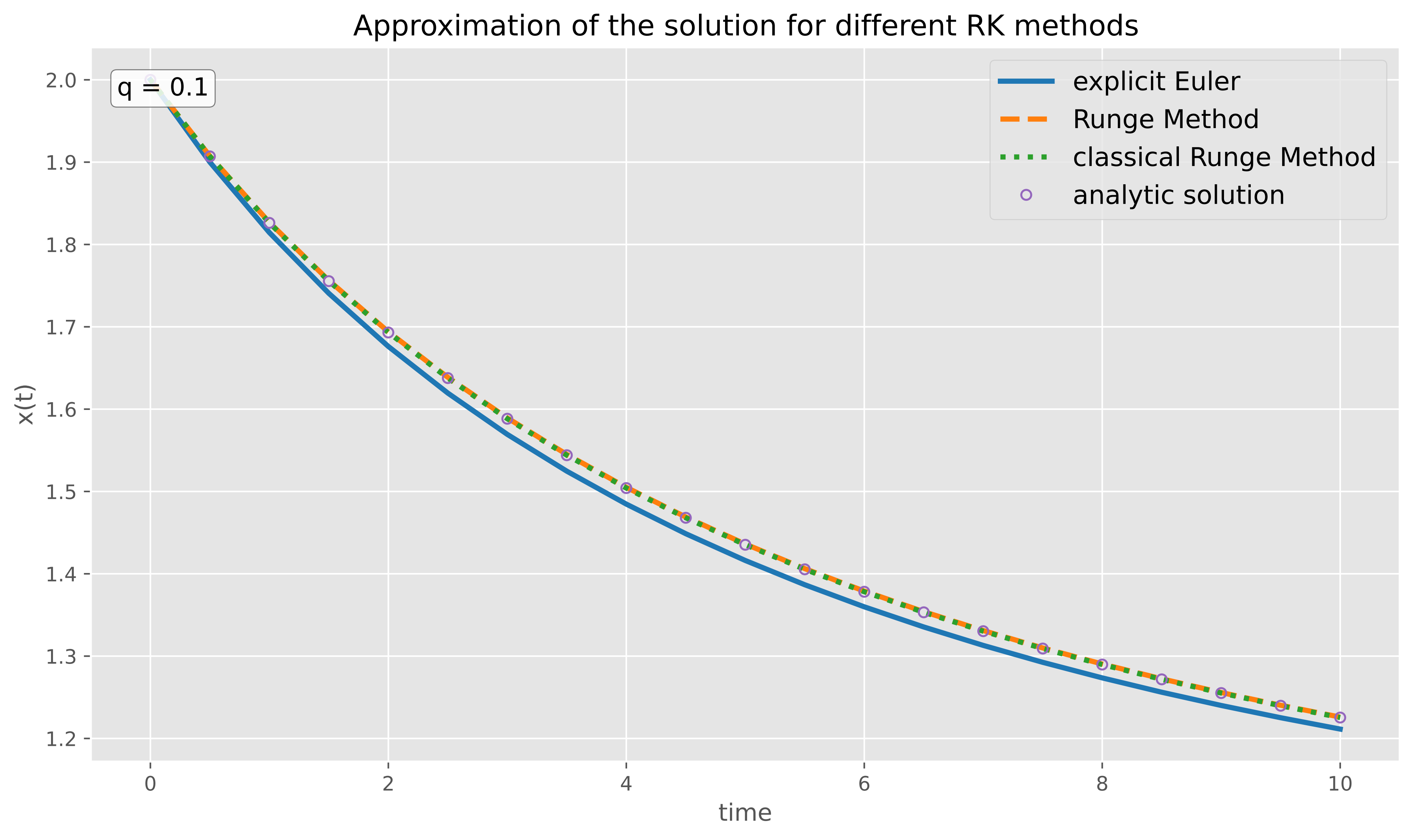
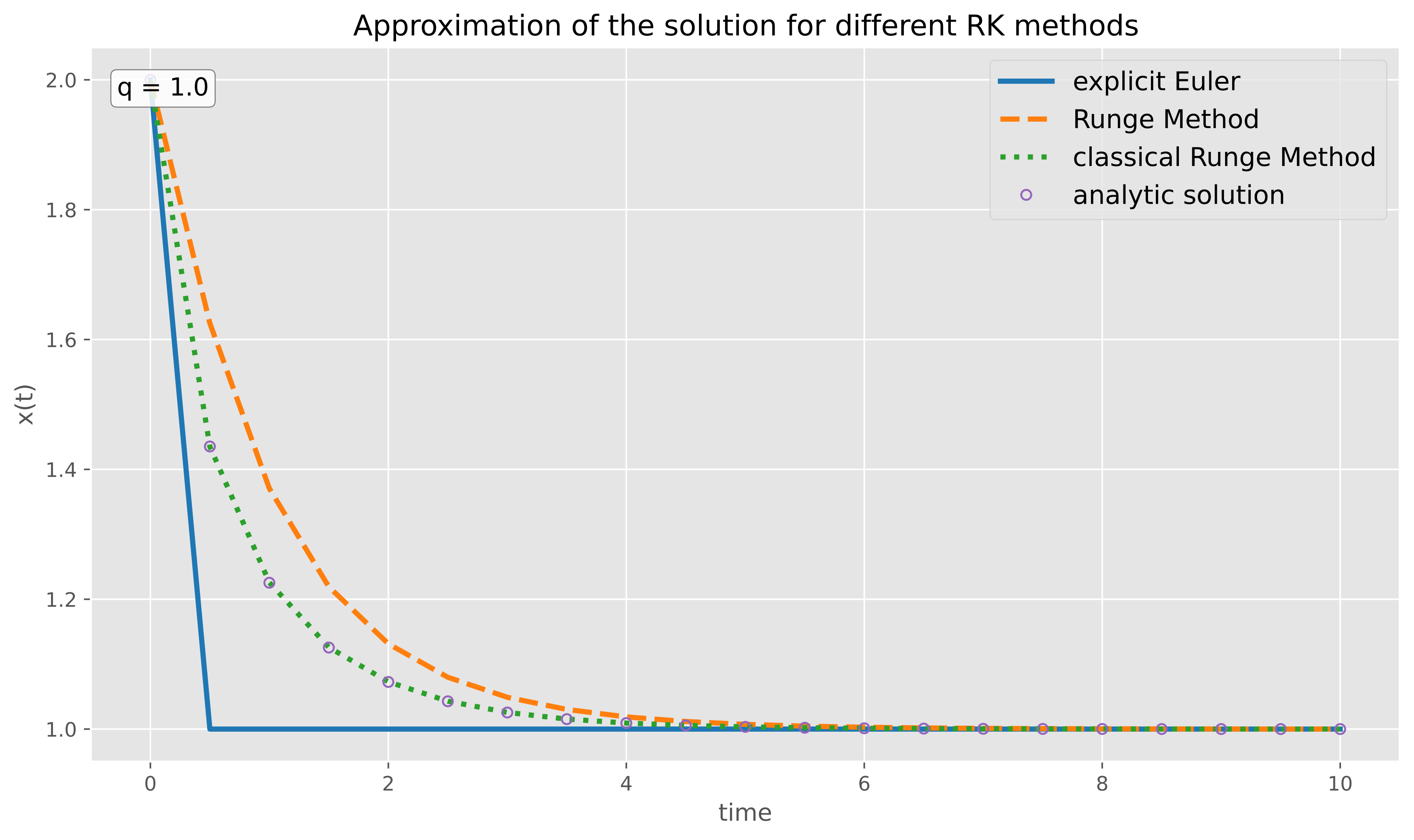
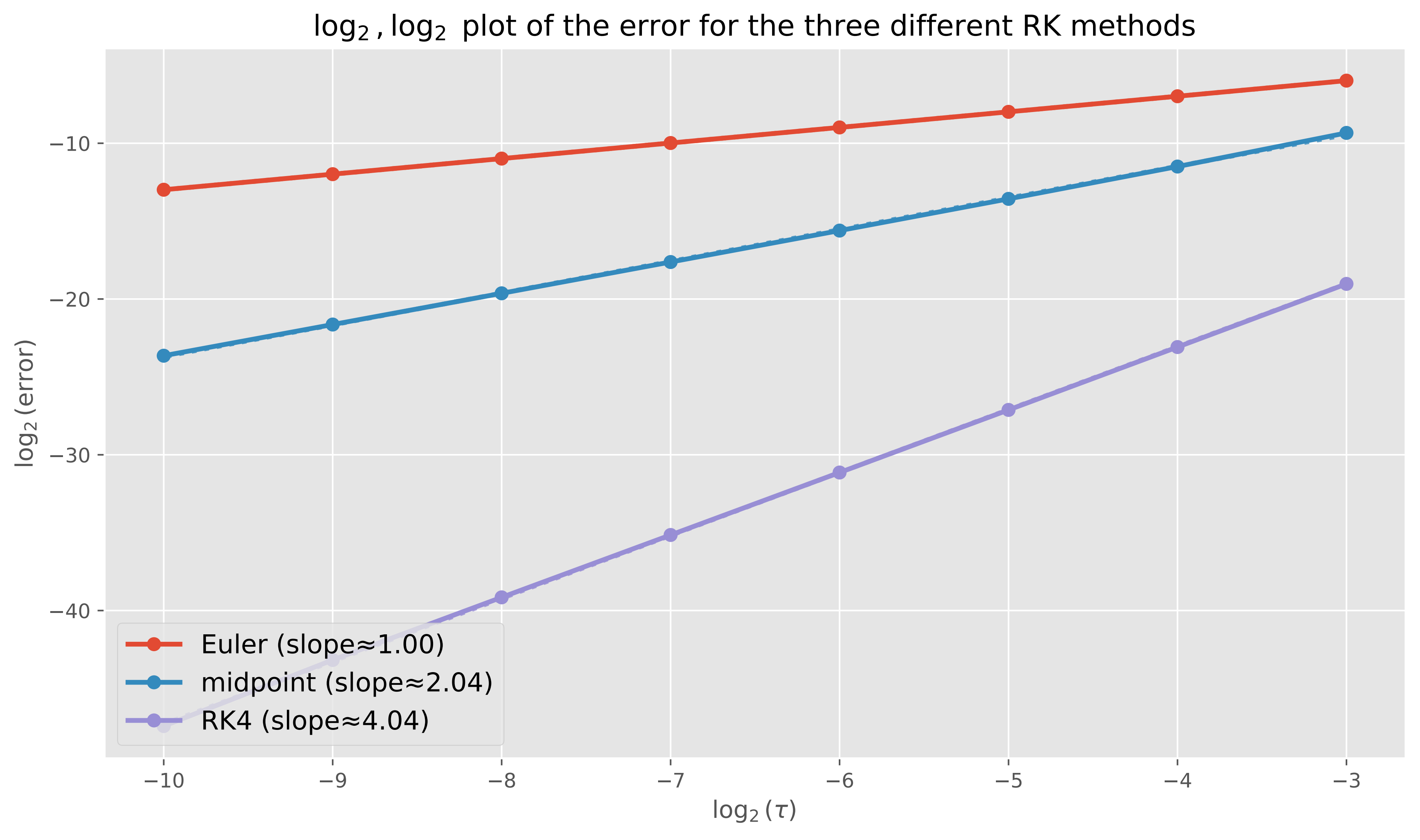


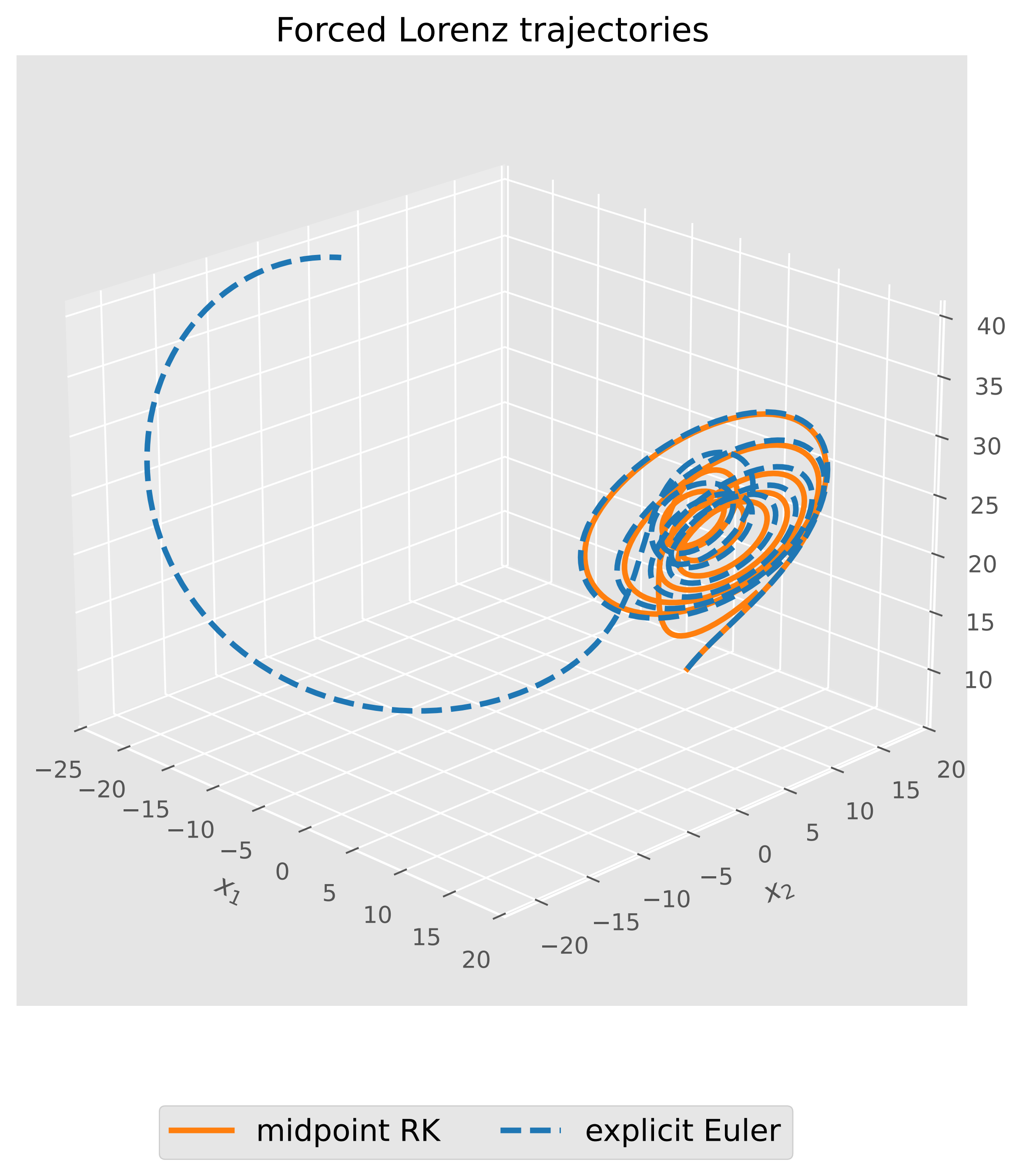
Approximation of the solution for different RK methods
q = 0.1
explicit Euler
Runge Method
classical Runge Method
analytic solution
x(t)
time
2.0
1.9
1.8
1.7
1.6
1.5
1.4
1.3
1.2
0
2
4
6
8
10



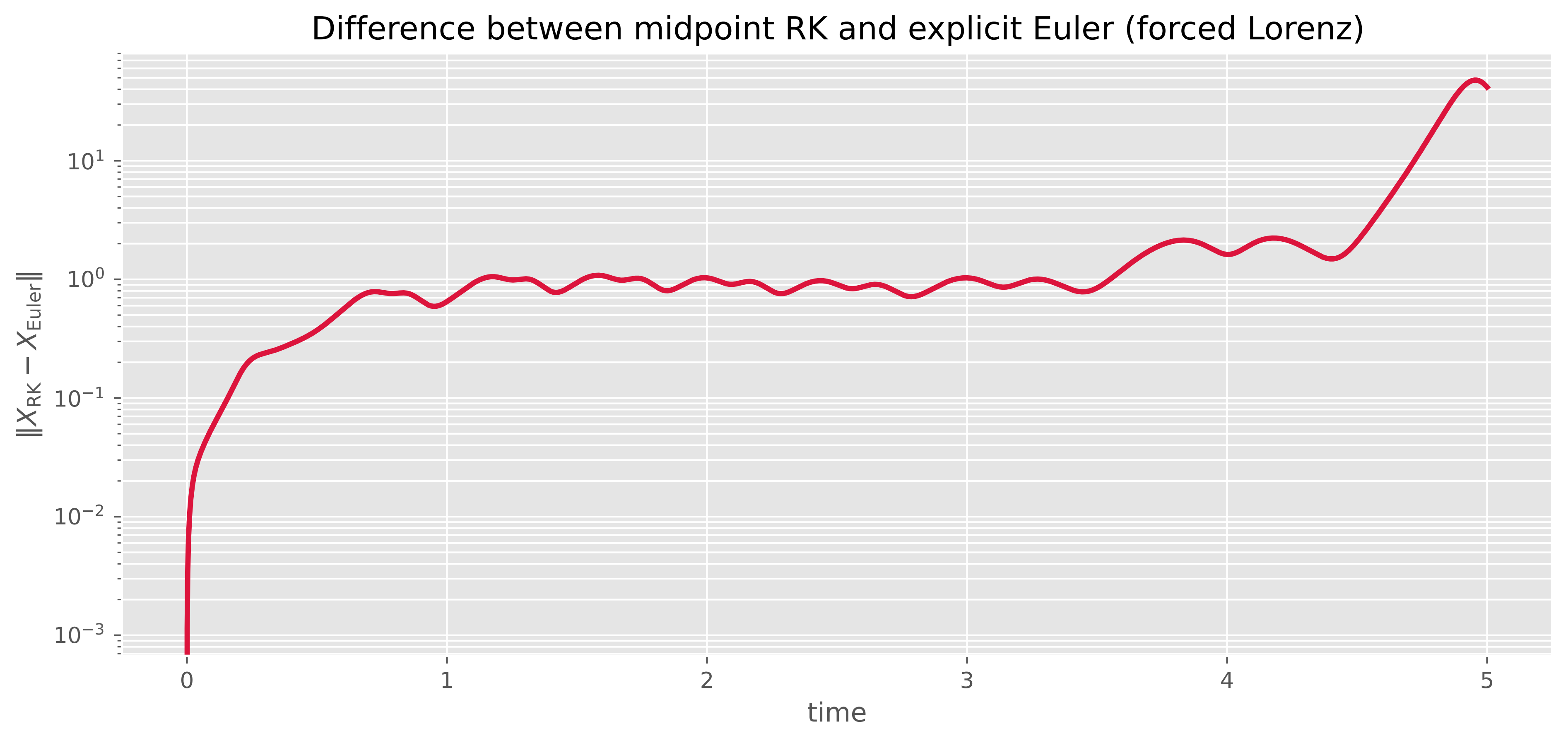
Approximation of the solution for different RK methods
q = 1.0
explicit Euler
Runge Method
classical Runge Method
analytic solution
x(t)
time
2.0
1.8
1.6
1.4
1.2
1.0
0
2
4
6
8
10



$\log_2, \log_2$ plot of the error for the three different RK methods
−10
−20
−30
−40
$\log_2$(error)
−10
−9
−8
−7
−6
−5
−4
−3
$\log_2(\tau)$
Euler (slope≈1.00)
midpoint (slope≈2.04)
RK4 (slope≈4.04)



Forced Lorenz trajectories
40
35
30
25
20
15
10
−25
−20
−15
−10
−5
0
5
10
15
20
x_1
20
15
10
5
0
−5
−10
−15
−20
x_2
midpoint RK
explicit Euler


Difference between midpoint RK and explicit Euler (forced Lorenz)
$\|X_{\mathrm{RK}} - X_{\mathrm{Euler}}\|$
10^{1}
10^{0}
10^{-1}
10^{-2}
10^{-3}
0
1
2
3
4
5
time

Numerical Methods for ODEs — Exercise 3

Solutions to Items 3) and 4)

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3) Convergence experiment: interpretation of the slope

Observation. In the $\log_2(\tau)$–$\log_2(\text{error})$ plot, the regression slopes are close to

$$\text{Euler} \approx 1, \qquad \text{midpoint} \approx 2, \qquad \text{RK4} \approx 4.$$

Meaning of the slope. For a one–step method of global order p the error on a fixed horizon satisfies

$$\|e(\tau)\| = C\,\tau^p + o(\tau^p), \qquad \tau \to 0.$$

Taking base–2 logarithms yields

$$\log_2(\|e(\tau)\|) = \log_2 C \; + \; p\,\log_2(\tau) + o(1).$$

Thus the slope in the $\log_2$–$\log_2$ diagram is an estimate of the convergence order p. In practice, a slope of p means that halving the step size reduces the error approximately by a factor 2^p.

Explanation of the observed slopes.

- Explicit Euler has local error $\mathcal{O}(\tau^2)$ and global error $\mathcal{O}(\tau)$, hence order $p = 1$.
- The two–stage midpoint method satisfies the second–order Runge–Kutta conditions, giving order $p = 2$.
- Classical RK4 (four stages) achieves order $p = 4$ with local error $\mathcal{O}(\tau^5)$.

Consequently, the observed slopes around 1, 2, and 4 match the theoretical global orders of the respective methods.

4) Forced Lorenz: comparison of midpoint and Euler

Observation. With $\tau = 0.001$ and $T = 5$, the midpoint method and the explicit Euler method produce trajectories that deviate from each other in the 3-D plot.

Explanation. The midpoint method is of order 2, while explicit Euler is only of order 1. Therefore, the truncation error per step is significantly smaller for the midpoint method. In addition, the Lorenz system amplifies small perturbations due to its chaotic nature. As a result, even small discrepancies from the discretization are magnified, and the Euler and midpoint trajectories separate visibly as time increases.