Question 1

(a) Suppose A is a $d \times d$ matrix, then we know that

$$e^A = \sum_{n=0}^{\infty} \frac{1}{n!} A^n.$$

(b) We make use of the exponential expansion we did just above. In particular, we see that

$$e^{-iUHU^\dagger t} = \sum_{n=0}^{\infty} \frac{(-it)^n}{n!} (UHU^\dagger)^n = \sum_{n=0}^{\infty} \frac{(-it)^n}{n!} \underbrace{UHU^\dagger \cdot UHU^\dagger \ldots UHU^\dagger}_{n \text{times}}$$

but we know that U is unitary, so we have that $U^\dagger = U^{-1}$, so we get that

$$e^{-iUHU^\dagger t} = \sum_{n=0}^{\infty} \frac{(-it)^n}{n!} UH^nU^\dagger = Ue^{-iHt}U^\dagger$$

as expected.

Question 2

(a) This is done with a direct computation. Notice,

$$ee^\dagger = \begin{pmatrix} \cos^2\frac{\theta}{2} & e^{-i\phi}\sin\frac{\theta}{2}\cos\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2}\cos\frac{\theta}{2} & \sin^2\frac{\theta}{2} \end{pmatrix}$$

and from inspection, we see that

$$(ee^\dagger)^\dagger = \begin{pmatrix} \cos^2\frac{\theta}{2} & e^{-i\phi}\sin\frac{\theta}{2}\cos\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2}\cos\frac{\theta}{2} & \sin^2\frac{\theta}{2} \end{pmatrix} = ee^\dagger$$

as expected.

(b) We can see b and c just from looking at the matrix. In particular, using the expansion of $e^{i\phi} = \cos\phi + i\sin\phi$ we see that we must have

$$b = \cos\phi\sin\frac{\theta}{2}\cos\frac{\theta}{2} \qquad c = \sin\phi\sin\frac{\theta}{2}\cos\frac{\theta}{2}.$$

On the other hand, we see that obtaining a and d is just a matter of solving the system,

$$a + d = \cos^2\frac{\theta}{2} \qquad \& \qquad a - d = \sin^2\frac{\theta}{2}$$

$$\implies 2a = \cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2} = 1 \implies a = \frac{1}{2}$$

$$\implies d = \cos^2\frac{\theta}{2} - \frac{1}{2}.$$

Hence, we have our real a, b, c and d.

Question 3

(a) We know that the eigenvectors will satisfy the eigenvalue problem when applied to X, and from inspection we can find that

$$f_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \& \qquad f_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

which are expressed in the basis of $\{e_0, e_1\}$.

(b) We know that the $+1$ eigenspace is simply $\{e_0\}$ for Z and $\{f_0\}$ for X, so we would expect the eigenbasis of a tensor space to be the tensor of the corresponding eigenspaces, that is $\{f_0 \otimes e_0 \otimes f_0\}$. Since we have all of these vectors in the standard basis representation, we can explicitly write out the vector that would be this tensor as

$$f_0 \otimes e_0 \otimes f_0 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

(c) Here we will avoid writing out the tensor product explicitly, since it is instead easier to compute using the multilinearity of the tensor. That is, the tensor product results in a tensor, and hence is multilinear and we can just consider the null spaces of each component of the tensor. Simply put, we need the null spaces of $I + X$ and $I + Z$, and then we just tensor them accordingly.

By inspection, we notice that the null space of $I + X$ better be f_1, since we want all vectors whose inverse is obtained by swapping components, which is spanned by f_1. Similarly, we see that for the null space of $I + Z$, we want all vectors that only have second component, which is spanned by e_1. Thus, we have that the basis of our null space, B_{null}, will be

$$B_{\text{null}} = \{f_1 \otimes e_1 \otimes f_1\}$$

as required.