ODEs and Dynamical Systems — Answers (b)–(d)

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(b) Long-term behaviour as a function of q

We solve $x'(t) = qx - x^3$ with $x(0) = 2$ and $q > 0$. The equilibria are 0 and $\pm\sqrt{q}$. For $q > 0$, $x = 0$ is unstable and $x = \pm\sqrt{q}$ are asymptotically stable since $f'(x) = q - 3x^2$ implies $f'(\pm\sqrt{q}) = -2q < 0$.
With $x(0) = 2 > 0$, the trajectory approaches the stable equilibrium $+\sqrt{q}$:

$$\begin{cases} q < 4: & \sqrt{q} < 2 \Rightarrow x(t) \text{ decreases monotonically to } \sqrt{q}, \\ q = 4: & x(t) \equiv 2 \text{ (equilibrium; adding this case gives a flat line at 2)}, \\ q > 4: & \sqrt{q} > 2 \Rightarrow x(t) \text{ increases monotonically to } \sqrt{q}. \end{cases}$$

This matches the parameter-sweep plot: for small q the approach to $\sqrt{q}$ is slower, so at $T = 10$ the solution can still be slightly above the limiting value.

(c) Method comparison (Euler vs. LSODA) and effect of q

We compare explicit Euler with step sizes $\tau = 0.1$ and $\tau = 0.01$ against an LSODA reference on $[0, 10]$.
Accuracy order. Explicit Euler is first order: the global error scales as $\mathcal{O}(\tau)$ for smooth problems on a fixed time horizon. Hence, reducing τ from 0.1 to 0.01 should reduce the error by about a factor of 10 (modulo transients).
Linear stability near the attractor. Linearizing at the stable equilibrium $x^* = \sqrt{q}$ gives $y' = f'(x^*)\,y = -2q\,y$. For the test equation $y' = \lambda y$ with $\lambda = -2q$, explicit Euler is stable iff

$$|1 + \tau\lambda| < 1 \quad \Longleftrightarrow \quad 0 < \tau < \frac{1}{q}.$$

Case $q = 10$. The stability bound is $\tau < 0.1$, so $\tau = 0.1$ lies *on the boundary* and yields visible phase/amplitude error and mild oscillation around the equilibrium; $\tau = 0.01$ is well inside the stable region and closely tracks LSODA. Empirically, the absolute error curve for $\tau = 0.1$ sits roughly an order of magnitude above that for $\tau = 0.01$ over most of $[0, 10]$, consistent with first-order convergence *and* the stability-edge effect at $\tau = 0.1$.
Case $q = 0.1$. The bound is $\tau < 10$, so both $\tau = 0.1$ and 0.01 are deep inside the stability region and the dynamics are slow. Both Euler solutions lie very close to LSODA; the $\tau = 0.01$ error is still smaller (by about the expected $\sim 10\times$ factor), but the difference is barely visible in the solution plot because all errors are small.

(d) Sensitivity for the Lorenz system

We consider the Lorenz system

$$\begin{aligned} x_1' &= a\,(x_2 - x_1), \\ x_2' &= b\,x_1 - x_2 - x_1 x_3, \qquad (a,b,c) = (10, 25, \tfrac{8}{3}), \\ x_3' &= -c\,x_3 + x_1 x_2, \end{aligned}$$

on $[0, 10]$ with initial data $x(0) = (10, 5, 12)$ and a perturbed initial data $\widetilde{x}(0) = (10, 5.01, 12)$. The right-hand side is smooth; therefore, by *Picard–Lindelöf* (Thm. 1.12) both IVPs admit unique solutions on $[0, 10]$.

Continuous dependence on initial data. Let $x(t)$ and $\widetilde{x}(t)$ denote the two solutions. With a (local) Lipschitz constant L in x, the script proves via *Gronwall*'s Lemma (Lemma 1.18) that

$$\|x(t) - \widetilde{x}(t)\| \;\le\; e^{L(t-t_0)}\,\|x(0) - \widetilde{x}(0)\| \qquad (t \in [0, 10]),$$

see *Theorem 1.19.* Hence a small perturbation in the initial state can grow (at most) exponentially in time. Consequently, it is expected that the two Lorenz trajectories, started 0.01 apart in the second component, initially stay close and then separate noticeably on $[0, 10]$.

Numerical method and faithfulness over finite time. We approximate by the explicit Euler method

$$x_{j+1} = x_j + \tau\, f(x_j), \qquad t_j = j\tau.$$

By the Taylor derivation in the script, the local truncation error is $O(\tau^2)$ (Example 2.8), hence Euler is *consistent of order 1* (Def. 2.4). The *Convergence Theorem* (Thm. 2.12) then gives a global error $O(\tau)$ on $[0, 10]$ under the same Lipschitz assumptions. Choosing a small step (e.g. $\tau = 0.001$) ensures that the discrete trajectories remain uniformly close to the continuous ones on the whole time interval. Therefore, the numerically observed separation between the two runs reflects the expected continuous-dependence behavior from Theorem 1.19, rather than a discretization artifact.

Conclusion. The script guarantees: (i) existence/uniqueness (Thm. 1.12), (ii) continuous dependence with at-most-exponential growth of perturbations (Lemma 1.18 & Thm. 1.19), and (iii) $O(\tau)$ global accuracy of explicit Euler on fixed horizons (Def. 2.4, Example 2.8, Thm. 2.12). Hence the clear divergence of two Lorenz trajectories started from $(10, 5, 12)$ and $(10, 5.01, 12)$ on $[0, 10]$ is precisely what the script's theory predicts.