Monte Carlo computation of L_p distance between two densities on the unit hypersphere

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Problem Statement

A d-dimensional unit hypersphere $\mathbb{S}^d = \{x \in \mathbb{R}^{d+1} \mid \|x\|_2^2 = \sum_{i=1}^{d+1} x_i^2 = 1\}$ is one of the standard mathematical spaces in the field of objected-oriented data analysis (Marron and Dryden 2021). Let $\mathcal{P}(\mathbb{S}^d)$ denote a space of probability densities on $\mathbb{S}^d$. For two densities $f, g \in \mathcal{P}(\mathbb{S}^d)$, it is frequently needed to measure dissimilarity between the two. Unfortunately, even for the most well-known distributions on the hypersphere, analytic formula for any discrepancy measure is rarely available, leading to require numerical schemes for approximation. Here we focus on L_p distance between the two densities,

$$L_p(f,g) = \left(\int_{\mathbb{S}^d} |f(x) - g(x)|^p \right)^{1/p} \tag{1}$$

and we show how to combine Monte Carlo way of integration by means of importance sampling to approximate Equation 1.

Computation

Importance sampling requires a proposal density. The easiest choice is to use uniform density $u(x)$ as an importance proposal since sampling from $u(x)$ is trivial. First, take a random sample from standard normal distribution $x \sim \mathcal{N}(0, I)$ in $\mathbb{R}^{d+1}$. Then, the rest is to take L_2 normalization, i.e., $x \leftarrow x/\|x\|_2$, which makes a sampled vector to have a unit norm. Given the sample generation process, we have the following

$$\begin{aligned}
L_p(f,g)^p &= \int_{\mathbb{S}^d} |f(x) - g(x)|^p dx \\
&= \int_{\mathbb{S}^d} \frac{|f(x) - g(x)|^p}{u(x)} u(x) dx \\
&= \mathbb{E}_{u(x)} \left[\frac{|f(x) - g(x)|^p}{u(x)} \right] \\
&\approx \frac{1}{N} \sum_{n=1}^{N} \frac{|f(x) - g(x)|^p}{u(x)} \text{ for } x_n \overset{iid}{\sim} u(x),
\end{aligned}$$

where the last term gets better approximation as $N \to \infty$.

References

Marron, James Stephen, and I. L. Dryden. 2021. *Object Oriented Data Analysis*. Boca Raton: Taylor & Francis Group, LLC.

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