Problem 1

(a) We parametrize the two curves. In particular, Γ_2 is simple, but we have to define Γ_1 as the sum of two smooth curves. Let the parameterizing constants be t_2 and t_1 respectively. Then,

$$\Gamma_1 = \begin{cases} e^{it_1} & 0 \le t_1 < \frac{\pi}{2} \\ i\sin(t_1) & \frac{\pi}{2} \le t_1 \le \pi \end{cases} \qquad\qquad \Gamma_2 = 1 - t_2$$

where $t_1 \in [0, \pi]$ and $t_2 \in [0, 1]$.

(b) The computation of the line integral of $f(z) = z^2$ is done using the definition, and by splitting Γ_1 into its parts. Thus,

$$\begin{aligned}
\int_{\Gamma_1} f(z)dz &= \int_0^{\frac{\pi}{2}} (e^{it_1})^2(ie^{it_1})dt_1 + \int_{\frac{\pi}{2}}^{\pi} (i\sin(t_1))^2(i\cos(t_1))dt_1 \\
&= i\int_0^{\frac{\pi}{2}} e^{3it_1}dt_1 - i\int_{\frac{\pi}{2}}^{\pi} \sin^2(t_1)\cos(t_1)dt_1 \\
&= i\frac{1}{3i}e^{3it_1}\Big|_0^{\frac{\pi}{2}} - i\frac{1}{3}\sin^3(t_1)\Big|_{\frac{\pi}{2}}^{\pi} \\
&= \frac{1}{3}(\cos(3t_1) + i\sin(3t_1))\Big|_0^{\frac{\pi}{2}} - \frac{i}{3}(-1) \\
&= \frac{1}{3}(0 - i - 1 - i(0)) + \frac{i}{3} = -\frac{1}{3} - \frac{i}{3} + \frac{i}{3} = -\frac{1}{3}
\end{aligned}$$

Similarly, we do the same computation for Γ_2,

$$\begin{aligned}
\int_{\Gamma_2} f(z)dz &= \int_0^1 (1 - t_2)^2(-1)dt_2 = -\int_0^1 (1 - 2t_2 + t_2^2)dt_2 \\
&= -\int_0^1 dt_2 + 2\int_0^1 t_2 dt_2 - \int_0^1 t_2^2 dt_2 \\
&= -1 + 2\frac{1}{2}(1) - \frac{1}{3}(1) = -\frac{1}{3}
\end{aligned}$$

(c) Similar to **(b)**, we apply the definition,

$$\begin{aligned}
\int_{\Gamma_1} g(z)dz &= \int_0^{\frac{\pi}{2}} (e^{it_1})(e^{-it_1})(ie^{it_1})dt_1 + i\int_{\frac{\pi}{2}}^{\pi} \sin^2(t_1)\cos(t_1)dt_1 \\
&= i\int_0^{\frac{\pi}{2}} e^{it_1}dt_1 + i\int_{\frac{\pi}{2}}^{\pi} \sin^2(t_1)\cos(t_1)dt_1 \\
&= i\frac{1}{i}e^{it_1}\Big|_0^{\frac{\pi}{2}} + i\frac{1}{3}\sin^3(t_1)\Big|_{\frac{\pi}{2}}^{\pi}
\end{aligned}$$

$$= (\cos(t_1) + i\sin(t_1))\Big|_0^{\frac{\pi}{2}} + \frac{i}{3}(-1)$$
$$= (0 + i - 1 - 0) - \frac{i}{3} = -1 + \frac{2i}{3}$$

and again for Γ_2 we see

$$\int_{\Gamma_2} g(z)dz = \int_0^1 |1 - t_2|^2(-1)dt_2 = -\int_0^1 (1 - t_2)^2 dt_2$$

But this is the same integral as before, so

$$\int_{\Gamma_2} g(z)dz = -\frac{1}{3}$$

(d) We expect the results we got in both **(b)** and **(c)** due to the Cauchy-Goursat theorem. In particular, since $f(z)$ is analytic, we know by theorem that the line integral along any closed contour on $\mathbb{C}$, which is open and simply connected, will be zero. However, we notice that in **(b)**, if we reversed the direction of one of our curves, say Γ_2, then $\Gamma = \Gamma_1 - \Gamma_2$ will be a curve that is closed. Hence, by theorem

$$0 = \oint_\Gamma f(z)dz = \int_{\Gamma_1} f(z)dz - \int_{\Gamma_2} f(z)dz \quad \Longrightarrow \quad \int_{\Gamma_1} f(z)dz = \int_{\Gamma_2} f(z)dz$$

which is reflected in our results. On the other hand, we already have shown in a previous assignment that $h(z) = |z|$ is not analytic, and hence neither is $g(z) = |z|^2$, and thus we wouldn't expect that the integral along different paths be the same.

Problem 2

(a) We can choose any contour Γ as long as $\Gamma(a) = i$ and $\Gamma(b) = \frac{i}{2}$, where $\Gamma = \alpha : [a, b] \to \mathbb{C}$, since we know that $f(z) = e^{\pi z}$ is an analytic function. We recall that since $f(z)$ is analytic, our integral value should be path independent. In particular, we could consider any closed path Γ. Then, by Cauchy's Integral theorem we have that

$$\oint_\Gamma f(z)dz = 0$$

Further, since the integral theorem doesn't require that we have a simple curve Γ, we can partition our curve Γ into two parts. Say $\Gamma = \alpha(t) : [a, b] \to \mathbb{C}$, then we can say $\Gamma_1 = \alpha_1(t) : [a, c] \to \mathbb{C}$ and $\Gamma_2 = \alpha_2(t) : [c, b] \to \mathbb{C}$ where $a, b, c \in \mathbb{R}$, $a < c < b$, $\alpha(a) = \alpha_1(a) = \alpha_2(b)$ and $\alpha_1(c) = \alpha_2(c)$. Then applying what we know from Cauchy's Integral theorem and that $\Gamma = \Gamma_1 + \Gamma_2$, we get

$$\oint_\Gamma f(z)dz = \int_{\Gamma_1} f(z)dz + \int_{\Gamma_2} f(z)dz = 0 \quad \Longrightarrow \quad \int_{\Gamma_1} f(z)dz = -\int_{\Gamma_2} f(z)dz$$

where the negative is simply a swap in the orientation of the curve Γ_2. Hence, we let $\Gamma = i(1 - \frac{1}{2}t)$ where we let $t \in [0, 1]$. Then

$$\int_\Gamma f(z)dz = \int_0^1 f(i(1 - \frac{1}{2}t))(-i\frac{1}{2})dt$$

$$= \int_0^1 e^{\pi(i-\frac{i}{2}t)}(-i\frac{1}{2})dt = -\frac{i}{2}\int_0^1 e^{i\pi}e^{-\frac{i\pi}{2}t}dt$$

$$= \frac{i}{2}\int_0^1 \left(\cos(\frac{t\pi}{2}) - i\sin(\frac{t\pi}{2})\right)dt = \frac{i}{2}\int_0^1 \cos(\frac{\pi}{2}t)dt + \frac{1}{2}\int_0^1 \sin(\frac{\pi}{2}t)dt$$

$$= \frac{i}{2}\frac{2}{\pi}\left(\sin(\frac{\pi}{2}t)\Big|_0^1\right) - \frac{1}{2}\frac{2}{\pi}\left(\cos(\frac{\pi}{2}t)\Big|_0^1\right) = \frac{i}{\pi}(1) - \frac{1}{\pi}(-1) = \frac{1}{\pi} + \frac{i}{\pi}$$

(b) We recall that composition of analytic functions will be analytic. Further, we see that none of the composed functions have discontinuities, and are hence analytic over all of $\mathbb{C}$, which is simply connected and open. We can apply Cuachy's Integral theorem, and hence can conclude that any line integral over a closed contour of $f(z) = \exp(\sin(\cos^2(z)))$ will be

$$\oint_\Gamma \exp(\sin(\cos^2(z)))dz = 0$$

$\forall \Gamma = \alpha : [a, b] \to \mathbb{C}$ such that $\alpha(a) = \alpha(b)$.

(c) Again, since our closed curve Γ is in the first quadrant we have that necessarily our closed contour *does not* include the point $z = -3$, which is the point at which the function $f(z) = \frac{1}{z+3}$ will not be analytic. This function can be thought of as being the composition between $\frac{1}{z}$ and $z + 3$ which are both analytic over $\mathbb{C} \setminus \{0\}$. Thus, since the first quadrant, defined explicitly by $Q_1 = \{z \in \mathbb{C} | \Im\{z\} > 0, \Re\{z\} > 0\}$ is open and simply connected, and that $f(z)$ is further analytic over this open and simply connected set, we can apply Cauchy's integral theorem,

$$\oint_\Gamma f(z)dz = 0$$

where $\Gamma \subset Q_1$.

(d) We have that $f(x + iy) = 2xy^3 + iy$, and to we can parameterize our curve with

$$\Gamma : \alpha(t) = \cos(t) - i\sin(t) \qquad t \in [0, 2\pi)$$

where we can let $\alpha_1(t) = \cos(t)$ and $\alpha_2(t) = -\sin(t)$. Then, in particular

$$f(\alpha(t)) = 2(\alpha_1(t))(\alpha_2(t))^3 + i\alpha_2(t) \quad \& \quad \alpha'(t) = -\sin(t) - i\cos(t)$$

then putting these together

$$\int_\Gamma f(z)dz = \int_0^{2\pi} (-2\cos(t)\sin^3(t) - i\sin(t))(-\sin(t) - i\cos(t))dt$$

$$= 2\int_0^{2\pi} \cos(t)\sin^4(t)dt + i\int_0^{2\pi} \sin^2(t)dt + 2i\int_0^{2\pi} \cos^2(t)\sin^2(t)dt - \int_0^{2\pi} \sin(t)\cos(t)dt$$

We notice that there are immediately 3 integrals that we can assume 0 since they are odd and over the period, thus we are left with,

$$\int_\Gamma f(z)dz = i\int_0^{2\pi} \sin^2(t)dt = i\left(\frac{1}{4}(2\pi)^2\right) = i\pi^2$$

(BONUS)

Assume that $f(z)$ has an anti-derivative, and call it $F(z)$ such that $F'(z) = f(z) \quad z \in \mathbb{C}$, which is from definition. Consider the differentiabilty of $f(z)$, we write out the Cuachy-Riemann equations for $f(x+iy) = x - 2xyi$ and see

$$u_x = 1 \qquad v_x = -2y$$

$$u_y = 0 \qquad v_y = -2x$$

However, by theorem, we have that if $f(z)$ were analytic at $z \in \mathbb{C}$, the C-R equations exist at z and are satisfied at this z. However, we see that this is not true $\forall z \in \mathbb{C}$, and hence by contra-positive, we can claim that $f(z)$ is not analytic over $\mathbb{C}$, which would imply that $F(z)$ is not analytic over $\mathbb{C}$, which is a direct contradiction. Thus, $f(z)$ does not have an anti-derivative over $\mathbb{C}$.