dx \\
&= \int_0^\infty x^N e^{-x}dx
\end{aligned}$$

(b) This relation can be shown by evaluating the integral first.

$$\begin{aligned}
\lim_{a\to 1}\left(-1\frac{\partial}{\partial a}\right)^N \int_0^\infty e^{-ax}dx &= \lim_{a\to 1}\left(-1\frac{\partial}{\partial a}\right)^N \frac{1}{a} \\
&= \lim_{a\to 1}(-1)^N\prod_{i=1}^N(-1)^i a^{-(i+1)} \\
&= \lim_{a\to 1}(-1)^N(-1)^N N!\prod_{i=1}^N a^{-(i+1)} = N!
\end{aligned}$$

(c) We define the substitution. Thus, let $x = N + y\sqrt{N}$, then, $dx = dy\sqrt{N}$. Now to get the bounds, as $x \to 0$ we see $y \to -\sqrt{N}$, and as $x \to \infty$, $y \to \infty$.

$$\begin{aligned}
\int_0^\infty x^N e^{-x}dx &= \int_{-\sqrt{N}}^\infty (N + y\sqrt{N})^N e^{-(N+y\sqrt{N})}\sqrt{N}dy \\
&= \int_{-\sqrt{N}}^\infty N^N\left(1 + \frac{y\sqrt{N}}{N}\right)^N e^{-(N+y\sqrt{N})}\sqrt{N}dy \\
&= N^N\sqrt{N}e^{-N}\int_{-\sqrt{N}}^\infty \left(1 + \frac{y\sqrt{N}}{N}\right)^N e^{-y\sqrt{N}}dy
\end{aligned}$$

We now use our approximation by replacing $1+\frac{y\sqrt{N}}{N}$ with $e^{\ln(1+\frac{y\sqrt{N}}{N})}$. Thus,

$$N^N\sqrt{N}e^{-N}\int_{-\sqrt{N}}^{\infty}\left(1+\frac{y\sqrt{N}}{N}\right)^N e^{-y\sqrt{N}}dy \approx N^N\sqrt{N}e^{-N}\int_{-\sqrt{N}}^{\infty}\left(e^{\frac{y\sqrt{N}}{N}-\frac{y^2}{2N}}\right)^N e^{-y\sqrt{N}}dy$$
$$\approx N^N\sqrt{N}e^{-N}\int_{-\sqrt{N}}^{\infty}e^{-\frac{y^2}{2}}dy$$

Thus,

$$N^N\sqrt{N}e^{-N}\int_{-\sqrt{N}}^{\infty}e^{-\frac{y^2}{2}}dy \approx N!$$

(d) We now have an expression for $N!$, so let us try plugging that in and seeing where we go,

$$\ln(N!) \approx \ln(N^N\sqrt{N}e^{-N}\int_{-\sqrt{N}}^{\infty}e^{-\frac{y^2}{2}}dy)$$

For $N >> 1$, the lower integral bound approaches $-\infty$. With this in mind, we can compute the integral,

$$\ln(N^N\sqrt{N}e^{-N}\int_{-\sqrt{N}}^{\infty}e^{-\frac{y^2}{2}}dy) \approx \ln(N^N\sqrt{N}e^{-N}\sqrt{2\pi N})$$
$$\approx \ln(N^N)+\ln(e^{-N})+\ln(\sqrt{2\pi N})$$

And thus we get,

$$\ln(N!) \approx N\ln(N) - N + \frac{1}{2}\ln(2\pi N)$$

Problem 2

(a) This is a fairly simple integral to compute,

$$\begin{aligned}\int_0^\infty e^{-wx}dx &= -\frac{1}{w}e^{-wx}\bigg|_0^\infty \\ &= -\frac{1}{w}(0-1) = \frac{1}{w}\end{aligned}$$

(b) This problem is a bit tougher. The first step we take is to apply the geometric series closed form,

$$\begin{aligned}\sum_{j=0}^\infty e^{-wj} &= \sum_{j=0}^\infty \left(e^{-w}\right)^j \\ &= \frac{1}{1-e^{-w}}\end{aligned}$$

Now we expand the taylor series for e^x to get,

$$\frac{1}{1-e^{-w}} = \left(1 - \sum_{i=0}^\infty \frac{(-w)^i}{i!}\right)^{-1}$$

We pull the first term from this series since it is 1,

$$\left(1 - \sum_{i=0}^\infty \frac{(-w)^i}{i!}\right)^{-1} = \left(\sum_{i=1}^\infty \frac{(-w)^i}{i!}\right)^{-1}$$

We pull another term for convenience and factor out a w,

$$\begin{aligned}\left(\sum_{i=1}^\infty \frac{(-w)^i}{i!}\right)^{-1} &= \left(w - \sum_{i=2}^\infty \frac{(-w)^i}{i!}\right)^{-1} \\ &= \frac{1}{w}\left(1 - \sum_{i=2}^\infty \frac{(-w)^{i-1}}{i!}\right)^{-1}\end{aligned}$$

Recognize that we can again use a taylor expansion on this new argument, as it takes a similar form to $(1+x)^{-1}$,

$$\begin{aligned}
\frac{1}{w}\left(1-\sum_{i=2}^{\infty}\frac{(-w)^{i-1}}{i!}\right)^{-1} &= \frac{1}{w}\sum_{k=0}^{\infty}\frac{(-1)^k k!}{k!}\left(\sum_{i=2}^{\infty}(-1)^{i-1}\frac{(w)^{i-1}}{i!}\right)^k \\
&= \frac{1}{w}\sum_{k=0}^{\infty}(-1)^k\left(\sum_{i=2}^{\infty}(-1)^{i-1}\frac{w^{i-1}}{i!}\right)^k
\end{aligned}$$

Expanding this double series will result in the terms we are looking for. We show this explicitly for the first few terms.

$$\frac{1}{w}\sum_{k=0}^{\infty}(-1)^k\left(\sum_{i=2}^{\infty}(-1)^{i-1}\frac{w^{i-1}}{i!}\right)^k = \frac{1}{w}\left(1-\left(-\frac{w}{2!}+\frac{w}{3!}-\dots\right)+\left(-\frac{w}{2!}+\frac{w}{3!}-\dots\right)^{2!}-\left(-\frac{w}{2!}+\frac{w}{3!}-\dots\right)^3+\dots\right)$$

$O(w)$ is exactly the first term in the first series, $\frac{w}{2!}$. Notice, if we expand the $\frac{1}{w}$ through, the first term matches our integral and the second term matches what we would expect, $\frac{1}{2}$.

Next, we look at terms of $O(w^2)$ which results in the following expansion,

$$\begin{aligned}
\frac{1}{w}\sum_{k=0}^{\infty}(-1)^k\left(\sum_{i=2}^{\infty}(-1)^{i-1}\frac{w^{i-1}}{i!}\right)^k &= \frac{1}{w}\left(1+\frac{w}{2!}-\frac{w^2}{6}+\frac{w^2}{4}+O(w^3)\right) \\
&= \frac{1}{w}\left(1+\frac{w}{2!}+\frac{w^2}{12}+O(w^3)\right)
\end{aligned}$$

And we continue to expand for higher order terms,

$$\begin{aligned}
\frac{1}{w}\sum_{k=0}^{\infty}(-1)^k\left(\sum_{i=2}^{\infty}(-1)^{i-1}\frac{w^{i-1}}{i!}\right)^k &= \frac{1}{w}\left(1+\frac{w}{2!}+\frac{w^2}{12}+\frac{w^3}{24}-2\frac{w^3}{2!3!}+\frac{w^3}{(2!)^3}+O(w^4)\right) \\
= \frac{1}{w}\left(1+\frac{w}{2!}+\frac{w^2}{12}+w^3\left(\frac{1}{24}-\frac{4}{24}+\frac{3}{24}\right)+w^4\left(-\frac{1}{5!}-\frac{1}{24}+\frac{1}{36}-\frac{1}{24}+\frac{1}{16}\right)+O(w^5)\right)
\end{aligned}$$

$$= \frac{1}{w}\left(1 + \frac{w}{2} + \frac{w^2}{12} - \frac{w^4}{720} + w^5\left(\frac{1}{6!} - \frac{1}{45} + \frac{7}{96} - \frac{1}{12} + \frac{1}{2^5}\right) + O(w^6)\right)$$

$$= \frac{1}{w}\left(1 + \frac{w}{2} + \frac{w^2}{12} - \frac{w^4}{720} + w^6\left(-\frac{1}{7!} + \frac{17}{2880} - \frac{137}{4320} + \frac{1}{16} - \frac{5}{96} + \frac{1}{2^6}\right) + O(w^7)\right)$$

$$= \frac{1}{w}\left(1 + \frac{w}{2} + \frac{w^2}{12} - \frac{w^4}{720} + \frac{w^6}{30240} + O(w^7)\right)$$

As required.