PHYS 444 Assignment 0
Discrete Groups

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Contents

Chapter 0

Discrete Groups

0.1 Introduction

A system has a symmetry[3] when we can make certain types of changes to the system and some properties of the system remain unchanged; we say that the property is **invariant** with respect to the symmetry. Symmetry is extremely important in physics because we can exploit symmetries to simplify the equations that describe the system.

A familiar example would be using polar coordinates to describe circular motion – instead of having two Cartesian variables, x & y, along with the constraint $x^2 + y^2 = a^2$, we have a single angular variable, θ (with $x = a\cos\theta$ and $y = a\sin\theta$).

Symmetry is described mathematically using group theory.[1] A group is a set of transformations along with a binary operation that combines any pair of transformations into a transformation that must also be symmetry transformations. The group is therefore a mathematical entity independent of whatever it was that was being transformed. This abstraction enables us to study group theory without reference to anything other than the set of transformations and the binary operation.

In this chapter I will introduce discrete groups and in a subsequent chapter I will describe continuous groups. Continuous groups generally have more interesting ramifications for physics due to Nöther's theorem: that for each continuous symmetry there is a conserved quantity.

Familiar examples from classical mechanics would be

- Translational symmetry leads to conservation of linear momentum.
- Rotational symmetry leads to conservation of angular momentum.
- Time-translation symmetry leads to conservation of energy.

As we shall see, the symmetries of special relativity lead to the internal conserved quantities of quantum mechanics known as quantum numbers.

[1] First developed by French mathematician Évariste Galois (1811-1832)

0.2 Definitions

0.2.1 Group

A group[4] is a set, $A = \{a_0, a_1, a_2, ...\}$, along with a binary operator, $\circ$, (known as the **group operation**) with the following four properties:

Identity: There is an element of the set called the identity, a_0, such that

$$\exists\ a_0 \in A \mid \forall\ a_i \in A, a_0 \circ a_i = a_i \circ a_0 = a_i.$$

Doing nothing is always a symmetry operation.

Inverse: For every element of the set, a_i, there is an element called the inverse of a_i, a_i^{-1}, such that

$$\forall\ a_i \in A\ \exists\ a_i^{-1} \in A \mid a_i^{-1} a_i = a_i a_i^{-1} = a_0.$$

Performing a symmetry operation and then undoing it is always a symmetry.

Closure: The result of the binary operation between two elements of the set A is also an element of the set A,

$$\forall\ a_i, a_j \in A, a_i \circ a_j = a_k \in A$$

Performing a symmetry operation then a second symmetry operation is also a symmetry operation.

Associativity: The order in which the binary operations are combined is immaterial

$$\forall\ a_i, a_j, a_k \in A, (a_i \circ a_j) \circ a_k = a_i \circ (a_j \circ a_k).$$

The order in which the symmetry operations are combined doesn't change the resulting symmetry operation.

Example 0.1 A Small Group

Consider the set of numbers, $\{+1, -1, +i, -i\}$, where $i = \sqrt{-1}$ is the unit imaginary number, along with the binary operation of multiplication. (a) Create the **Cayley table** (aka multiplication table) for this group. (b) Identify the identity. (c) Identify the inverse of each element. (d) Discuss the other two properties of a group. (e) Is this a group?

Solution

(a) This is just like grade school!

$\cdot$	$+1$	-1	$+i$	$-i$
$+1$	$+1$	-1	$+i$	$-i$
-1	-1	$+1$	$-i$	$+i$
$+i$	$+i$	$-i$	-1	$+1$
$-i$	-1	$+i$	$+1$	-1

$\cdot$	$+1$	-1	$+i$	$-i$	$+j$	$-j$	$+k$	$-k$
$+1$								
-1								
$+i$								
$-i$								
$+j$								
$-j$								
$+k$								
$-k$								

Table 1: The **Cayley table** to be completed for Example 0.2(a).

(b) The identity is $+1$ because $(+1) \cdot (a_i) = a_i \; \forall \; a_i \in A$.

(c) Reading from the **Cayley table** I have:

$$\begin{aligned}
(+1) \cdot (+1) = +1 \quad &\Rightarrow \quad (+1)^{-1} = +1 \\
(-1) \cdot (-1) = +1 \quad &\Rightarrow \quad (-1)^{-1} = -1 \\
(+i) \cdot (-i) = +1 \quad &\Rightarrow \quad (+i)^{-1} = -i \\
(-i) \cdot (+i) = +1 \quad &\Rightarrow \quad (-i)^{-1} = +i
\end{aligned}$$

(d) Because ordinary multiplication is associative the associativity property is met. And, again from the **Cayley table**, the closure property is also met.

(e) Because all of the properties in the definition of a group are met this is a group.

Example 0.2 Quaternions

Quaternions were discovered by Hamiliton and consist of the set $\{\pm 1, \pm i, \pm j, \pm k\}$ with ordinary multiplication and $ii = jj = kk = ijk = -1$. (a) Complete the **Cayley table**, Table 1, for this group. (b) Identify the identity. (c) Identify the inverse of each element. (d) Discuss the other two properties of a group. (e) Is this a group? (f) Are there any repeated elements in the rows or columns of the **Cayley table**? (g) Compare the top, left corner of the multiplication to that in Example 0.1.

Solution

(a) This is less like grade school. I have $ijk = -1$ so $(-i)(ijk) = (-ii)jk = jk = (-i)(-1)i$ $\Rightarrow jk = +i$, and similarly, $ij = +k$ and $ki = +j$.

$\cdot$	$+1$	-1	$+i$	$-i$	$+j$	$-j$	$+k$	$-k$
$+1$	$+1$	-1	$+i$	$-i$	$+j$	$-j$	$+k$	$-k$
-1	-1	$+1$	$-i$	$+i$	$-j$	$+j$	$-k$	$+k$
$+i$	$+i$	$-i$	-1	$+1$	$+k$	$-k$	$-j$	$+j$
$-i$	$-i$	$+i$	$+1$	-1	$-k$	$+k$	$+j$	$-j$
$+j$	$+j$	$-j$	$-k$	$+k$	-1	$+1$	$+i$	$-i$
$-j$	$-j$	$+j$	$+k$	$-k$	$+1$	-1	$-i$	$+i$
$+k$	$+k$	$-k$	$+j$	$-j$	$-i$	$+i$	-1	$+1$
$-k$	$-k$	$+k$	$-j$	$+j$	$+i$	$-i$	$+1$	-1

(b) The identity is $+1$ because $(+1) \cdot (a_i) = a_i \forall a_i$.

(c) Reading from the **Cayley table** I have:

$$\begin{aligned}
(\pm 1) \cdot (\pm 1) = +1 &\quad \Rightarrow \quad (\pm 1)^{-1} = \pm 1 \\
(\pm i) \cdot (\mp i) = +1 &\quad \Rightarrow \quad (\pm i)^{-1} = \mp i \\
(\pm j) \cdot (\mp j) = +1 &\quad \Rightarrow \quad (\pm j)^{-1} = \mp j \\
(\pm k) \cdot (\mp k) = +1 &\quad \Rightarrow \quad (\pm k)^{-1} = \mp k
\end{aligned}$$

(d) Because ordinary multiplication is associative the associativity property is met. And, again from the **Cayley table**, the closure property is also met.

(e) Because all of the properties in the definition of a group are met this is a group.

(f) Each element appears exactly once in each row or column of the **Cayley table**, just like a sudoku!

(g) The top left quadrant of the **Cayley table** is the same as that in Example 0.1.

0.2.2 Abelian Groups

The small group in Example 0.1 uses ordinary multiplication as its group operation; consequently the group operation is commutative, *i.e.*

$$a_i \circ a_j = a_j \circ a_i,$$

for all a_i, a_j in the group. Such groups are termed **Abelian**.[2]

For the quaternion group in Example 0.2 we have, for example, $ij = +k$, and $ji = -k$, so the group operation for this group is non-commutative. If the binary operation is not commutative then the group is **non-Abelian**.

Example 0.3 The Cyclic Group

[2]Named for the Norwegian mathematician Niels Henrik Abel (1802-1829).

$\circ$	a_0	a_1	a_2
a_0			
a_1			
a_2			

Table 2: The **Cayley table** to be completed for Example 0.3(a).

An important set of Abelian groups are the cyclic groups, C_n, which have as group operations the cyclic permutations of a set of n items. The rotational symmetries of a regular n-gon (*i.e.* an n sided regular polygon) is described by the cyclic group C_n. (a) Create the **Cayley table** for C_3 (*i.e.* the rotational symmetries of an equilateral triangle). Take a_1 and a_2 to correspond to rotations of 120° and 240° degrees, respectively. (b) Compare your **Cayley table** to that for integer addition modulus 3, Z/3Z.

Solution

(a) The **Cayley table** for C_3 is

$\circ$	a_0	a_1	a_2
a_0	a_0	a_1	a_2
a_1	a_1	a_2	a_0
a_2	a_2	a_0	a_1

(b) The **Cayley table** for Z/3Z is

$+$	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

The two groups are **isomorphic**.

0.2.3 Subgroups

A subgroup is a subset of the group elements that is also a group. We saw in Example 0.2(g) that the quaternion group contains the group from Example 0.1 as a subgroup. Every group has the identity, $\{a_0\}$, as a subgroup; this group is called the trivial group.

Example 0.4 A Tiny Group

Consider the tiny group $\{+1, -1\}$ with multiplication as the binary operation. The **Cayley table** is

$\circ$	$+1$	-1
$+1$	$+1$	-1
-1	-1	$+1$

Is it possible to have another group with two elements that has a different **Cayley table**?

0.2.4 Isomorphisms

Consider the group Z/2Z, consisting of the set of integers, $\mathbb{Z}$, modulo 2 with addition modulus two as the group operation (*i.e.* the result of any addition is "modulus 2" meaning that it is divided by two and the remainder is kept). The **Cayley table** is

$+$	0	1
0	0	1
1	1	0

and we can see that if the identity is $a_0 = 0$, then I can write the **Cayley table** as

$\circ$	a_0	a_1
a_0	a_0	a_1
a_1	a_1	a_0

where $a_0 = 0$ and $a_1 = 1$.

Now consider the **Cayley table** in Example 0.4. If I write $a_0 = +1$ and $a_1 = -1$ then the **Cayley table** is

$\circ$	a_0	a_1
a_0	a_0	a_1
a_1	a_1	a_0

and it has the exact same structure as Z_2. These two groups are said to be **isomorphic** – the group structure of the two groups are identical. In fact, any group with two elements will be isomorphic to these two groups.

0.3 The Symmetric Group, S_n

The symmetric group, S_n, consists of the $n!$ possible permutations of n objects which we label $1, 2, 3, ..., n$. For example, $\mathrm{S}_2 = (12), (21)$, has $2! = 2$ elements and is again and it has the exact same structure as Z_2. These two groups are said to be **isomorphic** to the tiny groups in Example 0.4 and subsection 0.2.4. Interestingly, the group operation is composition, one permutation follows the other; this is best seen through an example.

The symmetric group, S_n, consists of all possible permutations of n objects. A permutation group is a subgroup of the symmetric group.

Example 0.5 The Symmetric Group of three objects, S_3

Consider the $3! = 6$ possible permutations of three objects,

$$123, 231, 312, 132, 321, 213.$$

It is convenient to use Cauchy's two-line notation:

$$\sigma_0 = \binom{123}{123} \qquad \sigma_1 = \binom{123}{231} \qquad \sigma_2 = \binom{123}{312}$$
$$\sigma_3 = \binom{123}{132} \qquad \sigma_4 = \binom{123}{321} \qquad \sigma_5 = \binom{123}{213},$$

so that, for example, $\sigma_2 = \binom{123}{312}$ means that object 1 is replaced with object 3, 2 with 1, and 3 with 2. The advantage of this notation is that any pair of the columns can be exchanged without changing the meaning of the group element (recall that the elements of a set have no fixed order). For example,

$$\sigma_2 = \binom{123}{312} = \binom{231}{123} = \binom{312}{231}.$$

(a) Complete the **Cayley table**, Table 1, for this group. (b) Identify the identity. (c) Identify the inverse of each element. (d) Discuss the other two properties of a group. (e) Is this a group? (f) Is this group Abelian? (g) Are there any subgroups?

Solution

(a) I have

$$\sigma_1 \circ \sigma_2 = \binom{123}{231} \circ \binom{123}{312} = \binom{\cancel{312}}{123} \circ \binom{123}{\cancel{312}} = \binom{123}{123} = \sigma_0$$

so $\sigma_1 = \sigma_2^{-1}$. Also,

$$\sigma_2 \circ \sigma_1 = \binom{123}{312} \circ \binom{123}{231} = \binom{\cancel{231}}{123} \circ \binom{123}{\cancel{231}} = \binom{123}{123} = \sigma_0$$

so $\sigma_2 = \sigma_1^{-1}$, and $\sigma_2 \circ \sigma_1 = \sigma_1 \circ \sigma_2$.

Next,

$$\sigma_1 \circ \sigma_3 = \binom{123}{231} \circ \binom{123}{132} = \binom{\cancel{132}}{213} \circ \binom{123}{\cancel{132}} = \binom{123}{213} = \sigma_5,$$

but

$$\sigma_3 \circ \sigma_1 = \binom{123}{132} \circ \binom{123}{231} = \binom{\cancel{231}}{321} \circ \binom{123}{\cancel{231}} = \binom{123}{321} = \sigma_4,$$

so $\sigma_1 \circ \sigma_3 \neq \sigma_3 \circ \sigma_1$ and the group is **non-Abelian**.

Continuing in this manner the **Cayley table** is

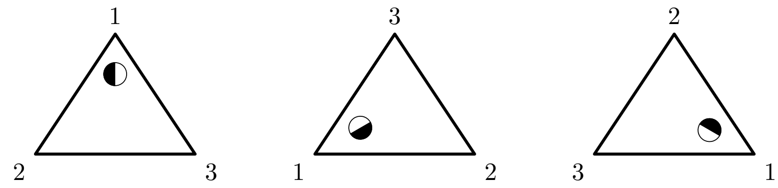

Figure 1: An equilateral triangle confined to a plane (*i.e.* no flipping) has symmetries described by the group C_3. The group element to transform from the first to the second is $\binom{123}{312}$, from the first to the third is $\binom{123}{231}$, and from the second to the third is $\binom{312}{231} = \binom{123}{312}$, the same as from the first to the second.

$\circ$	σ_0	σ_1	σ_2	σ_3	σ_4	σ_5
σ_0	σ_0	σ_1	σ_2	σ_3	σ_4	σ_5
σ_1	σ_1	σ_2	σ_0	σ_4	σ_5	σ_3
σ_2	σ_2	σ_0	σ_1	σ_5	σ_3	σ_4
σ_3	σ_3	σ_5	σ_4	σ_0	σ_2	σ_1
σ_4	σ_4	σ_3	σ_5	σ_1	σ_0	σ_2
σ_5	σ_5	σ_4	σ_3	σ_2	σ_1	σ_0

(b) Clearly, σ_0 is the identity.

(c) I have σ_1 and σ_2 as each other's inverses and σ_3, σ_4, and σ_5 are their own inverses.

(d) Since the group consists of all the possible permutations it must be closed. The associativity property is trickier to check...

0.3.1 The Cyclic Group, C_n

The cyclic group, C_n, is the set of cyclic permutations of n objects. For example, if $n = 3$ and I label the objects $\{1, 2, 3\}$ then the cyclic permutations are

$$123, 231, 312$$

and using Cayley's two line notation I have the group elements

$$\{\binom{123}{123}, \binom{123}{231}, \binom{123}{312}\}$$

with composition as the **group operation**. This is clearly a subgroup of the symmetric group, S_3, which we saw in Example 0.5, containing elements $\{\sigma_0, \sigma_1, \sigma_2\}$ – that group has every possible permutation of 3 items while this group only has the cyclic permutations, $1 \to 2 \to 3 \to 1$.

The cyclic group, C_n, describe the symmetries of a regular n-sided polygon confined to a plane (*i.e.* rotations but no flips). It is symmetric for rotations of multiples of $2\pi/n$ about its centre. For example, an equilateral triangle confined to a plane has C_3 symmetry group (see Figure 1).

Example 0.6 The nth root of One

Consider the solutions to the equation $z^n - 1 = 0$. This is an nth order polynomial so it will have n roots. (a) Find the roots for $n = 2$. (b) Find the roots for $n = 3$. [**Hint:** Use Euler's formula to write $e^{2\pi i} = 1$.] (c) Using the set of roots from (b) and regular multiplication as the binary operation show that this is a group. (d) Show that this group is **isomorphic** to C_3 by creating the **Cayley table**. (e) Argue that this group is isomorphic to C_n by sketching the roots in the complex plane.

0.4 Representation

A group, G, is a mathematical entity defined by its set of elements, $\{a_i\}$, and the results of its binary operator, $\circ$, its Cayley table. A **representations** of a group $A = \{a_0, a_1, \cdots\}$ is a mapping, f, onto a set of linear operators which act upon a vector space with the properties

- $f(a_0) = I$, the identity of the vector space, and
- $f(a_i)f(a_j) = f(a_i \circ a_j)$,

This mapping may be an isomorphism but it need not be; for example a trivial mapping would be $f(a_i) = I$. Later we will see that two-to-one mappings occur in particle physics. We have examined several mathematical systems are representations of groups; we need to keep in mind that the group itself is an abstract mathematical concept and is not actually any specific representation.

0.4.1 Regular Representation

For a finite group of order n we can take the group elements as the basis for a n dimensional vector space,

$$|\sigma_0\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \end{bmatrix}, |\sigma_1\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ \vdots \end{bmatrix}, |\sigma_2\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ \vdots \end{bmatrix}, \cdots$$

and then use the Cayley table to construct $n \times n$ matrices, M_i, with elements either zero or one, such that the set of matrices form a representation of the group, with matrix multiplication as the binary operation – *i.e.* create each M_i so that when it is multiplied by each basis vector it gives the correct result,

$$M_i = \left[|\sigma_i \circ \sigma_0\rangle, |\sigma_i \circ \sigma_1\rangle, |\sigma_i \circ \sigma_2\rangle, \cdots \right]$$

This is easier to see with an example.

Example 0.7 Regular Representation of C_3

(a) Construct the Cayley table for C_3. (b) Construct the 3×3 matrices $f(\sigma_i)$ such that they obey the same multiplication table. (c) Show that $f(\sigma_1)f(\sigma_2) = f(\sigma_1 \circ \sigma_2) = I$.

Solution

(a) The Cayley table is the first quadrant of that of S_3 given in Example 0.5,

$\circ$	σ_0	σ_1	σ_2
σ_0	σ_0	σ_1	σ_2
σ_1	σ_1	σ_2	σ_0
σ_2	σ_2	σ_0	σ_1

(b) I have $\sigma_0 \circ \sigma_i = \sigma_i$ so σ_0 is, of course, the identity,

$$f(\sigma_0) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

which maps each basis vector onto itself.

Taking $|\sigma_1\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, along with the Cayley table from (a) I see that

$$M_1 |\sigma_0\rangle = |\sigma_1\rangle \quad \Rightarrow M_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

so the first column of M_1 is $|\sigma_1\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. Similarly,

$$M_1 |\sigma_1\rangle = |\sigma_2\rangle \quad \Rightarrow M_1 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

and the second column of M_1 is $|\sigma_2\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. Finally,

$$M_1 |\sigma_2\rangle = |\sigma_0\rangle \quad \Rightarrow M_1 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

and the third and final column of M_1 is $|\sigma_0\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. The representation of σ_1 is therefore

$$f(\sigma_1) = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \big[|\sigma_1\rangle, |\sigma_2\rangle, |\sigma_0\rangle\big].$$

I can see the pattern,

$$f(\sigma_1) = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \big[|\sigma_1\rangle, |\sigma_2\rangle, |\sigma_0\rangle\big] = \big[M_1 |\sigma 0\rangle, M_1 |\sigma 1\rangle, M_1 |\sigma 2\rangle\big] = \big[|\sigma_1 \circ \sigma_0\rangle, |\sigma_1 \circ \sigma_1\rangle, |\sigma_1 \circ \sigma_2\rangle\big].$$

From the Cayley table I have

$$f(\sigma_2) = \big[|\sigma_2 \circ \sigma_0\rangle, |\sigma_2 \circ \sigma_1\rangle, |\sigma_2 \circ \sigma_2\rangle\big].\big[|\sigma_2\rangle, |\sigma_0\rangle, |\sigma_1\rangle\big] = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

(c)

$$f(\sigma_1)f(\sigma_2) = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = f(\sigma_0) = f(\sigma_1 \circ \sigma_2),$$

as required.

Bibliography

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Index