Question 1

(a) Since $\alpha \in K$ is algebraic over F, and $F \subset E$ as a subfield, then α is algebraic over E. By definition, $\exists\, f(x) \in F[x]$ minimial polynomial of α over F. There are two cases to consider for this polynomial:

(i) Suppose $f(x)$ is irreducable in $E[x]$ where the coeffecients of $f(x)$ are sent under the canonical inclusion map. Then, since $f(\alpha) = 0$ and it is monic by construction, $f(x)$ is the minimal polynomial over E of α. With this determined

$$[E(\alpha) : E] = \deg_E(f(x)) = \deg_F(f(x)) = [F(\alpha) : F]\,.$$

(ii) Suppose $f(x)$ is reducable in $E[x]$ where the coeffecients are mapped as before. Then, $\exists\, h(x), g(x) \in E[x]$ such that $\deg_E(h(x)) \geq 1$, $g(x)$ is the minimal polynomial of α over E and $f(x) = h(x)g(x)$. Therefore,

$$[E(\alpha) : E] = \deg_E(g(x)) \leq \deg_E(h(x)g(x)) = \deg_F(f(x)) = [F(\alpha) : F]\,.$$

(b) We first note that $E(\alpha)/E$, $E(\alpha)/F$, $F(\alpha)/F$ and $E(\alpha)/F(\alpha)$ are all finite extensions. So, we are motivated to apply the Tower Theorem:

$$[E(\alpha) : F] = [E(\alpha) : F(\alpha)][F(\alpha) : F] \geq [E(\alpha) : F(\alpha)][F(\alpha) : E]$$

where the inequality comes from **(a)**. Then, with another application of the Tower theorem

$$[E(\alpha) : F(\alpha)] \leq \frac{[E(\alpha) : F]}{[E(\alpha) : E]} = \frac{[E(\alpha) : E][E : F]}{[E(\alpha) : E]} = [E : F]$$

as required.

Question 2

Since $F(\alpha,\beta)/F(\alpha), F(\alpha,\beta)/F(\beta), F(\alpha)/F, F(\beta)/F$, and $F(\alpha,\beta)/F$ are all finite extensions, we can apply the Tower Theorem. Let $\deg_F(\alpha) = m$, and $\deg_F(\beta) = n$, then

$$[F(\alpha,\beta) : F] = [F(\alpha,\beta) : F(\alpha)][F(\alpha) : F] = [F(\alpha,\beta) : F(\alpha)]m$$

$$[F(\alpha,\beta) : F] = [F(\alpha,\beta) : F(\beta)][F(\beta) : F] = [F(\alpha,\beta) : F(\beta)]n\,.$$

So we clearly have that $n \mid [F(\alpha,\beta) : F]$ and $m \mid [F(\alpha,\beta) : F]$, and since these are coprime, $nm \mid [F(\alpha,\beta) : F]$.