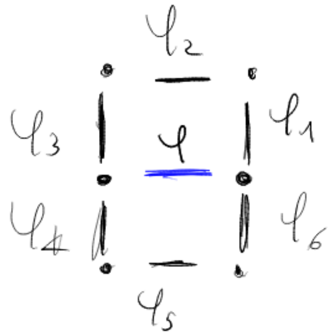
Fact sheet: Heat bath

$$\underset{\substack{\uparrow \\ \text{Action fixed part}}}{S-\hat{S}} = -\beta\Big[\cos\big(\varphi + \underbrace{\varphi_1-\varphi_2-\varphi_3}_{\varphi_a}\big) + \cos\big(\varphi + \underbrace{\varphi_4-\varphi_5-\varphi_6}_{\varphi_b}\big)\Big]$$



Metropolis step: $\varphi^n \in (-\pi, \pi)$ $\quad \xi \in (0,1)$

$$\varphi \to \varphi^n \quad \text{if} \quad \xi < e^{-\Delta S} = \exp\Big\{\beta\Big[\cos(\varphi^n+\varphi_a) + \cos(\varphi^n+\varphi_b) + \\ -\cos(\varphi+\varphi_a) - \cos(\varphi+\varphi_b)\Big]\Big\}$$

Since $e^{\cos\varphi}$ is similar to $e^{-\frac{\varphi^2}{2}}$ in $\varphi \in (-\pi, \pi)$, I could use a Metropolis-Hastings step with a normal distributed proposal.
to find the best gaussian, I can expand $-(S-\hat{S})$ up to φ^2 terms and complete the square.

$$-(S-\hat{S}) \simeq \ldots - \frac{\beta}{2}\Big[(\varphi+\varphi_a)^2 + (\varphi+\varphi_b)^2\Big] = \qquad \cos x \simeq 1 - \frac{x^2}{2}$$

$$\simeq \ldots - \beta\Big[\varphi^2 + \varphi(\varphi_a+\varphi_b)\Big] =$$

$$\simeq \ldots - \beta\Big(\varphi + \underbrace{\frac{\varphi_a+\varphi_b}{2}}_{\bar{\varphi}}\Big)^2 = \ldots - \beta\varphi^{*2} \qquad \varphi^* = \varphi + \bar{\varphi}$$

$$e^{-\beta(\varphi^{*2} + \underset{\substack{\uparrow \\ \text{dummy}}}{f^2})}\, d\varphi^* df = e^{-\beta r^2} r\, dr\, d\theta \qquad \begin{cases} \varphi^* = r\cos\theta \\ f = r\sin\theta \end{cases}$$

$$\propto dE\, d\theta$$

$$E = e^{-\beta r^2}$$

$$\varphi = \sqrt{-\frac{1}{\beta}\log E}\,\cos\theta - \bar{\varphi}$$

$$dE \propto e^{-\beta r^2} r\, dr$$

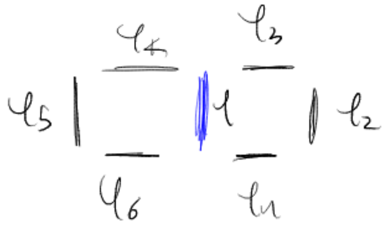
$$\theta \in (0, 2\pi) \qquad E \in (e^{-\beta\pi^2}, 1)$$

Metropolis-Hastings step: $\xi \in (0,1)$

$\varphi \to \varphi^n$, $\quad \varphi^n = \sqrt{-\frac{1}{\beta}\log E}\,\cos\theta - \bar{\varphi}$ $\quad$ if

$$\xi < \exp\Big\{\beta\Big[\cos(\varphi^n+\varphi_a) + \cos(\varphi^n+\varphi_b) + (\varphi^n+\bar{\varphi})^2 + $$
$$ - \cos(\varphi+\varphi_a) - \cos(\varphi+\varphi_b) - (\varphi+\bar{\varphi})^2\Big]\Big\}$$

$$S - \hat{S} = -\beta\Big[\cos\big(\varphi - \overbrace{(\varphi_1+\varphi_2-\varphi_3)}^{\varphi_a}\big) + \cos\big(\varphi - \overbrace{(\varphi_4+\varphi_5-\varphi_6)}^{\varphi_b}\big)\Big]$$



Metropolis step: $\quad \xi \in (0,1) \quad \varphi^n \in (-\pi, \pi)$

$\varphi \to \varphi^n$ $\quad$ if

$$\xi < \exp\Big\{\beta\Big[\cos(\varphi^n-\varphi_a) + \cos(\varphi^n-\varphi_b) + $$
$$ - \cos(\varphi-\varphi_a) - \cos(\varphi-\varphi_b)\Big]\Big\}$$

$$-(S-\hat{S}) \simeq \ldots - \frac{\beta}{2}\Big[(\varphi-\varphi_a)^2 + (\varphi-\varphi_b)^2\Big] =$$
$$= \ldots - \beta\Big[\varphi^2 - \varphi(\varphi_a+\varphi_b)\Big] =$$
$$= \ldots - \beta\Big(\varphi - \underbrace{\frac{\varphi_a+\varphi_b}{2}}_{\bar{\varphi}}\Big)^2$$

$E \in (e^{-\beta\pi^2}, 1)$

Metropolis-Hastings step: $\quad \theta \in (0, 2\pi)$

$\varphi \to \varphi^n \quad \varphi^n = \sqrt{-\frac{1}{\beta}\log E}\,\cos\theta + \bar{\varphi}$ $\quad \xi \in (0,1)$

$$\text{if } \quad \xi < \exp\Big\{\beta\Big[\cos(\varphi^n-\varphi_a) + \cos(\varphi^n-\varphi_b) + (\varphi^n-\bar{\varphi})^2 + $$
$$ - \cos(\varphi-\varphi_a) - \cos(\varphi-\varphi_b) - (\varphi-\bar{\varphi})^2\Big]\Big\}$$