Problem 1

For this problem, every part will simply be applying the boundary condition to the general solution. In particular, for the PDE,

$$\frac{\partial^2 u}{\partial x^2} = -\lambda u$$

we recognize that

$$u = a\cos(\sqrt{\lambda}x) + b\sin(\sqrt{\lambda}x)$$

from here we apply the boundary conditions.

(a) The boundary conditions give us,

$$u(0) = a\cos(0) + b\sin(0) = a = 0 \quad \rightarrow \quad u(L) = b\sin(\sqrt{\lambda}L) = 0$$

For non-trivial solutions, $b \neq 0$, we get,

$$\lambda = \left(\frac{n\pi}{L}\right)^2 \qquad n \in \mathbb{N} \cup \{0\}$$

(b) Apply BCs,

$$\frac{\partial u}{\partial x}(0) = -a\sqrt{\lambda}\sin(0) + b\sqrt{\lambda}\cos(0) = b\sqrt{\lambda} = 0 \quad \rightarrow \quad \frac{\partial u}{\partial x}(L) = -a\sqrt{\lambda}\sin(\sqrt{\lambda}L)$$

avoiding the trivial case,

$$\sqrt{\lambda}L = n\pi \quad \rightarrow \quad \lambda = \left(\frac{n\pi}{L}\right)^2 \qquad n \in \mathbb{N} \setminus \{0\}$$

(c) Apply BCs

$$u(0) = a\cos(0) + b\sin(0) = a = 0 \quad \rightarrow \quad \frac{\partial u}{\partial x}(L) = b\sqrt{\lambda}\cos(\sqrt{\lambda}L) = 0$$

avoiding the non-trivial solution,

$$\sqrt{\lambda}L = \left(n + \frac{1}{2}\right)\pi \quad \rightarrow \quad \lambda = \left(\frac{(n+\frac{1}{2})\pi}{L}\right)^2 \qquad n \in \mathbb{N} \setminus \{0\}$$

(d) Apply BCs,

$$\frac{\partial u}{\partial x}(0) = -a\sqrt{\lambda}\sin(0) + b\sqrt{\lambda}\cos(0) = b\sqrt{\lambda} = 0 \quad \rightarrow \quad u(L) = a\cos(\sqrt{\lambda}L) = 0$$

non-trivial solution,

$$\sqrt{\lambda}L = \left(n + \frac{1}{2}\right)\pi \quad \rightarrow \quad \lambda = \left(\frac{(n+\frac{1}{2})\pi}{L}\right)^2 \qquad n \in \mathbb{N} \setminus \{0\}$$

(e) Apply the BCs,

$$u(0) = a\cos(0) + b\sin(0) = a = 0 \quad \rightarrow \quad u(L) + \beta\frac{du}{dx}(L) = b\sin(\sqrt{\lambda}L) + \beta b\sqrt{\lambda}\cos(\sqrt{\lambda}L) = 0$$

$$\tan(\sqrt{\lambda}L) = -\sqrt{\lambda}\beta$$

where λ satisfies the above equation.

(f) Apply BCs,

$$u(0) - \beta\frac{du}{dx}(0) = a\cos(0) + b\sin(0) + \beta a\sqrt{\lambda}\sin(0) - \beta b\sqrt{\lambda}\cos(0) = 0 \quad \rightarrow \quad a = \beta b\sqrt{\lambda}$$

$$u(L) = a\cos(\sqrt{\lambda}L) + b\sin(\sqrt{\lambda}L) = \beta b\sqrt{\lambda}\cos(\sqrt{\lambda}L) + b\sin(\sqrt{\lambda}L) = 0 \quad \rightarrow \quad \tan(\sqrt{\lambda L}) = -\beta\sqrt{\lambda}$$

(g) Apply BCs

$$\frac{du}{dx}(0) = -a\sqrt{\lambda}\sin(0) + b\sqrt{\lambda}\cos(0) = b\sqrt{\lambda} = 0 \quad \rightarrow \quad u(L) + \beta\frac{du}{dx}(L) = a\cos(\sqrt{\lambda}L) - \beta a\sqrt{\lambda}\sin(\sqrt{\lambda}L) = 0$$

$$\tan(\sqrt{\lambda}L) = \frac{1}{\beta\sqrt{\lambda}}$$

(h)

$$u(0) - \beta\frac{du}{dx}(0) = a\cos(0) + b\sin(0) + \beta a\sqrt{\lambda}\sin(0) - \beta b\sqrt{\lambda}\cos(0) = 0 \quad \rightarrow \quad a = \beta b\sqrt{\lambda}$$

$$\frac{du}{dx}(L) = -a\sqrt{\lambda}\sin(\sqrt{\lambda}L) + b\sqrt{\lambda}\cos(\sqrt{\lambda}L) = -\beta\lambda\sin(\sqrt{\lambda}L) + \sqrt{\lambda}\cos(\sqrt{\lambda}L) = 0$$

$$\tan(\sqrt{\lambda}L) = \frac{\sqrt{\lambda}}{\lambda L}$$

Problem 2

Substitute $u(x,y) = M(x)N(y)$ into the PDE,

$$u_{xx} + u_{yy} - au = 0 \quad \rightarrow \quad M''N + MN'' - aMN = 0 \quad \rightarrow \quad M''N = -M(N'' - aN)$$

$$-\frac{M''}{M} = \frac{N'' - aN}{N}$$

This is only possible if both the LS and RS are equivalent to a constant, say λ. Then the two ODEs are,

$$-\frac{M''}{M} = \lambda \qquad \frac{N'' - aN}{N} = \lambda$$

Notice that the ODE with M is the same as in **Q1**, and the BCs are the same as in part **(b)**, so we already know that,

$$M_n(x) = a\cos(\sqrt{\lambda_n}x) \quad \& \quad \lambda_n = \left(\frac{n\pi}{L}\right)^2$$

For N we see,

$$N'' - aN = N\lambda \quad \rightarrow \quad N'' - (\lambda + a)N = 0$$

We recognize this ODE, and conclude,

$$N_n(y) = Ae^{\sqrt{\lambda_n}y} + Be^{-\sqrt{\lambda_n}y}$$

Problem 3

Let $u(r,\theta) = R(r)\Theta(\theta)$, subbing this into the PDE we get,

$$\begin{aligned}
\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} &= 0 \\
\frac{1}{r}\frac{\partial}{\partial r}\left(rR'\Theta\right) + \frac{1}{r^2}R\Theta'' &= 0 \\
\frac{1}{r}(R'\Theta + rR''\Theta) + \frac{1}{r^2}R\Theta'' &= 0 \\
r\Theta(R' + rR'') &= -R\Theta'' \\
r\frac{R' + R''}{R} &= -\frac{\Theta''}{\Theta}
\end{aligned}$$

Thus, $\exists \lambda \in \mathbb{R}$ such that,

$$r\frac{R' + R''}{R} = \lambda \qquad \& \qquad -\frac{\Theta''}{\Theta} = \lambda$$

as required.

Problem 4

We have that $u(r,\phi) = R(r)\Phi(\phi)$, hence,

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{1}{\sin\phi}\frac{\partial}{\partial \phi}\left(\sin\phi\frac{\partial u}{\partial \phi}\right) = 0 \quad \rightarrow \quad \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 R'\Phi\right) + \frac{1}{r^2}\frac{1}{\sin\phi}\frac{\partial}{\partial \phi}\left(\sin\phi R\Phi'\right) = 0$$

$$\frac{1}{r^2}\left(2rR'\Phi + r^2R''\Phi\right) + \frac{1}{r^2}\frac{1}{\sin\phi}\left(\cos\phi R\Phi' + \sin\phi R\Phi''\right) = 0$$

$$\Phi\left(2rR' + r^2R''\right) + R\left(\cot\phi\Phi' + \Phi''\right) = 0$$

$$\frac{2rR' + r^2R''}{R} = -\frac{\cot\phi\Phi' + \Phi''}{\Phi}$$

Thus, $\exists\lambda$ such that,

$$\frac{2rR' + r^2R''}{R} = \lambda \qquad \& \qquad -\frac{\cot\phi\Phi' + \Phi''}{\Phi} = \lambda$$

as required.

Problem 5

(a) First multiply the ODE through by the integrating factor,

$$\frac{r}{a_0}(-a_0u'' - a_1u' + a_2u) = \frac{r}{a_0}\lambda u$$

$$-ru'' - r\frac{a_1}{a_0}u' + r\frac{a_2}{a_0}u = r\frac{\lambda}{a_0}u$$

however, notice that $r' = r\frac{a_1}{a_0}$, and thus,

$$-ru'' - r'u' + r\frac{a_2}{a_0}u = r\frac{\lambda}{a_0}u$$

$$-(ru')' + r\frac{a_2}{a_0}u = r\frac{\lambda}{a_0}u$$

which is exactly in the form of a Sturm-Liouville problem. In particular we notice the following conditions on the constants; $a_0 > 0$, $a_2 > 0$ and $a_1 \in \mathbb{R}$.

(b) We notice that

$$a_0 = x^2 \qquad a_1 = ax \qquad a_2 = -b$$

Further, notice

$$r = \int \frac{a_1}{a_0}dx = \int \frac{ax}{x^2}dx = \int \frac{a}{x}dx = a\ln(x)$$

Subbing this info into the solution from **(a)**

$$-(x^a u')' + x^a\frac{-b}{x^2}u = x^a\frac{\lambda}{x^2}u$$

$$-(x^a u')' - bx^{a-2}u = \lambda x^{a-2}u$$

which is our Sturm-Liouville problem.