Question 1

1.1 The speed of light is exactly 299792456 m/s, which rounded to 1% give us $c \approx 3.00 \times 10^8$. We also know $\hbar \approx 1.05 \times 10^{-34}\,\mathrm{m^2kg/s} = 6.58 \times 10^{-16}\,\mathrm{eV \cdot s}$ up to 1% error.

1.2

(1.2.1) The given mass is only in units of energy, and we want a dimension of mass. So, we recall that energy is $\frac{[M][L]^2}{[T]^2}$, where $[M], [L], [T]$ are dimensions of mass, length and time respectively. Then, we see that we only need to get rid of the length/time dimensions twice, which is just our dimensions for c, so

$$938\,\mathrm{MeV} \to \frac{938\,\mathrm{MeV}}{c^2}$$

will be the true mass.

(1.2.2) We recall that a unit of energy is the eV, so to get a length from this quantity that has units of energy, we recognize $\hbar$ has units of energy-time and we can get length from the speed of light. That is,

$$\lambda = \frac{2\pi}{E_\gamma} \to \frac{2\pi}{E_\gamma} \cdot \frac{\hbar}{c}$$

will be the true wavelength.

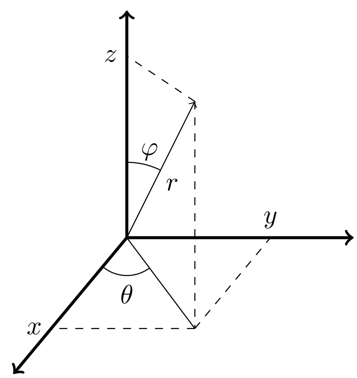
(1.2.3) We recall that the dimensions of the inverse square-root gravitational constant are $\frac{[M]^{1/2}[T]}{[L]^{3/2}}$, and we want dimensions of $[M]$. So, we can see that the corresponding factor we need to multiply by to get the units back is $\sqrt{\hbar c}$, since this will have dimensions of $\frac{[M]^{1/2}[L]^{3/2}}{[T]}$ and thus

$$m_{Pl} = \frac{1}{\sqrt{G}} \to \sqrt{\frac{\hbar c}{G}}$$

as required.

Question 2

2.1 We first draw what our spherical coordinates will look like relative to the standard euclidean $\{x, y, z\}$ coordinates:



which we will convert into the (r, φ, θ) spherical coordinates. Notice we get

$$x = r\sin(\varphi)\cos(\theta) \quad y = r\sin(\varphi)\sin(\theta) \quad z = r\cos(\varphi)$$

for our conversion. Then, we recall that the metric is $ds^2 = dx^2 + dy^2 + dz^2$, so checking each component, we get

$$\begin{aligned}
dx &= \sin(\varphi)\cos(\theta)dr + r\cos(\varphi)\cos(\theta)d\varphi - r\sin(\varphi)\sin(\theta)d\theta \\
dy &= \sin(\varphi)\sin(\theta)dr + r\cos(\varphi)\sin(\theta)d\varphi + r\sin(\varphi)\cos(\theta)d\theta \\
dz &= \cos(\varphi)dr - r\sin(\varphi)d\varphi
\end{aligned}$$

and we can compute the square to get

$$\begin{aligned}
dx^2 &= \sin^2(\varphi)\cos^2(\theta)dr^2 + r\sin(\varphi)\cos(\varphi)\cos^2(\theta)drd\varphi - r\sin^2(\varphi)\cos(\theta)\sin(\theta)drd\theta \\
&\quad + r\cos(\varphi)\sin(\varphi)\cos^2(\theta)drd\varphi + r^2\cos^2(\varphi)\cos^2(\theta)d\varphi^2 - r^2\cos(\varphi)\sin(\varphi)\cos(\theta)\sin(\theta)d\varphi d\theta \\
&\quad - r\sin^2(\varphi)\sin(\theta)\cos(\theta)drd\theta - r^2\sin(\varphi)\cos(\varphi)\sin(\theta)\cos(\theta)d\varphi d\theta + r^2\sin^2(\varphi)\sin^2(\theta)d\theta^2 \\
dy^2 &= \sin^2(\varphi)\sin^2(\theta)dr^2 + r\sin(\varphi)\cos(\varphi)\sin^2(\theta)d\varphi dr + r\sin^2(\varphi)\sin(\theta)\cos(\theta)drd\theta \\
&\quad + r\cos(\varphi)\sin(\varphi)\sin^2(\theta)drd\varphi + r^2\cos^2(\varphi)\sin^2(\theta)d\varphi^2 + r^2\cos(\varphi)\sin(\varphi)\sin(\theta)\cos(\theta)d\varphi d\theta \\
&\quad + r\sin^2(\varphi)\cos(\theta)\sin(\theta)drd\theta + r^2\sin(\varphi)\cos(\varphi)\cos(\theta)\sin(\theta)d\varphi d\theta + r^2\sin^2(\varphi)\cos^2(\theta)d\theta^2 \\
dz^2 &= \cos^2(\varphi)dr^2 - 2r\sin(\varphi)\cos(\varphi)drd\varphi + r^2\sin^2(\varphi)d\varphi^2\,.
\end{aligned}$$

So, adding these together and simplifying

$$\begin{aligned}
ds^2 &= dx^2 + dy^2 + dz^2 \\
&= dr^2 + r^2 d\varphi^2 + r^2\sin^2(\varphi)d\theta^2
\end{aligned}$$

and the matrix will be

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2(\varphi) \end{bmatrix}.$$

2.2 To see this, we recall how the vectors will change under a lorentz transform

$$A^\mu \to \bar{A}^\mu = \Lambda^\mu{}_\nu A^\nu \quad A_\mu \to \bar{A}_\mu = g_{\mu\nu}\bar{A}^\nu = g_{\mu\nu}\Lambda^\nu{}_\eta A^\eta\,.$$

Then, we see that

$$A^\mu B_\mu \to \bar{A}^\mu \bar{B}_\mu = \Lambda^\mu{}_\nu A^\nu g_{\mu\sigma}\Lambda^\sigma{}_\eta B^\eta = \underbrace{\Lambda^\mu{}_\nu g_{\mu\sigma}\Lambda^\sigma{}_\eta}_{g_{\nu\eta}} A^\nu B^\eta = A^\nu B_\nu$$

and hence we have that the contraction is indeed Lorentz invariant. Moreover, we see

$$A^\mu B_\mu = A_\nu g^{\nu\mu} B_\mu = A_\nu B^\nu$$

and so the two quantities are the same, as we would expect.

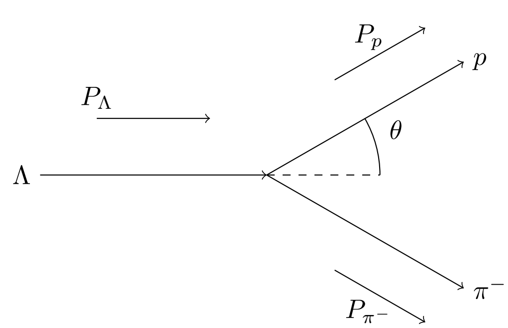
Question 3

3.1 Consider the following table, as required.

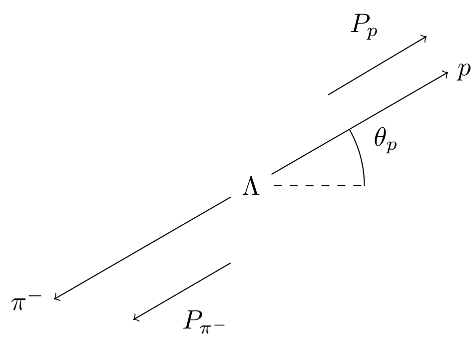
	mass MeV	**Composition**
p	938.27	uud
Λ	1115.68	uds
π^-	139.57	$d\bar{u}$

3.2

3.2.1 In the Lab Frame we will see



but in the COM frame we would see



3.2.2 In the COM frame, we know that $\vec{p}_p = \vec{p}$ and $\vec{p}_{\pi^-} = -\vec{p}$, and that $\vec{p}_\Lambda = \vec{0}$. Using this, we see that

$$\begin{aligned} s_i &= (P_\Lambda)^\mu (P_\Lambda)_\mu = E_\Lambda^2 \\ s_f &= (P_p + P_{\pi^-})^\mu (P_p + P_{\pi^-})_\mu = (E_p + E_{\pi^-})^2 - (\vec{p} - \vec{p})^2 = (E_p + E_{\pi^-})^2 \ . \end{aligned}$$

Moreover, we know that

$$E_p^2 = m_p^2 + p^2 \quad \& \quad E_{\pi^-}^2 = m_{\pi^-}^2 + p^2$$

which gives

$$\begin{aligned} E_p^2 - E_{\pi^-}^2 &= m_p^2 - m_{\pi^-}^2 \\ (E_p - E_{\pi^-})(E_p + E_{\pi^-}) &= m_p^2 - m_{\pi^-}^2 \\ (E_p - m_\Lambda + E_p) m_\lambda &= m_p^2 - m_{\pi^-}^2 \\ E_p = \frac{m_p^2 - m_{\pi^-}^2 + m_\Lambda^2}{2m_\Lambda} \end{aligned}$$

and in a similar manner we get

$$E_{\pi^-} = \frac{-m_p^2 + m_{\pi^-}^2 + m_\Lambda^2}{2m_\Lambda} \ .$$

The expressions for our momentum come from the relationship used before, that is

$$p_p = \sqrt{E_p^2 - m_p^2} \quad \& \quad p_{\pi^-} = \sqrt{E_{\pi^-}^2 - m_{\pi^-}^2} \ .$$

Plugging in some numbers, we get

$$E_p \approx 943.645\,\text{MeV} \quad E_{\pi^-} \approx 172.035\,\text{MeV} \quad p_p \approx 100.58\,\text{MeV} \approx p_{\pi^-} \ .$$

3.2.3 To compute the boost we would need, we only need the velocity that the Λ Hyperon is traveling at, as that will uniquely determine the Lorentz boost transformation. First, we get a nice relation we can use:

$$\begin{aligned} E^2 &= p^2c^2 + m^2c^4 \\ &= \gamma^2 m^2 v^2 c^2 + m^2 c^4 \\ &= m^2c^4\left(\gamma^2\beta^2 + 1\right) \\ &= m^2\left(\frac{\beta^2}{1-\beta^2} + 1\right) \qquad (\text{set } c = 1) \\ E^2 &= m^2\gamma^2 \,. \end{aligned}$$

Notice that the information for the velocity is cooked into γ, so this will make computing the boost even easier. We see

$$\gamma_\Lambda^2 = \frac{E_\Lambda^2}{m_\Lambda^2} = \frac{m_\Lambda^2 + p_\Lambda^2}{m_\Lambda^2} \approx 4.214 \quad \beta^2 = 1 - \frac{1}{\gamma^2} \approx 0.763\,.$$

Then plugging this into our Lorentz Transform,

$$\Lambda = \begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \sqrt{\frac{m_\Lambda^2+p_\Lambda^2}{m_\Lambda^2}} & -\frac{p_\Lambda}{m_\Lambda} & 0 & 0 \\ -\frac{p_\Lambda}{m_\Lambda} & \sqrt{\frac{m_\Lambda^2+p_\Lambda^2}{m_\Lambda^2}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2.053 & -1.566 & 0 & 0 \\ -1.566 & 2.053 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

where we have taken the positive root of β^2 since we suppose a boost in the positive direction.

3.2.4 We start by expressing the 4-momentum using θ_p, and what we get is

$$\begin{aligned} P_p^\mu &= (E_p, \vec{p}) = (E_p, p\cos(\theta_p), p\sin(\theta_p), 0) = (E_p, \sqrt{3}/2p, p/2, 0) \\ P_{\pi^-}^\mu &= (E_{\pi^-}, -p\cos(\theta_p), -p\sin(\theta_p), 0) = (E_{\pi^-}, -\sqrt{3}/2p, -p/2, 0) \end{aligned}$$

where I have abused my notation, but it is understood that the vectors are column vectors. Then, we see that

$$\bar{P}_p^\mu = (\Lambda^\nu{}_\mu)^{-1} P_p^\mu = \begin{bmatrix} \sqrt{\frac{m_\Lambda^2+p_\Lambda^2}{m_\Lambda^2}} & \frac{p_\Lambda}{m_\Lambda} & 0 & 0 \\ \frac{p_\Lambda}{m_\Lambda} & \sqrt{\frac{m_\Lambda^2+p_\Lambda^2}{m_\Lambda^2}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} E_p \\ \frac{\sqrt{3}}{2}p \\ p/2 \\ 0 \end{bmatrix} = \begin{bmatrix} E_p\sqrt{\frac{m_\Lambda^2+p_\Lambda^2}{m_\Lambda^2}} + \frac{\sqrt{3}}{2}\frac{p_\Lambda}{m_\Lambda}p \\ \frac{p_\Lambda}{m_\Lambda}E_p + \sqrt{\frac{m_\Lambda^2+p_\Lambda^2}{m_\Lambda^2}}\frac{\sqrt{3}}{2}p \\ p/2 \\ 0 \end{bmatrix}$$

$$\bar{P}_{\pi^-}^\mu = (\Lambda^\nu{}_\mu)^{-1} P_{\pi^-}^\mu = \begin{bmatrix} \sqrt{\frac{m_\Lambda^2+p_\Lambda^2}{m_\Lambda^2}} & \frac{p_\Lambda}{m_\Lambda} & 0 & 0 \\ \frac{p_\Lambda}{m_\Lambda} & \sqrt{\frac{m_\Lambda^2+p_\Lambda^2}{m_\Lambda^2}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} E_{\pi^-} \\ -\frac{\sqrt{3}}{2}p \\ -p/2 \\ 0 \end{bmatrix} = \begin{bmatrix} E_{\pi^-}\sqrt{\frac{m_\Lambda^2+p_\Lambda^2}{m_\Lambda^2}} - \frac{\sqrt{3}}{2}\frac{p_\Lambda}{m_\Lambda}p \\ \frac{p_\Lambda}{m_\Lambda}E_{\pi^-} - \sqrt{\frac{m_\Lambda^2+p_\Lambda^2}{m_\Lambda^2}}\frac{\sqrt{3}}{2}p \\ -p/2 \\ 0 \end{bmatrix}.$$

This isn't the most intuitive form, but all the values are known, so we can compute it explicitly, which will help us find θ. Computing $\bar{P}_p^\mu$ explicitly yields

$$\bar{P}_p^\mu \approx \begin{bmatrix} 2073.71\,\text{MeV} \\ 1656.57\,\text{MeV} \\ 50.29\,\text{MeV} \\ 0 \end{bmatrix}$$

and so $\theta \approx \arctan\left(\frac{50.29}{1656.57}\right) \approx 1.74^o$ Degrees. For π^- we find through a similar computation $\theta_{\pi^-} \approx \arctan\left(-\frac{50.29}{95.28}\right) \approx -27.83^o$ Degrees.

Question 4

4.1 To show that $\mathbf{O}(n)$ is a group under multiplication, we need only show the definition of a group is satisfied. In particular, if $M, N \in \mathbf{O}(n)$, notice

$$MN(MN)^t = MN(N^tM^t) = MNN^tM^t = MM^t = I \implies MN \in \mathbf{O}(n)\,,$$

which is closure (Notice we don't have to show $(MN)^tMN = I$ since we showed the inverse of MN is it's transpose and inverses are unique from linear algebra). Next, since $II^t = II = I$, we have an identity $I \in \mathbf{O}(n)$. Matrix multiplication is associative, and since $\mathbf{O}(n) \subset M_{n\times n}(\mathbb{R})$, we have associativity for free. Finally, we show inverses are also orthogonal. We know they exist, since

$$\det(MM^t) = det(I) \implies (\det(M))^2 = 1 \implies \det(M) = \pm 1\,.$$

But, since M is orthogonal, by definition $M^{-1} = M^t$, so

$$M^{-1}(M^{-1})^t = M^t(M^t)^t = M^tM = I \implies M^{-1} \in \mathbf{O}(n)\,.$$

So, we can conclude that $\mathbf{O}(n)$ is indeed a group.

4.2 To show that $\mathbf{SO}(n)$ is a group, we need only show that it is a subgroup, so our criterion aren't as restrictive. In particular, we get associativity for free, since $\mathbf{SO}(n) \subset \mathbf{O}(n)$, and since $\det(I) = 1$, $I \in \mathbf{SO}(n)$, and so we have the identity as well. All we need is closure and inverses. Well, notice if $M, N \in \mathbf{SO}(n)$, then

$$\det(MN) = \underbrace{\det(M)}_{1}\underbrace{\det(N)}_{1} = 1 \implies MN \in \mathbf{SO}(n)\,.$$

For inverses, we note

$$\det(M^{-1}) = \det(M^t) = \det(M) = 1 \implies M^{-1} \in \mathbf{SO}(n)\,.$$

Thus, we have shown $\mathbf{SO}(n)$ is indeed a subgroup of $\mathbf{O}(n)$.