Assignment 2

Due Friday Mar 22nd before the end of the day

1 Non-coordinated efforts (Marked by Erickson)

Recall that a manifold $\mathcal{M}$ is a topological space (a set of points where open sets are defined) together with an atlas, i.e. a collection of charts $\{(U_\alpha, \psi_\alpha)\}$ which provides us with local coordinate systems. Consider a familiar differential manifold, namely a 2-sphere $\mathcal{M} = S^2$, which is a good approximation of the surface of the Earth[1]. The surface of the Earth is given in Cartesian coordinates as a submanifold of $\mathbb{R}^3$ (an embedding) given by the equation

$$x^2 + y^2 + z^2 = 1\,. \tag{1.1}$$

There is another coordinate system which is very convenient for S^2: *spherical coordinates*, given by (for fixed radius r)

$$x = r\sin\theta\cos\phi\,, \quad y = r\sin\theta\sin\phi\,, \quad z = r\cos\theta\,. \tag{1.2}$$

A chart (U, ψ) gives a local coordinate system: if $p \in U \subseteq \mathcal{M}$ where U is an open subset, then $\psi(p) = (x^1, ..., x^n)_p$ gives local coordinates to point p. For example, in Cartesian coordinates, $(x, y, z) = (1, 0, 0)$ is a point on S^2 and in spherical coordinates, this same point would be denoted $(\theta, \phi) = (\pi/2, 0)$. The angle θ is called *polar angle* and ϕ is the *azimuthal angle*. We will study spherical coordinates in detail.

(a) Consider a chart (U, ψ) which gives us the spherical coordinates above, i.e. for $p \in U \subseteq S^2$, we have $\psi(p) = (\theta, \phi)_p$. **Show that** this chart (i.e. spherical coordinate system) does *not* cover the whole sphere, and **provide** the points (in Cartesian coordinates) not covered by the chart.

Hint: find the points in S^2 (in terms of Cartesian coordinates) which makes ψ (or the transition function between Cartesian and spherical coordinates) to not be a diffeomorphism.

[1] What we call "world map" is essentially an atlas (used to be a book, now it's called Google map) whose charts can be, for example, made of maps of countries (with grids) and oceans around the globe.

(b) Vectors (vector fields) are important geometric objects: they can specify, for example, wind velocity on Earth surface. If $p \in S^2$, then $\mathsf{V}_p \in T_p S^2$ is a vector at p which lives on tangent space[2] at p. Since $\dim S^2 = 2$, we have $\dim T_p S^2 = 2$ and hence $\mathsf{V} = V^a \Upsilon_a$, where $a = \theta, \phi$. The spherical coordinates (θ, ϕ) provide us with a set of "natural" basis vectors $\{\Upsilon_\mu\}$, where $\Upsilon_1 = \partial_\theta, \Upsilon_2 = \partial_\phi$.

Show that $\{\Upsilon_\mu\}$ forms a coordinate basis[3].

Hint: consider an arbitrary smooth function $f(\theta, \phi)$ which the vectors can act on.

(c) By first finding the metric tensor of S^2 in spherical coordinates, show that the spherical coordinate basis vectors $\{\Upsilon_\mu\}$ are *orthogonal but not orthonormal*: that is, you cannot rescale them simultaneously by a common fixed constant C to make them both unit length.

Note: You should use the tools learnt in Block 2 instead of vector calculus to do this. Also, you should perform the coordinate transformation explicitly and not quoting the metric online or otherwise.

Hint: you can find the metric by first finding the metric for $\mathbb{R}^3$ in spherical coordinates, using the fact that the line element of $\mathbb{R}^3$ in Cartesian coordinates is

$$\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 \,. \tag{1.3}$$

(d) Construct the orthonormal basis $\{\Upsilon'_a\}$ out of the coordinate basis of the spherical coordinates and show that $\{\Upsilon'_a\}$ *is a non-coordinate basis*[4].

(e) A function $\psi(x^\nu)$ satisfies the *wave equation* on a curved manifold if

$$\nabla_\mu \nabla^\mu \psi = 0 \,. \tag{1.4}$$

Find the explicit form of the wave equation on S^2 in spherical coordinates, where the line element (including time component) is

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}s^2\big|_{S^2} \,, \tag{1.5}$$

where $\mathrm{d}s^2\big|_{S^2}$ is the line element on S^2 which can be found from part (c).

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[2] An important thing is that vectors live on tangent space: a vector X pointing from origin to the surface of a sphere is *not* a (tangent) vector associated *intrinsically* to the sphere, but associated *extrinsically* as an embedding in $\mathbb{R}^3$. An ant living on S^2 can never "see" this vector X.

[3] The name "coordinate basis" comes from the fact that it is naturally given by the coordinate system, in the form of "partial derivatives".

[4] When you have used the usual orthonormal polar basis vectors (that is, the calculus textbook spherical coordinates) you have been using non-coordinate bases all along and you didn't even know it.

2 Wrapping donuts (Marked by Pipo)

Let us consider the torus $S_1 \times S_1$ (a fancy name for donuts). This set can be understood as a differential manifold with charts that go from the torus to the subset of $\mathbb{R}^2$, $\{\theta^1, \theta^2\}$ $\theta_i \in (0, 2\pi)$. Let us pick the chart $\phi(p)$ that maps points p of the manifold to the coordinates (θ^1, θ^2) corresponding to the angular variable of each circumference for each point. Now consider the following vector field in this manifold:

$$\mathsf{v} = \partial_1 + \lambda \partial_2,$$

where $\{\partial_i\}_{i=1,2}$ are the tangent vectors to each circumference (corresponding to partial derivatives with respect to θ_1,θ_2 respectively, and $\lambda \in \mathbb{R} - \{0\}$.

We define the integral curves $\gamma(s)$ generated by an arbitrary vector field w, whose parametrization in coordinates θ_1, θ_2 in $\mathbb{R}^2$ is $\phi[\gamma(s)] = \boldsymbol{x}(s) \in \mathbb{R}^2$ through the set of differential equations

$$\frac{\mathrm{d}x^\mu}{\mathrm{d}s} = w^\mu(s).$$

(a) Show that the curves generated by $\mathsf{v} = \partial_1 + \lambda \partial_2$ have the form

$$\phi[\gamma(s)] = \boldsymbol{x}(s) = \big(\theta^1(s), \theta^2(s)\big) = \big([\theta^1(0) + s], [\theta^2(0) + \lambda s]\big),$$

where the square brackets in the right hand side of the equality mean that we take the value of these angles modulo 2π: $[\theta] = \theta \mod 2\pi$.

(b) Determine the values of λ for which the curves

$$\gamma : \{\gamma(s), \forall s \in \mathbb{R}\}$$

densely[5] cover the torus.

(c) Given a continuous function f defined in the torus, show that $\mathsf{v}(f) = 0$ implies that f is constant[6] if λ is irrational. Hint: You can use without proof that given a topological space, a continuous function that is constant in a **dense** subset is also constant in the whole topological space.

(d) Given the same function f, show $\mathsf{v}(f) = 0$ implies that $\bar{f}(\theta_1, \theta_2) = h(\lambda\theta_1 - \theta_2)$ if λ is rational[7]. Here h is an arbitrary smooth function from $\mathbb{R}$ to $\mathbb{R}$.

[5] You don't need to prove that the cover of the torus is dense for some values of λ, but at least you need to argue when the curves will definitely NOT cover the whole torus and why. Hint: think about periodicity.

[6] a constant function is a map from all points in its domain in the manifold to the same real number.

[7] Remember that $\bar{f} = f \circ \phi^{-1}$.

3 ⏭ and ⏮ in vector fields (Marked by Pipo)

Let $\varphi : \mathcal{M} \to \mathcal{M}'$ be a smooth map between two differential manifolds. It naturally induces a smooth map between functions defined over the manifolds $\varphi^* : \mathcal{F}_{\mathcal{M}'} \to \mathcal{F}_{\mathcal{M}}$, namely the pull-back of φ, defined as

$$\varphi^* f(p) = f(\varphi(p)), \tag{3.1}$$

or more briefly $\varphi^* f = f \circ \varphi$.

Further, it also induces a map from elements of the tangent space at a point in $\mathcal{M}$ to the tangent space at a point in $\mathcal{M}'$, i.e. $\varphi_* : T_p\mathcal{M} \to T_{\varphi(p)}\mathcal{M}'$, called the push-forward of φ. It is defined by means of its action over functions in $\mathcal{F}_{\mathcal{M}'}$ as

$$\varphi_* \mathsf{v}|_{\varphi(p)} f = \mathsf{v}|_p \varphi^* f. \tag{3.2}$$

How do these relations apply to vector fields? Recall that a vector field is a map that assigns a single vector to each point in the manifold. Namely, a vector field is a section of the so-called tangent bundle of $\mathcal{M}$, denoted by $T\mathcal{M} = \bigcup_{p\in\mathcal{M}} T_p\mathcal{M}$.

Since a vector at a point maps functions defined over $\mathcal{M}$ to real values, a vector field maps a function defined over $\mathcal{M}$ to another function defined by

$$\mathsf{v}[f](p) = \mathsf{v}|_p f. \tag{3.3}$$

Therefore, given a smooth map φ, one could be tempted to write the push-forward of a vector field as

$$\varphi_* \mathsf{v}[f](\varphi(p)) = \varphi_* \mathsf{v}|_{\varphi(p)} f = \mathsf{v}|_p \varphi^* f. \tag{3.4}$$

However, as we will see, the situation in this case is more involved.

Finally, a diffeomorphism is a smooth map φ such that its inverse is also smooth. If the map is a diffeomorphism, both push-forward and pull-back can be defined in all cases. Indeed, we can write

$$\varphi_* f(p) = (\varphi^{-1})^* f(p) \tag{3.5}$$

and

$$\varphi^* \mathsf{v}|_p f = \mathsf{v}|_{\varphi^{-1}(p)} \varphi_* f. \tag{3.6}$$

(a) Let $\gamma : \mathbb{R} \to \mathbb{R}^2$, defined by the equation $\gamma(t) = (\cos(t), \sin(t))$. If $f(x, y) = 2xy$, calculate $\gamma^* f(t)$. Since each manifold can be described with a single chart, you may forget about the transition functions.

(b) Let γ be defined as in part (a). Calculate $\gamma_* \partial_t|_{\gamma(t_0)}$ in the basis ∂_x, ∂_y at an arbitrary point $x, y \in \gamma(\mathbb{R})$.

(c) Show that if φ is not injective, then the push-forward as defined in (3.4) does not define a vector field in general.

(d) Two vector fields v and v' are said to be related by a map φ if

$$\mathsf{v}'|_{\varphi(p)} f = \mathsf{v}|_p \varphi^* f. \tag{3.7}$$

for all $p \in \mathcal{M}$ and for all $f \in \mathcal{F}_{\mathcal{M}'}$. Show that then $\varphi_* \mathsf{v} = \mathsf{v}'$ in the sense of (3.4) and that

$$\mathsf{v} \circ \varphi^* = \varphi^* \circ \mathsf{v}', \tag{3.8}$$

where $\circ$ denotes composition of maps.

(e) Show that if φ is a diffeomorphism, then the push-forward of vector fields in the sense of (3.4) is always well defined and that

$$\varphi_* \mathsf{v} = \varphi_* \circ \mathsf{v} \circ \varphi^*. \tag{3.9}$$

4 Killing me softly (Marked by Richard)

Find all the Killing vectors[8] of the following two-dimensional metric:

$$\mathrm{d}s^2 = -\mathrm{d}u^2 + \cosh^2(u) \mathrm{d}\phi^2.$$

Hint:

- The Killing equations that you will get is a system of coupled linear differential equations. However they are easy to solve with the methods that you already know, but for part of the answer there is also much easier methods if you are a bit clever about it. For example, are there any cyclic coordinates? That will give you one Killing already.
- In case you have an equation of the form $\partial_a v_b + \partial_b v_a = f(b) v_a$, it might be of help to differentiate with respect to b.
- $\partial_b \frac{1}{\cosh^2 b} = -2 \frac{\tanh(b)}{\cosh^2 b}$ and $\int \mathrm{d}b \frac{1}{\cosh^2 b} = \tanh(b) + const.$

[8] Kind of obvious hint: A Killing vector satisfies the Killing equation

5 The Weyl tensor (Marked by Pipo)

In this exercise we are getting our feet into the muddy waters of the Weyl tensor, how it is constructed and what properties it has.

(a) In a spacetime of dimension n ($n-1$ spatial dimensions and 1 time dimension), you are given the following tensor:

$$C_{abcd} = R_{abcd} + A(g_{a[d}R_{c]b} + g_{b[c}R_{d]a}) + BRg_{a[c}g_{d]b},$$

where R_{abcd} is the Riemann tensor, R_{ab} is the Ricci tensor, R is the curvature scalar, g_{ab} is the metric tensor and finally A and B are two arbitrary constants. Show that C_{abcd} has the same symmetries as the Riemann tensor, that is:

- $C_{abcd} = -C_{bacd}$.
- $C_{abcd} = -C_{abdc}$.
- $C_{[abc]d} = 0$.

Hint: For the last symmetry $C_{[abc]d} = 0$, you can assume that given two symmetric tensors, K_{da} and L_{bc}, they fulfill

$$K_{d[a}L_{bc]} = 0.$$

Also, you may find useful that the second term can be rearranged in the following way:

$$A(g_{a[d}R_{c]b} + g_{b[c}R_{d]a}) = A(g_{d[a}R_{b]c} - R_{d[a}g_{b]c}).$$

(b) Calculate the two constants A and B such that this tensor is traceless, that is ${C_{abc}}^{b} = 0$. Do you obtain for C_{abcd} the components of the Weyl tensor as you know from the lecture notes?

(c) We saw in class that the Riemann tensor has $(n^2(n^2-1))/12$ independent components because of its symmetries. In addition, the Weyl tensor has the following constraints:

$${C_{abc}}^{b} = 0.$$

These are $[n(n+1)]/2$ constraints. How many non-zero components does the Weyl tensor have? (Prove it with the information given. It is not enough to Google it and state it here.)

BONUS non-question: You don't need to do anything here, but if you want, for no marks, use the answer of c) to show explicitly that in 3 dimensions the Riemann is fully determined by the Ricci tensor.