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Problem: A binary tree is a rooted tree in which each node has at most two children. Show by induction that in any binary tree the number of nodes with two children is exactly one less than the number of leaves.

Let B(n) represent a binary tree with n nodes.
Let f represent the number of nodes with two children (full nodes).
Let l represent the number of nodes with zero children (leaf nodes).
Let P(n) be the statement that

$$For\ B(n),\ f = l - 1 \tag{1}$$

Proof: We will prove P(n) true for all integer $n \geq 1$
Base Case: $n = 1$. $P(1)$: We have $f = 0$ (no nodes have two children) and $l = 1$ (the single root node). Thus

$$\begin{aligned} &f = l - 1 \\ &0 = 1 - 1 (substitution) \\ &0 = 0 \end{aligned}$$

Therefore P(1) is true.
Inductive step: Assume that P(k) for some integer $k > 0$. This means that $f_k = l_k - 1$. We now prove that $P(k) \implies P(k+1)$.
We prove this by cases:
Let c represent the node we add (child node).
Let p represent the node we add to (parent node).
Case 1: We add c to a leaf node (no children)
Case 2: We add c to a node with one children
In Case 1: $f_{k+1} = f_k$ because p has only one child so f remains unchanged. Additionally $l_{k+1} = l_k$ because p is no longer a leaf node (it has c as a child) but c is a leaf node (no children) so $l_{k+1} = l_k + 1 - 1 = l_k$. Thus because

$$\begin{aligned} &f_k = l_k - 1\ (inductive\ hypothesis) \\ &f_{k+1} = l_{k+1} - 1\ (substitution) \\ &P(k+1)\ is\ true \end{aligned}$$

In Case 2: $f_{k+1} = f_k + 1$ because p now has 2 children, and $l_{k+1} = l_k + 1$ because c is a leaf node (no children). Thus because

$$\begin{aligned} &f_k = l_k - 1\ (inductive\ hypothesis) \\ &f_k + 1 = l_k + 1 (symmetric\ addition) \\ &f_{k+1} = l_{k+1} - 1\ (substitution) \\ &P(k+1)\ is\ true. \end{aligned}$$

Thus, P(n) holds for n = k + 1, and the proof of the induction step is complete.
Conclusion: By the principle of induction, P(n) is true for all integer $n \geq 1$.