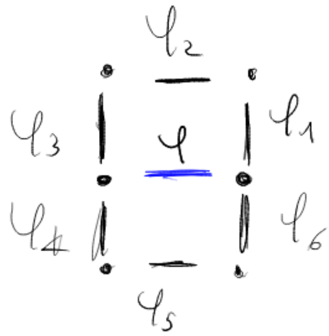
Fact sheet: Heat bath

Naive Metropolis implementation:

$$S-\hat{S} = -\beta\left[\cos\left(\varphi + \underbrace{\varphi_1-\varphi_2-\varphi_3}_{\varphi_a}\right) + \cos\left(\varphi + \underbrace{\varphi_4-\varphi_5-\varphi_6}_{\varphi_b}\right)\right]$$

Action fixed part

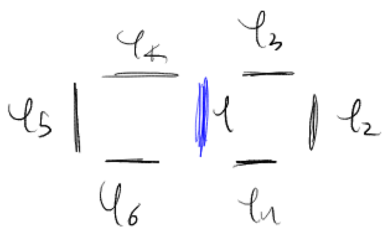


Metropolis step: $\varphi^n \in (-\pi, \pi)$ $\quad \xi \in (0,1)$

$\varphi \to \varphi^n$ if $\quad \xi < e^{-\Delta S} = \exp\left\{\beta\left[\cos(\varphi^n+\varphi_a) + \cos(\varphi^n+\varphi_b) + \right.\right.$

$$\left.\left. -\cos(\varphi+\varphi_a) - \cos(\varphi+\varphi_b)\right]\right\}$$

$$S-\hat{S} = -\beta\left[\cos\left(\varphi - \left(\overbrace{\varphi_1+\varphi_2-\varphi_3}^{\varphi_a}\right)\right) + \cos\left(\varphi - \left(\overbrace{\varphi_4+\varphi_5-\varphi_6}^{\varphi_b}\right)\right)\right]$$



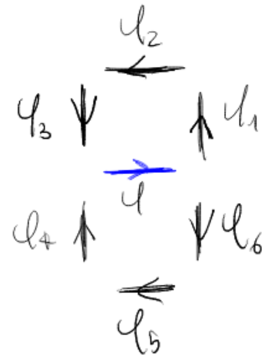
Metropolis step: $\quad \xi \in (0,1) \quad \varphi^n \in (-\pi, \pi)$

$\varphi \to \varphi^n$ if

$$\xi < \exp\left\{\beta\left[\cos(\varphi^n-\varphi_a) + \cos(\varphi^n-\varphi_b) + \right.\right.$$

$$\left.\left. -\cos(\varphi-\varphi_a) - \cos(\varphi-\varphi_b)\right]\right\}.$$

Heat bath with Metropolis-Hastings step



$$W_c(\underset{\text{link}}{U}|\hat{U}) \propto \exp(\beta \,\mathrm{Re}(U\underset{\text{staple}}{S}))$$

$$S = e^{i(\underbrace{\varphi_1-\varphi_2-\varphi_3}_{\varphi_A})} + e^{i(\underbrace{\varphi_4-\varphi_5-\varphi_6}_{\varphi_B})}$$

$$|S| \equiv k \qquad U(1) \ni U_0 \equiv \frac{S}{k}U \qquad US = k\,U_0$$

$$W_c(U|\hat{U})dU = W_c(U_0|\hat{U}_0)dU_0 = \exp(\beta k \,\mathrm{Re}\, U_0)\, dU_0 = e^{\beta k \cos\varphi_0} d\varphi_0$$

$$\varphi = \arg\frac{U_0}{S} = \varphi_0 - \arg S \quad \mathrm{mod}\ (-\pi, \pi]$$

To sample $e^{\beta k \cos\varphi_0}$, I can use the fact that $e^{\beta k\cos\varphi_0} \sim e^{-\frac{\beta k}{2}\varphi_0^2}$ making proposals gaussian distributed and accepting/rejecting with Metropolis-Hastings algorithm:

$$\varphi \to \varphi^n \quad \text{if} \quad \xi < \exp\left[\beta k\left(\cos\varphi_0^n + \frac{\varphi_0^{n2}}{2} - \cos\varphi_0 - \frac{\varphi_0^2}{2}\right)\right]$$

$$\xi \in (0,1) \qquad \varphi_0 = \varphi + \arg S \qquad \varphi^n = \varphi_0^n - \arg S$$

For sampling the gaussian:

$$e^{-\frac{\beta k}{2}(\varphi_0^{n2} + \underset{\text{dummy}}{t^2})} d\varphi_0^n dt = e^{-\frac{\beta k}{2}r^2} r\,dr\,d\theta$$

$$\varphi_0^n = r\cos\theta$$

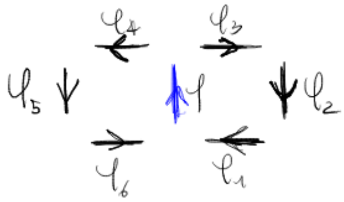
$$t = r\sin\theta$$

$$E = e^{-\frac{\beta k}{2}r^2}$$

$\propto dE\,d\theta \leftarrow$ Uniform $\theta \in (0, 2\pi)$ and since I want to be $\varphi_0^n \in (-\pi, \pi)$, E is uniform in $(e^{-\frac{\beta k}{2}\pi^2}, 1)$

$$\varphi_0^n = \sqrt{-\frac{2}{\beta k}\log E}\,\cos\theta$$

$$W_c(U|\hat{U}) = \exp\left[\beta\, \mathrm{Re}(US)\right]$$



$$S = e^{-i\underbrace{(\varphi_1+\varphi_2-\varphi_3)}_{\varphi_A}} + e^{-i\underbrace{(\varphi_4+\varphi_5-\varphi_6)}_{\varphi_B}}$$

$$W_c(U|\hat{U})\,dU = e^{\beta k \cos\varphi_0}\, d\varphi_0$$

$$E \in \left(e^{-\frac{\beta k}{2}\pi^2}, 1\right) \qquad \theta \in (0, 2\pi) \qquad \varphi_0^n = \sqrt{-\frac{2}{\beta k}\log E}\,\cos\theta$$

$$\varphi_0 = \varphi + \arg S \qquad \varphi^n = \varphi_0^n - \arg S \qquad \xi \in (0,1)$$

$$\varphi \to \varphi^n \quad \text{if} \quad \xi < \exp\left\{\beta k\left[\cos\varphi_0^n + \frac{\varphi_0^{n\,2}}{2} - \cos\varphi_0 - \frac{\varphi_0^2}{2}\right]\right\}$$