Question 1

Let $X = \{(x, y) \in \mathbb{R}^2 | y = \pm 1\}$, and we define the equivalence relation $\sim$ where $(x, 1) \sim (x, -1)$ for $x \neq 0$. Then $Y = X/\sim$ is the set of equivalence classes of x values and two zeros, $(0, -1)$ and $(0, +1)$.

We first see that X is a topological space by the relative topology. To see that X is also Hausdorff, consider $(x_1, y_1), (x_2, y_2) \in X$ distinct. Then, suppose $U_1, U_2 \subset X$ as open sets. By the relative topology of $\mathbb{R}^2$, we can always choose open balls in $\mathbb{R}^2$ small enough so that the intersection of X with that open set is not disjoint. In particular, this means that the open set will be an open interval on either the line $y = 1$ or $y = -1$.

Then, if $y_1 \neq y_2$, we have that $U_1 \cap U_2 = \emptyset$ by construction. Suppose that $y_1 = y_2$, then we need to consider only the distinct points x_1 and x_2 on $\mathbb{R}$, but since the relative topology here will just be the standard topology of $\mathbb{R}$, we get Hausdorff for free. So X is Hausdorff for sure.

Now consider Y. In particular, to see that Y is not Hausdorff, we choose the distinct points $(0, -1)$ and $(0, +1)$. We note that open sets are defined in Y by the quotient topology, and we use open sets in X under the equivalence relation to get our topology, $\bar{\tau}$.

In particular, suppose again we can pick open balls small enough in $\mathbb{R}^2$ around the two points so that the intersection of the open ball with X is restricted to a single line. Call these two open sets $U_1 = B_r((0, 1)) \cap X$ and $U_2 = B_r((0, -1)) \cap X$, where $r < 2$. Suppose that $p_1 \in U_1$, then $\exists x \in \mathbb{R} \setminus \{0\}$ such that $p_1 = (x, 1) \sim (x, -1) \in U_2$. So, in Y, the open sets will never be disjoint around $(0, 1)$ and $(0, -1)$.

Thus Y is not Hausdorff.

Question 2

Question 3

(a) Suppose $x \in \mathbb{S}^n \setminus \{N\}$, then we can parameterize the line connecting x and N by the following equation,

$$y = t(x - N) + N \quad t \in \mathbb{R}$$

In particular, we can find the t for which this line intersects the subspace defined by $x_{n+1} = 0 \subset \mathbb{R}^{n+1}$,

$$(y_1, \ldots, y_n, 0) = tx + N(1-t) = (tx_1, \ldots, tx_n, tx_{n+1} + 1 - t)$$

$$\implies t(x_{n+1} - 1) + 1 = 0 \iff t = \frac{1}{1 - x_{n+1}}$$

Hence, we see that the intersection of the line connecting N and x with the subspace setting $x_{n+!} = 0$ is just,

$$\frac{(x_1, \ldots, x_n, 0)}{1 - x_{n+1}} = (u, 0) = (\sigma(x), 0)$$

as expected. We do the same for S, and we see that

$$y = t(x - S) + S$$

and again, for the subspace defined by $x_{n+1} = 0$, we get,

$$(y_1, \ldots, y_n, 0) = (tx_1, \ldots, tx_n, tx_{n+1} - 1 + t)$$

$$\implies tx_{n+1} - 1 + t = 0 \iff t = \frac{1}{1 + x_{n+1}}$$

Hence, the intersection point will be,

$$\frac{(x_1, \ldots, x_n, 0)}{1 + x_{n+1}} = -\frac{(-x_1, \ldots, -x_n, 0)}{1 - (-x_{n+1})} = (-\sigma(-x), 0) = (\tilde{\sigma}(x), 0)$$

again as we would expect.

(b) First we show injection. To see this, suppose that $x, y \in \mathbb{S}^n \setminus \{N\}$. Then, as we showed in **(a)**, we can relate the image under σ of these points to the intersection of the line connecting the point and N with the subspace that sets the $n^{th} + 1$ component to 0. That is to say, if $\sigma(x) = \sigma(y)$, then $(\sigma(x), 0) = (\sigma(y), 0)$, which is to say that the two lines would intersect at that point in the plane defined by setting the $n^{th} + 1$ component to 0. Yet, these two lines necessarily intersect at N aswell, so they must be the same line.

Then, we have that this line intersects $\mathbb{S}^n$ at N, by necessity, and both x and y. But this is impossible, since a straight line in R^{n+1} will only intersect a sphere at most twice. Further, by supposition, $x \neq N$ and $y \neq N$, and hence it must be that $x = y$. Thus we have that σ is injective.

Now we check surjection. The image lies in $\mathbb{R}^n$, so suppose $u \in \mathbb{R}^n$. In particular, we can again use the fact that σ is simply the intersection of the line connecting N and a point on $\mathbb{S}^n$ with the plane defined by setting the last component to 0. In particular, we then associate u with $(u, 0)$. Then, the line connecting this point and N is defined by,

$$y = t((u,0) - N) + N = (tu, 0) + N(1-t) = (tu, 1-t)$$

for $t \in \mathbb{R}$. To find the point $y \in \mathbb{S}^n \setminus \{N\}$ with which this line intersects, we use the definition of $\mathbb{S}^n$

$$1 = \sum_{i=1}^{n+1} y_i^2 = \sum_{i=1}^{n} (tu_i) + (1-t)^2$$
$$1 - (1 - 2t + t^2) = t^2|u|^2$$
$$2 - t = t|u|^2 \iff t = \frac{2}{|u|^2 + 1}$$

Thus, we have that

$$y = \left(\frac{2u}{|u|^2+1}, 1 - \frac{2}{|u|^2+1}\right) = \frac{(2u_i, \ldots, 2u_n, |u|^2 - 1)}{|u|^2 + 1}$$

Hence we have that every $u \in \mathbb{R}^n$ has a preimage on $\mathbb{S}^n \setminus \{N\}$, and thus σ is surjective.

We can now conclude that σ is a bijection, with the inverse stated above.

(c) We note that $\tilde{\sigma} \circ \sigma^{-1} : \mathbb{R}^n \to \mathbb{R}^n$, since both stereographic projections are bijective. Then, suppose that $u \in \mathbb{R}^n$, and from the inverse computed in **(b)** we see

$$\tilde{\sigma}(\sigma^{-1}(u)) = \tilde{\sigma}\left(\frac{(2u_i, \ldots, 2u_n, |u|^2 - 1)}{|u|^2 + 1}\right)$$

and by definition, we recall that $\tilde{\sigma}(x) = -\sigma(-x)$, thus,

$$\tilde{\sigma}(\sigma^{-1}(u)) = -\sigma\left(-\frac{(2u_i, \ldots, 2u_n, |u|^2 - 1)}{|u|^2 + 1}\right) = -\sigma\left(\frac{(-2u_i, \ldots, -2u_n, -|u|^2 + 1)}{|u|^2 + 1}\right)$$

and applying the definition of σ we see,

$$\tilde{\sigma}(\sigma^{-1}(u)) = -\sigma\left(\frac{(-2u_i, \ldots, -2u_n, -|u|^2 + 1)}{|u|^2 + 1}\right) = -\frac{(-2u_i, \ldots, -2u_n)}{(|u|^2+1)(1 + |u|^2 - 1)} = \frac{(2u_i, \ldots, 2u_n)}{(|u|^2+1)|u|^2}$$

Which is smooth except at the origin, which makes sense considering the stereographic projections get the origin from opposite poles. Further, we recall that both σ and $\tilde{\sigma}$ are invertible, and inverse will thus also be smooth. Then, we have a diffeomorphism and hence the two charts are smoothly compatible, and hence the atlas $\{(\sigma, \mathbb{S}^n \setminus \{N\}), (\tilde{\sigma}, \mathbb{S}^n \setminus \{S\})\}$ is a smooth atlas and gives a smooth structure on $\mathbb{S}^n$.

(d) To see how this smooth structure is the same as the standard smooth structure on $\mathbb{S}^n$ we consider the charts $(U_i^{\pm}, \varphi_i^{\pm})$ and note that these are exactly our charts when $i = n+1$. Further, since every smooth atlas has a maximal atlas, we can see that the standard smooth structure then must contain the atlas we made above. Hence we have the same smooth structure on the two manifolds.

Question 4

(a) Denote $\pi_1 : \mathbb{R}^{n+1} \to \mathbb{P}^n$ and $\pi_2 : \mathbb{S}^n \to \mathbb{S}^n/\sim$ the respective projection maps. We know that these two maps are homeomorphisms, so we know to expect their composition to also be a homeomorphism. In particular, consider the map

$$f = \pi_1 \circ \pi_2^{-1} : \mathbb{S}^n/\sim \to \pi_1(\mathbb{S}^n) \subset \mathbb{P}^n$$

which is a homeomorphism due to π_1 and π_2 being homeomorphisms, and further is a homeomorphism from $\mathbb{S}^n/\sim$ to a subset of $\mathbb{P}^n$.

Now all we need to do is convince ourselves that this subset is actually all of $\mathbb{P}^n$. To see this, suppose that $p \in \mathbb{P}^n$, then $\pi_1^{-1}(p) \subset \mathbb{R}^{n+1}$ is a real line through the origin such that all those points are sent to p under π_1. But the intersection of this line with S^n will correspond with two antipodal points. In particular, $\pi_2\left(\pi_1^{-1}(p) \cap S^n\right) \subset \mathbb{S}^n/\sim$, and hence every point in $\mathbb{P}^n$ has a preimage in $\mathbb{S}^n/\sim$.

(b) To see that $\mathbb{P}^n$ is compact and connected, we use the fact that it is homeomorphic to $\mathbb{S}^n/\sim$, in particular we know that $\mathbb{S}^n/\sim$ is the quotient of the compact and connected set $\mathbb{S}^n \subset \mathbb{R}^{n+1}$. Then, $\mathbb{S}^n/\sim$ must also be compact and connected, and hence $\mathbb{P}^n$ is compact and connected since it is homeomorphic to a compact and connected set.

(c) We will start with the sets and corresponding maps $\{U_i, \varphi_i\}_i$.

First, we note that φ_i is injective by construction. Suppose $[x], [y] \in U_i \subset \mathbb{P}^n$ and $\varphi_i([x]) = \varphi_i([y])$, but then by definition of φ_i we must have that $x_j = y_j \ \forall j = 1, \ldots, n+1$ and hence φ_i is injective.

(i) We need that $\varphi_i(U_i)$ is open in $\mathbb{R}^n$. To see this, we recall the projection map $\pi : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{P}^n$ and denote the trivially continuous map $\iota : \mathbb{R}^n \to \mathbb{R}^{n+1} \setminus \{0\}$ where $\iota(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{n+1}) = (x_1, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_{n+1})$. Then, we notice that $\varphi^{-1} = \pi \circ \iota$, and since both ι and π are continuous, so is φ^{-1}. Finally, we know that U_i is open in $\mathbb{P}^n$ and hence $\varphi_i(U_i)$ is open.

(ii) Fix $\alpha, \beta \in \{1, \ldots, n+1\}$, then consider $\varphi_\alpha(U_\alpha \cap U_\beta)$ and $\varphi_\beta(U_\alpha \cap U_\beta)$: we wish to show these are open in $\mathbb{R}^n$. To see this, we only need that $U_\alpha \cap U_\beta$ is open in $\mathbb{P}^n$, but this follows from the quotient topology of $\mathbb{P}^n$. Then, since $U_\alpha \cap U_\beta \subset U_\alpha$, and we already showed prior that φ_i^{-1} is continuous, we get that $\varphi_\alpha(U_\alpha \cap U_\beta)$ is open. A similar argument with α swapped for β shows the same for $\varphi_\beta(U_\alpha \cap U_\beta)$.

(iii) Suppose that $U_\alpha \cap U_\beta \neq \emptyset$. We want

$$\varphi_\alpha \circ \varphi_\beta^{-1} : \varphi_\beta(U_\alpha \cap U_\beta) \subset \mathbb{R}^n \to \varphi_\alpha(U_\alpha \cap U_\beta) \subset \mathbb{R}^n$$

to be a diffeomorphism. To see this, suppose that $p \in U_\alpha \cap U_\beta$, then if $p = [x_1, \dots, x_{n+1}]$, we know that $\varphi_\beta(p) = \left(\frac{x_1}{x_\beta}, \dots, \frac{x_{\beta-1}}{x_\beta}, \frac{x_{\beta+1}}{x_\beta}, \dots, \frac{x_{n+1}}{x_\beta}\right) \in \mathbb{R}^n$. Applying our map to this, we see,

$$\begin{aligned}\varphi_\alpha(\varphi_\beta^{-1}(\varphi_\beta(p))) &= \varphi_\alpha\left(\varphi_\beta^{-1}\left(\left(\frac{x_1}{x_\beta}, \dots, \frac{x_{\beta-1}}{x_\beta}, \frac{x_{\beta+1}}{x_\beta}, \dots, \frac{x_{n+1}}{x_\beta}\right)\right)\right)\\ &= \left(\frac{x_1}{x_\alpha}, \dots, \frac{x_{\alpha-1}}{x_\alpha}, \frac{x_{\alpha+1}}{x_\alpha}, \dots, \frac{x_{n+1}}{x_\alpha}\right)\end{aligned}$$

and since necessarily we have $x_\alpha \neq 0$ and $x_\beta \neq 0$, we have that this is indeed a diffeomorphism. Indeed the inverse would carry the same property of being smooth, and the bijective properties come from each φ_i being a homeomorphism.

(iv) We need next that U_i covers $\mathbb{P}^n$ for countably many U_i. This follows directly from the construction of the problem. In particular, we only need the set $\{U_i\}_i$ where $i = 1, \dots, n+1$. To see this, suppose a point $p \in \mathbb{P}^n$, then since we recall that $\mathbb{P}^n = \mathbb{R}^{n+1} \setminus \{0\}/\sim$ where $x \sim y \iff x = ky$ for $k \in \mathbb{R} \setminus \{0\}$, we can conclude that $[0] \notin \mathbb{P}^n$. Hence, p must have atleast one non-zero element, suppose it is the i^{th} element, then we know that $p \in U_i$. Hence all of $\mathbb{P}^n$ is covered by this open cover, and it is countable since it is finite.

(v) Finally suppose $p, q \in \mathbb{P}^n$ distinct. Then, without loss of generality, suppose that the i^{th} element of p is zero. Further assume it is non-zero for q, then we can see that $\exists j = 1, \dots, i-1, i+1, \dots, n+1$ such that this component of p is non-zero, and hence $p \in U_j$ and $q \in U_i$ but $p \notin U_i$. Now assume that p and q are non-zero in all of the same components, then we have that necessarily $p, q \in U_i$ for all such non-zero components.

Hence we have that $\{U_i, \varphi_i\}_i$ induces a smooth structure on $\mathbb{P}^n$.

Now we can look at the second set of maps and sets, $\{V_i, \psi_i\}_i$.