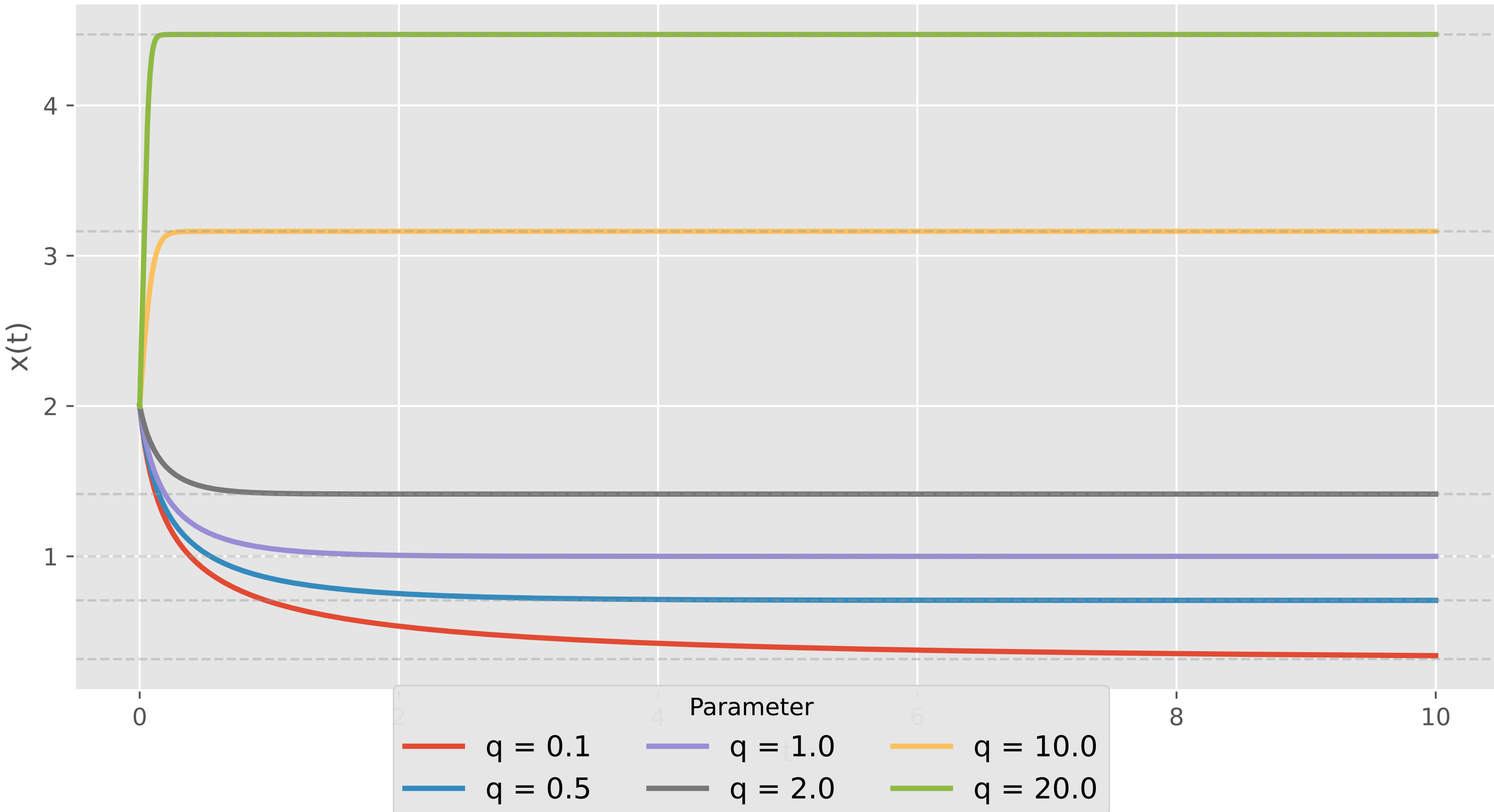
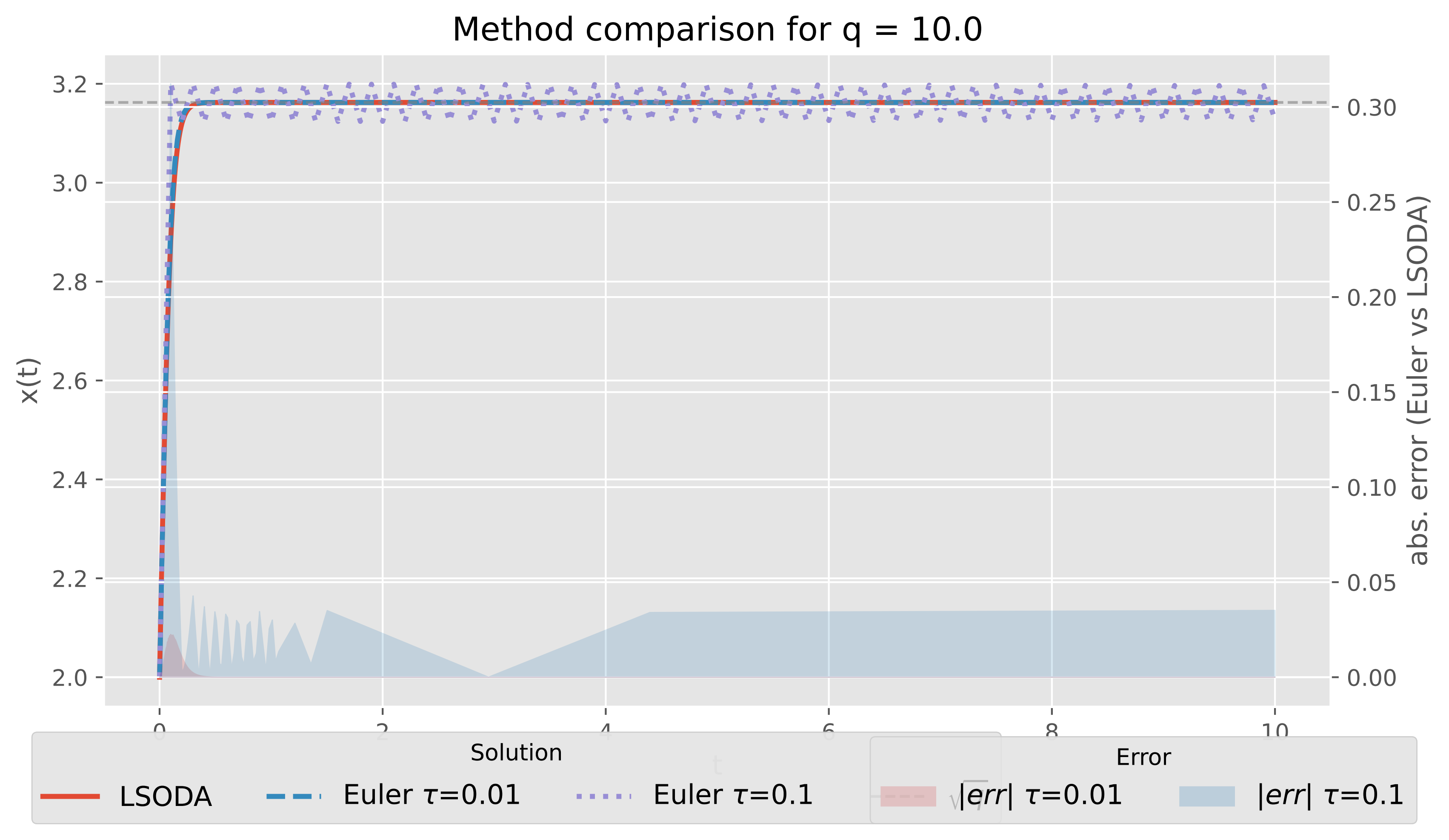
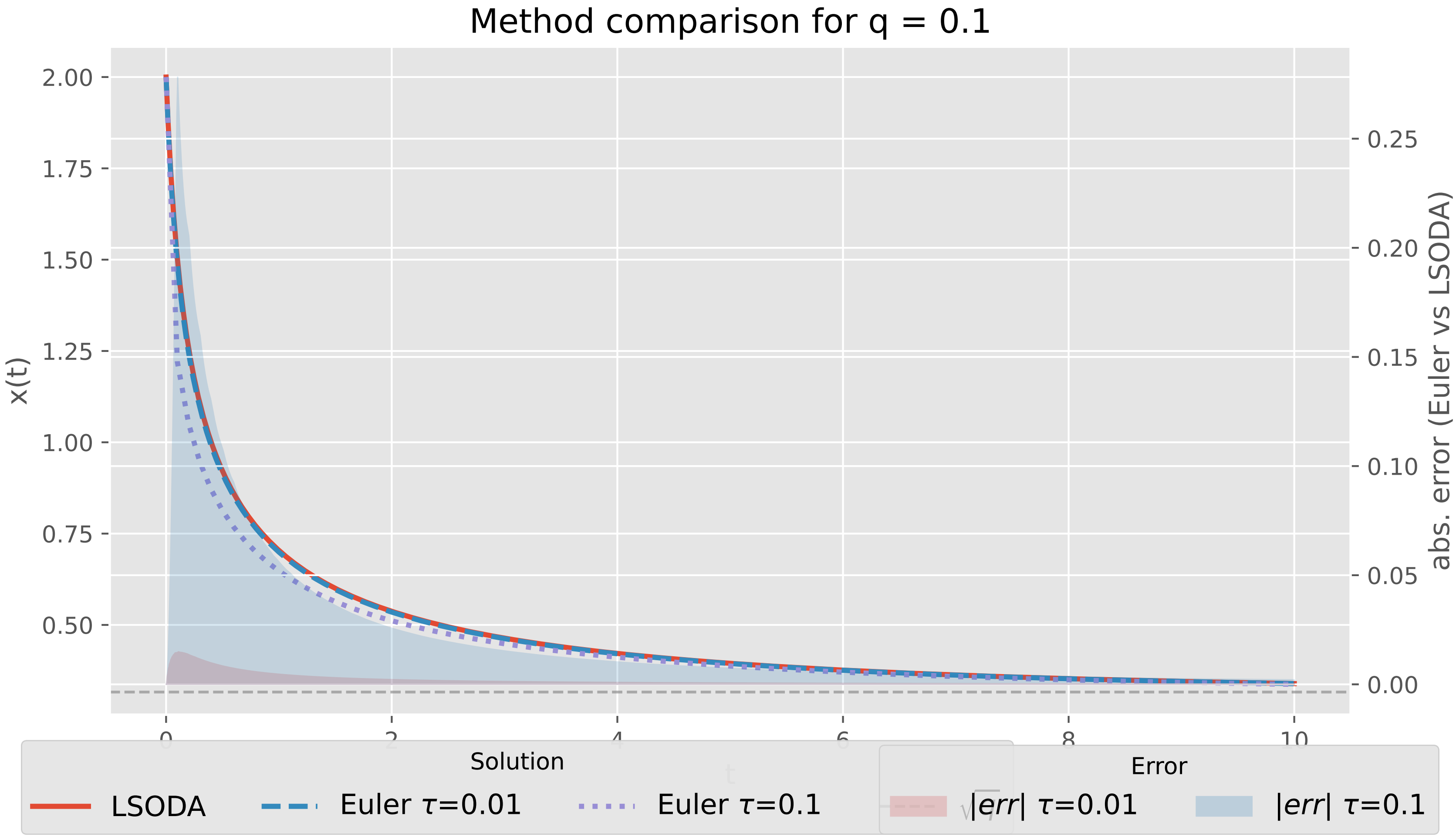
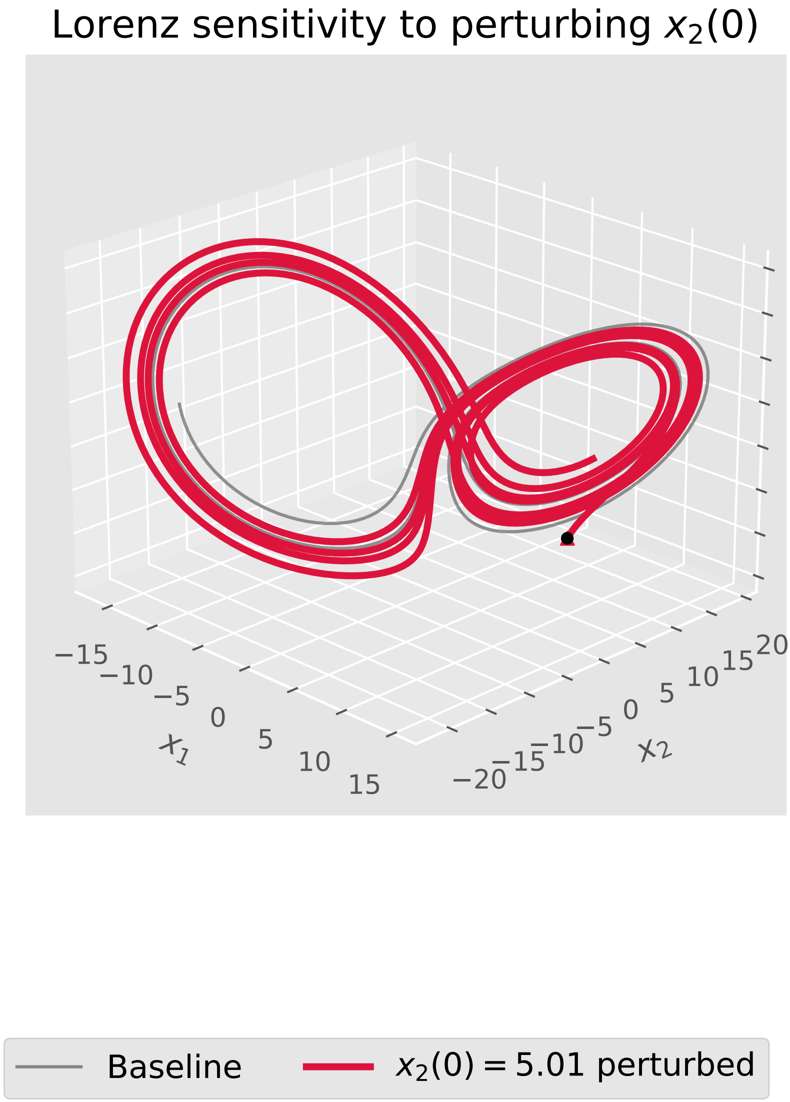
Cubic ODE parameter sweep (equilibria shown)

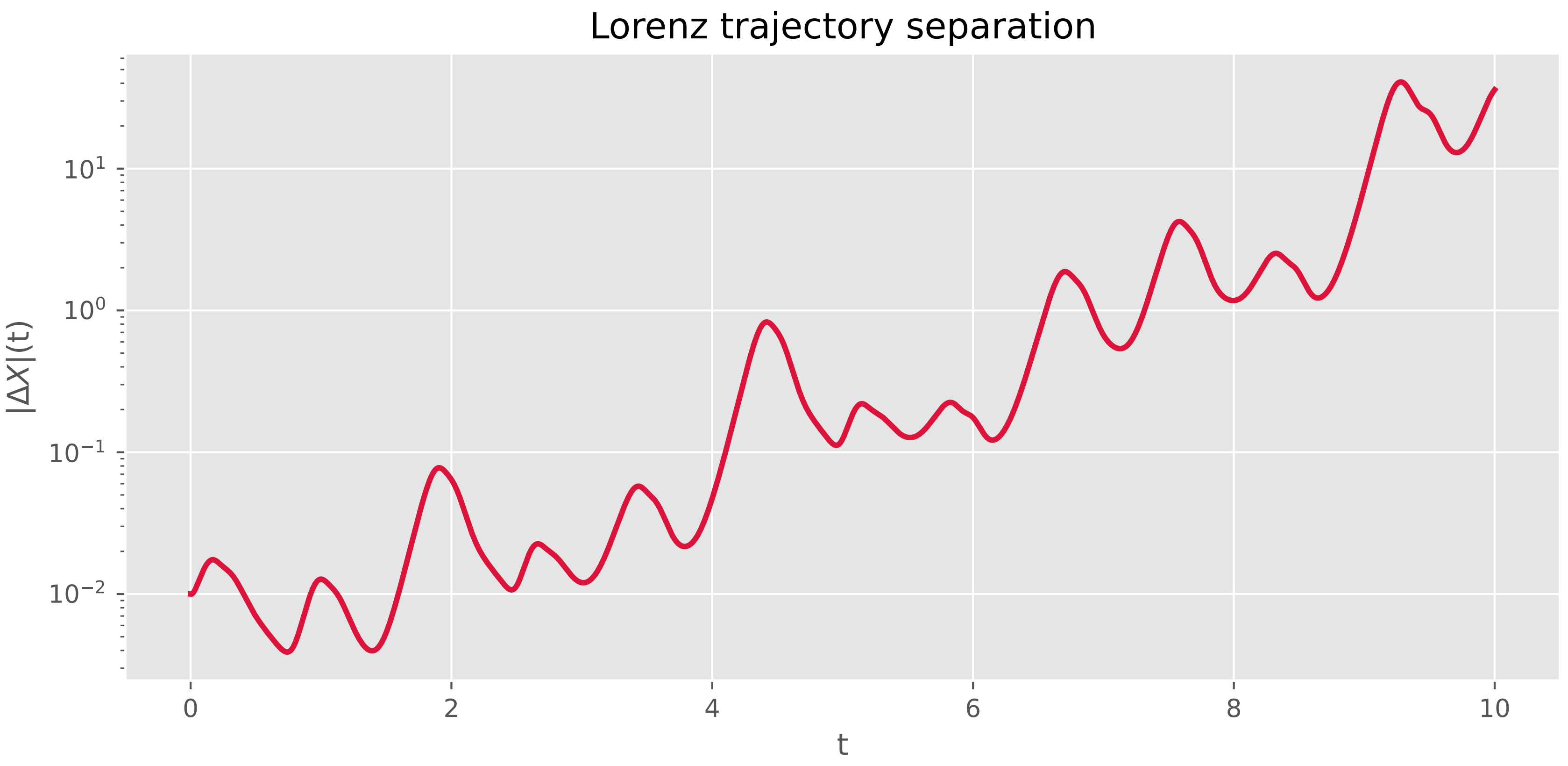




Method comparison for q = 10.0
x(t)
abs. error (Euler vs LSODA)
t
3.2
3.0
2.8
2.6
2.4
2.2
2.0
0.30
0.25
0.20
0.15
0.10
0.05
0.00
0
2
4
6
8
10
Solution
LSODA
Euler τ=0.01
Euler τ=0.1
Error
|err| τ=0.01
|err| τ=0.1


Method comparison for q = 0.1
x(t)
abs. error (Euler vs LSODA)
t
2.00
1.75
1.50
1.25
1.00
0.75
0.50
0.25
0.20
0.15
0.10
0.05
0.00
0
2
4
6
8
10
Solution
LSODA
Euler τ=0.01
Euler τ=0.1
Error
|err| τ=0.01
|err| τ=0.1


Lorenz sensitivity to perturbing $x_2(0)$
x_1
x_2
−15
−10
−5
0
5
10
15
−20
−15
−10
−5
0
5
10
15
20
5
10
15
20
25
30
35
40
Baseline
$x_2(0) = 5.01$ perturbed



Lorenz trajectory separation
$|\Delta X|(t)$
10^{1}
10^{0}
10^{-1}
10^{-2}
0
2
4
6
8
10
t

Answers

(b) Long-term behaviour as a function of q

\textbf{(b) Long-term behaviour as a function of q.}

We solve $x'(t) = qx - x^3$ with $x(0) = 2$ and $q > 0$. The equilibria are 0 and $\pm\sqrt{q}$. For $q > 0$, $x = 0$ is unstable and $x = \pm\sqrt{q}$ are asymptotically stable since $f'(x) = q - 3x^2$ implies $f'(\pm\sqrt{q}) = -2q < 0$.

With $x(0) = 2 > 0$, the trajectory approaches the stable equilibrium $+\sqrt{q}$:

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\[
\begin{cases}
q<4\!: & \sqrt{q}<2 \Rightarrow x(t)\ \text{decreases monotonically to}\ \sqrt{q},\\[2pt]
q=4\!: & x(t)\equiv 2\ \text{(equilibrium; adding this case gives a flat line at
2)},\\[2pt]
q>4\!: & \sqrt{q}>2 \Rightarrow x(t)\ \text{increases monotonically to}\ \sqrt{q}.
\end{cases}
\]
```

This matches the parameter-sweep plot: for small q the approach to $\sqrt{q}$ is slower, so at $T = 10$ the solution can still be slightly above the limiting value.

(c) Method comparison (Euler vs. LSODA) and effect of q

\textbf{(c) Method comparison (Euler vs.\ LSODA) and effect of q.}

We compare explicit Euler with step sizes $\tau = 0.1$ and $\tau = 0.01$ against an LSODA reference on $[0, 10]$.

\emph{Accuracy order.} Explicit Euler is first order: the global error scales as $\mathcal{O}(\tau)$ for smooth problems on a fixed time horizon. Hence, reducing τ from 0.1 to 0.01 should reduce the error by about a factor of 10 (modulo transients).

\emph{Linear stability near the attractor.} Linearizing at the stable equilibrium $x^* = \sqrt{q}$ gives $y' = f'(x^*)y = -2qy$. For the test equation $y' = \lambda y$ with $\lambda = -2q$, explicit Euler is stable iff

```
\[
|1+\tau\lambda|<1 \quad\Longleftrightarrow\quad 0<\tau<\frac{1}{q}.
\]
```

\emph{Case $q = 10$.} The stability bound is $\tau < 0.1$, so $\tau = 0.1$ lies \emph{on the boundary} and yields visible phase/amplitude error and mild oscillation around the equilibrium; $\tau = 0.01$ is well inside the stable region and closely tracks LSODA. Empirically, the absolute error curve for $\tau = 0.1$ sits roughly an order of magnitude above that for $\tau = 0.01$ over most of $[0, 10]$, consistent with first-order convergence \emph{and} the stability-edge effect at $\tau = 0.1$.

\emph{Case $q = 0.1$.} The bound is $\tau < 10$, so both $\tau = 0.1$ and 0.01 are deep inside the stability region and the dynamics are slow. Both Euler solutions lie very close to LSODA; the $\tau = 0.01$ error is still smaller (by about the expected $\sim 10\times$ factor), but the difference is barely visible in the solution plot because all errors are small.

(d) Sensitivity for the Lorenz system

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With standard parameters $(a, b, c) = (10, 25, 8/3)$ the Lorenz system exhibits sensitive dependence on initial conditions (positive largest Lyapunov exponent). We integrate on $[0, 10]$ with explicit Euler ($\tau = 0.001$) from $(x_1(0), x_2(0), x_3(0)) = (10, 5, 12)$ and from the perturbed $(10, 5.01, 12)$.

The two trajectories coincide initially but separate clearly after a short time, ultimately exploring different parts of the attractor. This is the expected behaviour for a chaotic system: for a small perturbation $\|\delta x(0)\|$ the separation typically grows like $\|\delta x(t)\| \approx \|\delta x(0)\| e^{\lambda t}$ with $\lambda > 0$.

\emph{Conclusion.} Yes, the solution changes significantly when $x_2(0)$ is perturbed to 5.01; the 3D plot makes this divergence clearly visible.