PHYS 825 Advanced Quantum Mechanics
Queen's University, Fall 2019
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Assignment 1
Due *9:30 (Beginning of class) Monday Sept 23rd 2019*

1. (3) Given that the time evolution operator is a function of the Hamiltonian, prove that an observable that commutes with the hamiltonian yields a a conserved quantity when acting on a state $|\psi(t)\rangle$.

2. (3) Consider two operators A and B whose commutator is constant $[A, B] = c$.
   (a) Show that $e^A B e^{-A} = B + c$.
   (b) Evaluate $e^A B^n e^{-A}$. (Hint: $e^A e^{-A} = 1$)
   (c) Show that $e^A e^B e^{-A} e^{-B} = e^c$. This is a simple version of the Baker-Campbell-Hausdorff formula.

3. (3) The matrix operator $\sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$ corresponds to a spin component of an electron, in units of $\hbar/2$. For a state represented by the vector $|\Psi\rangle = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$, where α and β are complex numbers, calculate the probability that the spin component is positive. Check your answer by finding the corresponding probability if $|\Psi\rangle$ is prepared with spin orientations a) $s_y = \pm\hbar/2$; b) $s_z = \pm\hbar/2$.

4. (6) In vacuum, the Hamiltonian for propagating neutrinos is simply the free particle Hamiltonian, with eigenvalues E_i and eigenvectors $|\nu_i\rangle$, corresponding to eigenstates with mass m_i. For simplicity, consider the two neutrino case $i = 1, 2$. There exists another basis, called the flavor basis $|\nu_{e,\mu}\rangle$, which diagonalizes *interaction* operators: $|\nu_e\rangle$ interacts with electrons, and $|\nu_\mu\rangle$ interacts with muons. These are related by a "rotation" θ:

$$|\nu_e\rangle = \cos\theta|\nu_1\rangle - \sin\theta|\nu_2\rangle \quad (1)$$
$$|\nu_\mu\rangle = \sin\theta|\nu_1\rangle + \cos\theta|\nu_2\rangle. \quad (2)$$

Noting that, since the neutrinos are relativistic with $E_i \gg m_i$, their kinetic energy for a fixed momentum is $E = \sqrt{p^2c^2 + m^2c^4} \simeq pc\,(1 + m^2c^2/2p^2)$, show that the *survival probability* of electron neutrinos propagating a distance $L = ct$ is:

$$P(\nu_e \to \nu_e) = 1 - \sin^2 2\theta \sin^2\left(\Delta m^2 c^4 \frac{L}{4E\hbar c}\right), \quad (3)$$

where $\Delta m^2 \equiv m_1^2 - m_2^2$ and $E \simeq pc$ in the final expression. Explain why the observation of neutrino oscillations *implies* that neutrinos have mass. Assuming the three-flavor case looks similar (it does), and knowing that we have detected oscillations between all three flavors, how many of the energy eigenstates *must* be massive?

5. (10) This problem will require a tiny amount of programming and the use of software capable of graphing numerical results, such as Matlab, Python, Mathematica or Maple. You may work as a team but each student must submit their own copy of the graph. For simplicity let us work

in units of frequency, rather than energy, such that $\omega = E/\hbar$. Consider a system of N energy eigenstates $\omega_i = 1/N, 2/N, ...1$ s^{-1}. Suppose you have prepared your state $|\alpha\rangle = \sum_k^N c_k|\omega_k\rangle$ with a mean energy ω_0 and a width σ_ω such that:

$$c_k \propto \exp[-(\omega_i - \omega_0)^2/2\sigma_\omega^2]. \tag{4}$$

(a) Write down the correct expressions for the normalized state and for the correlation amplitude $C(t)$.

(b) Now suppose $N = 100$, $\omega_0 = 0.5$ s^{-1} and $\sigma_\omega = 0.1$ s^{-1}. On the same graph, plot: 1) $|C(t)|$ for $t \in \{0, 100\}$ s; 2) the real part of each $c_i(t)$ for the same period.

(c) Explain how the correlation time relates to the relative phases of the different eigenkets that you have plotted.

(d) Assuming $C(t)$ is a (half)-Gaussian with variance σ_t^2, evaluate $\sigma_t\sigma_E$. Explain the relationship with a "classical limit".

6. (15) An alternative (and more general) way of describing a quantum system is through the density operator:

$$\rho \equiv \sum_i w_i|\alpha^{(i)}\rangle\langle\alpha^{(i)}|, \;\; \sum_i w_i = 1 \tag{5}$$

where the states $|\alpha^{(i)}\rangle$ need not be orthogonal. A state where only one $w_i \neq 0$ is called a *pure* state. Otherwise it is called a *mixture*. Note that w_i is not parametrizing a superposition of states – the $|\alpha\rangle$'s themselves could be superpositions.

(a) Show that the trace of the density matrix $\mathrm{Tr}(\rho) = 1$. (it is helpful to remember the definition of the matrix elements)

(b) Show that, for a pure state, $\rho^2 = \rho$ and $\mathrm{Tr}(\rho^2) = 1$.

(c) Define the ensemble average of an operator $[A]$:

$$[A] \equiv \sum_i w_i\langle\alpha^{(i)}|A|\alpha^{(i)}\rangle \tag{6}$$

Now show that this may be written $[A] = \mathrm{Tr}(\rho A)$.

(d) Show that, generally, $\mathrm{Tr}(ABC...)$ is invariant under cyclic permutations. Use this to demonstrate that the trace does not depend on the basis used to represent the matrix.

(e) Given a spin 1/2 particle, show that an equal superposition of states $|+\rangle + |-\rangle$, where $|\pm\rangle \equiv |S_z = \pm\hbar/2\rangle$ is not equivalent to the mixture of states proportional to $|+\rangle\langle+| + |-\rangle\langle-|$. Do this by explicitly evaluating $[S_x]$, where $S_x = \hbar/2(|+\rangle\langle-| + |-\rangle\langle+|)$. Do not forget to normalize your states and mixtures.

(f) Given that the states $|\alpha^{(i)}\rangle$ obey the time-dependent Schrödinger equation, derive the equation of motion for ρ in the case where $w_i(t)$ are also time-dependent.