Module 4 Problem Set

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Problem 1

$T(n) = 3T(\frac{n}{2}) + n^2$

According to the master theorem,

$a = 3, b = 2, k = 2, p = 0$

Since $3 < 2^2$ and $0 \geq 0$, $T(n) = \mathcal{O}(n^k \log^p(n)) = \mathcal{O}(n^2)$

Problem 2

$T(n) = 4T(\frac{n}{2}) + n^2$

According to the master theorem,

$a = 4, b = 2, k = 2, p = 0$

Since $4 = 2^2$ and $0 > -1$, $T(n) = \mathcal{O}(n^{\log_b^{(a)}} \log^{p+1}(n)) = \mathcal{O}(n^2 \log(n))$

Problem 3

$T(n) = T(\frac{n}{2}) + n^2$

According to the master theorem,

$a = 1, b = 2, k = 2, p = 0$

Since $1 < 2^2$ and $0 \geq 0$, $T(n) = \mathcal{O}(n^k \log^p(n)) = \mathcal{O}(n^2)$

Problem 4

$T(n) = 16T(\frac{n}{4}) + n$

According to the master theorem,

$a = 16, b = 4, k = 1, p = 0$

Since $16 > 4^1$, $T(n) = \mathcal{O}(n^{\log_b^{(a)}}) = \mathcal{O}(n^2)$

Problem 5

$T(n) = 2T(\frac{n}{2}) + n \log(n)$

According to the master theorem,

$a = 2, b = 2, k = 1, p = 1$

Since $2 = 2^1$ and $1 > -1$, $T(n) = \mathcal{O}(n^{\log_b^{(a)}} \log^{p+1}(n)) = \mathcal{O}(n \log^2(n))$

Problem 6

$T(n) = 2T(\frac{n}{2}) + \frac{n}{\log(n)}$

According to the master theorem,

$a = 2, b = 2, k = 1, p = -1$

Since $2 = 2^1$ and $-1 = -1$, $T(n) = \mathcal{O}(n^{\log_b^{(a)}} \log(\log(n)) = \mathcal{O}(n \log(\log(n)))$

Problem 7

$T(n) = 2T(\frac{n}{4}) + n^{0.51}$

According to the master theorem,

$a = 2, b = 4, k = 0.51, p = 0$

Since $2 < 4^{0.51}$ and $0 \geq 0$, $T(n) = \mathcal{O}(n^k \log^p(n)) = \mathcal{O}(n^{0.51})$

Problem 8

$T(n) = 6T(\frac{n}{3}) + n^2 \log(n)$

According to the master theorem,

$a = 6, b = 3, k = 2, p = 1$

Since $6 < 3^2$ and $1 \geq 0$, $T(n) = \mathcal{O}(n^k \log^p(n)) = \mathcal{O}(n^2 \log(n))$

Problem 9

$T(n) = 7T(\frac{n}{3}) + n^2$

According to the master theorem,

$a = 7, b = 3, k = 2, p = 0$

Since $7 < 3^2$ and $0 \geq 0$, $T(n) = \mathcal{O}(n^k \log^p(n)) = \mathcal{O}(n^2)$

Problem 10

$T(n) = \sqrt{2}T(\frac{n}{2}) + \log(n)$

According to the master theorem,

$a = \sqrt{2}, b = 2, k = 0, p = 1$

Since $\sqrt{2} > 2^0$, $T(n) = \mathcal{O}(n^{\log_b^{(a)}}) = \mathcal{O}(n^{\frac{1}{2}})$

Problem 11

$T(n) = \frac{5}{2}n - \frac{1}{2}$

According to the master theorem,

$a = 3, b = 3, k = 0, p = 0$

Since $3 < 3^0$, $T(n) = \theta(n^{\log_b^{(a)}}) = \theta(n)$

Thus, $T(n) = c_1 n + c_0$

Since $T(1) = 2$, then $T(1) = c_1 + c_0 = 2$

So first set $c_1 = 2$ and $c_0 = 0$,

Then, $T(n) = 2n$

However, confirm with coefficients, $3T(\frac{n}{3}) + 1 = 3(\frac{2n}{3}) + 1 \neq 2n$

$T(n) = c_1 n + c_0$

$3T(\frac{n}{3}) + 1 = 3(c_1 \frac{n}{3} + c_0) + 1 = c_1 n + c_0 = c_1 n + 3c_0 + 1$

Then, $-1 = 2c_0$, $c_0 = -\frac{1}{2}$

Since $c_1 + c_0 = 2$, then $c_1 = \frac{5}{2}$

Confirm with coefficients, try with $c_0 = -\frac{1}{2}$, $c_1 = \frac{5}{2}$

$T(n) = \frac{5}{2}n - \frac{1}{2}$

$3T(\frac{n}{3}) + 1 = 3(\frac{5}{2} \times \frac{n}{3} - \frac{1}{2})$

$= \frac{5}{2}n - \frac{3}{2} + 1$

$= \frac{5}{2}n - \frac{3}{2} + 1 = T(n)$

Thus, $T(n) = \frac{5}{2}n - \frac{1}{2}$

Problem 12

$T(n) = 75n!$

Try cn^k

$T(n-1) = c(n-1)^k$, $nT(n-1) = cn(n-1)^k$

No, polynomials different degree.

Then, try $T(n) = cn!$

$nT(n-1) = nc(n-1)!$

$T(n) = nc(n-1)!$

Since $T(2) = 150$

$T(2) = 2c = 150$

$c = 75$

Confirm with coefficient $c = 75$

Then, $T(n) = 75n!$

$nT(n-1) = n \times 75 \times (n-1)! = 75n!$

Confirmed, thus $T(n) = 75n!$

Problem 13

$T(n) = \mathcal{O}(n\log(n))$

Main recurence is true for all n, so substitute '9n/10' for s in the template, then substitude 'n/10' for s, add together to give a new recurrence

$T(\frac{9n}{10}) = T(\frac{81n}{100}) + T(\frac{9n}{100}) + \frac{9n}{10}$

$T(\frac{9n}{100}) = T(\frac{81n}{100}) + T(\frac{9n}{100}) + \frac{9n}{10}$

$T(\frac{n}{10}) = T(\frac{9n}{100}) + T(\frac{n}{100}) + \frac{n}{10}$

$T(n) = T(\frac{9n}{10}) + T(\frac{n}{10}) + n$

$= T(\frac{81n}{100}) + T(\frac{9n}{100}) + \frac{9n}{10} + T(\frac{9n}{100}) + T(\frac{n}{100}) + \frac{n}{10} + n$

$= T(\frac{81n}{100}) + 2T(\frac{9n}{100}) + T(\frac{n}{100}) + \frac{9n}{10} + \frac{n}{10} + n$

$= T(\frac{81n}{100}) + 2T(\frac{9n}{100}) + T(\frac{n}{100}) + 2n$

$(\frac{9}{10})^k(n) \leq 1$, $k - \log_{\frac{10}{9}}(n) \geq 0$, and $\mathcal{O}(n\log_{\frac{10}{9}}(n))$

Thus, $T(n) = \mathcal{O}(n\log(n))$

Problem 14

$T(n) = \mathcal{O}(n\log(n))$

According to Problem 13

$T(n) = cn\log(n)$

$T(\frac{9n}{10}) = c(\frac{9n}{10})\log(\frac{9n}{10})$

$= \frac{9n}{10}cn\log(\frac{9n}{10})$

$= \frac{9n}{10}cn(\log(n) + \log(\frac{9n}{10}))$

$= \frac{9n}{10}cn\log(n) + \frac{9}{10}cn\log(\frac{9n}{10})$

$T(\frac{n}{10}) = c(\frac{n}{10})\log(\frac{n}{10})$

$= (\frac{1}{10})cn\log(\frac{1}{10}n)$

$= (\frac{1}{10})cn(\log(n) + \log(\frac{1}{10}))$

$= (\frac{1}{10})cn\log(n) + (\frac{1}{10})cn\log(\frac{1}{10}))$

$T(\frac{9n}{10}) + T(\frac{n}{10}) = cn\log(n) + cn(\frac{9}{10})\log(\frac{9}{10}) + cn(\frac{1}{10})\log(\frac{1}{10})$

Let $A = (\frac{9}{10})\log(\frac{9}{10}) + (\frac{1}{10})\log(\frac{1}{10})$

Then, $T(\frac{9n}{10}) + T(\frac{n}{10}) = cn\log(n) + cAn$

So, $T(\frac{9n}{10}) + T(\frac{n}{10}) + n = cn\log(n) + (cA+1)n$

This means that,

$cA + 1 = 0$, $c = -\frac{1}{A}$

Solution is $T(n) = -\frac{1}{A}n\log(n)$, where A as above, also, since A is a negative number, $T(n) = \mathcal{O}(n\log(n))$