1. Qubits

1-1. Vector Spaces

Complex Numbers

$$z = x + yi, \quad i^2 = -1$$

Addition and Multiplicatiom of Complex Number

$$\begin{aligned} z_1 + z_2 &= (x_1 + x_2) + (y_1 + y_2)i \\ z_1 z_2 &= (x_1 x_2 - y_1 y_2) + (x_1 y_2 + x_2 y_1)i \end{aligned}$$

Conjugate and Size of Complex Number

$$\bar{z} = x - yi, \quad |z| = \sqrt{x^2 + y^2}$$

Example of Arithmetics on Complex Numbers

Compute $|z_1|$, $z_1 + z_2$, and $z_1 z_2$ for

$$z_1 = 1 + 2i, \quad z_2 = 3 - 4i$$

$$\begin{aligned} z_1 \bar{z}_1 &= (1+2i)(1-2i) = 1 - 2i + 2i - 4i^2 = 1 + 4 = 5 \Longrightarrow |z_1| = \sqrt{5} \\ z_1 + z_2 &= 1 + 2i + 3 - 4i = (1+3) + (2i - 4i) = 4 - 2i \\ z_1 z_2 &= (1+2i)(3-4i) = 3(1+2i) + (-4i)(1+2i) \\ &= 3 + 6i - 4i - 8i^2 = 3 + 2i - 4i + 8 = 11 - 2i \end{aligned}$$

Polar Form of Complex Number

$$z = r\exp(\theta i) = r\cos\theta + r\sin\theta i$$

Euler's Formula

$$\exp(\theta i) = \cos(\theta) + \sin(\theta) i$$

Arithmetics of Complex Numbers in Polar Form

$$\exp(z_1 + z_2) = \exp(z_1)\exp(z_2)$$

If $z_1 = r_1\exp(\theta_1 i)$ and $z_2 = r_2\exp(\theta_2 i)$, then

$$z_1 \cdot z_2 = r_1 r_2 \exp((\theta_1 + \theta_2)i)$$

Complex Vector Space

A set V of objects, called **complex vectors**, is said to be a **complex vector space** if the following conditions are satisfied:

1. Any vectors $u \in V$ can be added to another vector $v \in V$:
    - $u + v = v + u$
    - $(u + v) + w = u + (v + w)$

2. The **zero vector $\mathbf{0}$** exists:
    - $u + \mathbf{0} = u$
    - $u + (-u) = \mathbf{0}$
3. Any complex number k can be multiplied to any vector $u \in V$:
    - $k \cdot (u + v) = k \cdot u + k \cdot v$
    - $(kl) \cdot u = k \cdot (l \cdot u)$
    - $0 \cdot u = \mathbf{0}$

Example: Complex Numbers as Complex Vectors

$$\mathbb{C} = \{z = x + yi \mid x, y \in \mathbb{R}\}$$

- Vector Addition: $z_1 + z_2 = (x_1 + x_2) + (y_1 + y_2)i$
- Scalar Product: $c \cdot z = cx + cyi$
- Zero Vector: $0 = 0 + 0i$
- Inverse: $-z = -x - yi$

Example: Two-dimensional Complex Vector Space

$$\mathbb{C}^2 = \{(z_1, z_2) \mid z_1, z_2 in \mathbb{C}\}$$

- Vector Addition: $(z_1, z_2) + (w_1, w_2) = (z_1 + w_1, z_2 + w_2)$
- Scalar Product: $c \cdot (z_1, z_2) = (cz_1, cz_2)$
- Zero Vector $(0, 0)$
- Inverse $(-z_1, -z_2)$

Inner Product

A map $\circ : V \times V \to \mathbb{C}$ on a complex vector space V is called an **inner product** if it satisfies the following conditions.

1. Conjugacy:

$$u \circ v = \overline{v \circ u}$$

2. Bilinearilty:

$$\begin{aligned} (c_1 u_1 + c_2 u_2) \circ v &= c_1 (u_1 \circ v) + c_2 (u_2 \circ v) \\ u \circ (c_1 v_1 + c_2 v_2) &= c_1 (u \circ v_1) + c_2 (u \circ v_2) \end{aligned}$$

- Positive Definiteness:

$$u \circ u \geq 0$$

  Equality holds only when $u = \mathbf{0}$

Example: Inner Product on $\mathbb{C}$

$$z \circ w = \bar{z}w$$

Example: Inner Product on $\mathbb{C}^2$

$$(z_1, z_2) \circ (w_1, w_2) = \bar{z}_1 w_1 + \bar{z}_2 w_2$$

Linear Combination

$$c_1u_1 + c_2u_2 + \cdots + c_nu_n$$

Linear Independence

The set of vector $\{u_1, u_2, \cdots, u_n\}$ is said to be **linearly independent** if $c_1u_1 + c_2u_2 + \cdots + c_nu_n = \mathbf{0}$ if and only if $c_1 = c_2 = \cdots = c_n = 0$.

Example

Check if the following sets of vectors are linearly independent. If not, find a non-trivial linear combination that gives the zero vector.

1. $u = \left(1 - \sqrt{3}i, -i\right), \quad v = \left(2, \sqrt{2}i\right)$
2. $u = \left(1, \frac{1+\sqrt{2}i}{\sqrt{5}}\right), \quad v = \left(1 + \sqrt{2}i, 1\right), \quad w = \left(\frac{1}{\sqrt{2}}, \frac{i}{\sqrt{2}}\right)$

Basis

A set of vectors $\mathcal{B} = \{u_1, u_2, \cdots, u_n\}$ is called as a **basis** for a vector space V if

1. $\mathcal{B}$ is linearly independent;
2. every vector $v \in V$ can be expressed as a linear combination of vectors in $\mathcal{B}$.

Example

Show that the following vectors form a basis for $\mathbb{C}^2$.

$$u_1 = (1, 1), \quad u_2 = (1, -1)$$

Orthogonality

Two vectors u and v are said to be **orthogonal** if their inner product is zero: $u \circ v = 0$.

Orthonormal Basis

A basis $\{u_1, u_2, \cdots, u_n\}$ is called **orthonormal** if they are **orthogonal** and **normalized** (i.e. each is of length 1).

Example

Show that the following vectors form an orthonormal basis for $\mathbb{C}^2$.

$$u_1 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \quad u_2 = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$$