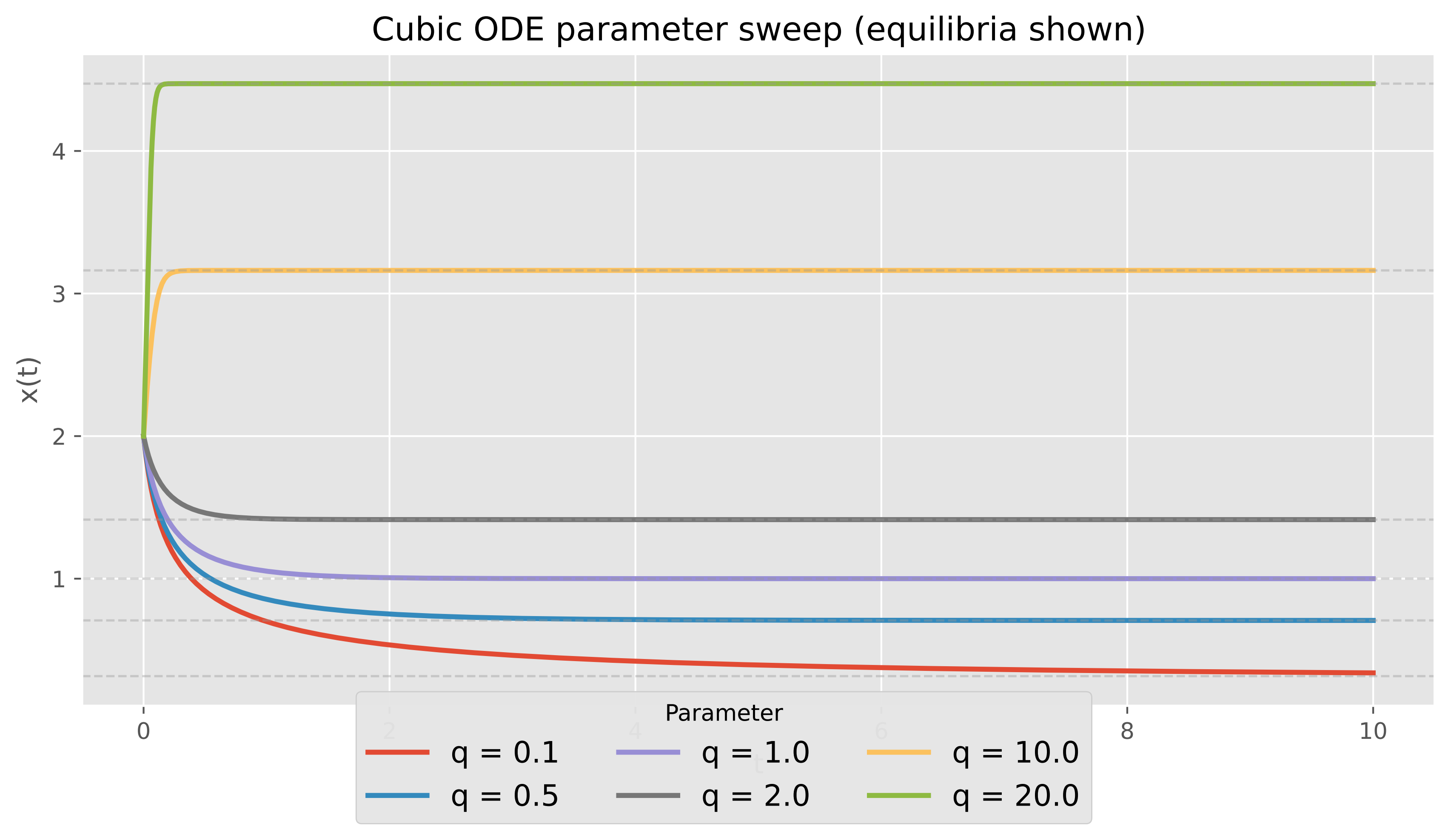
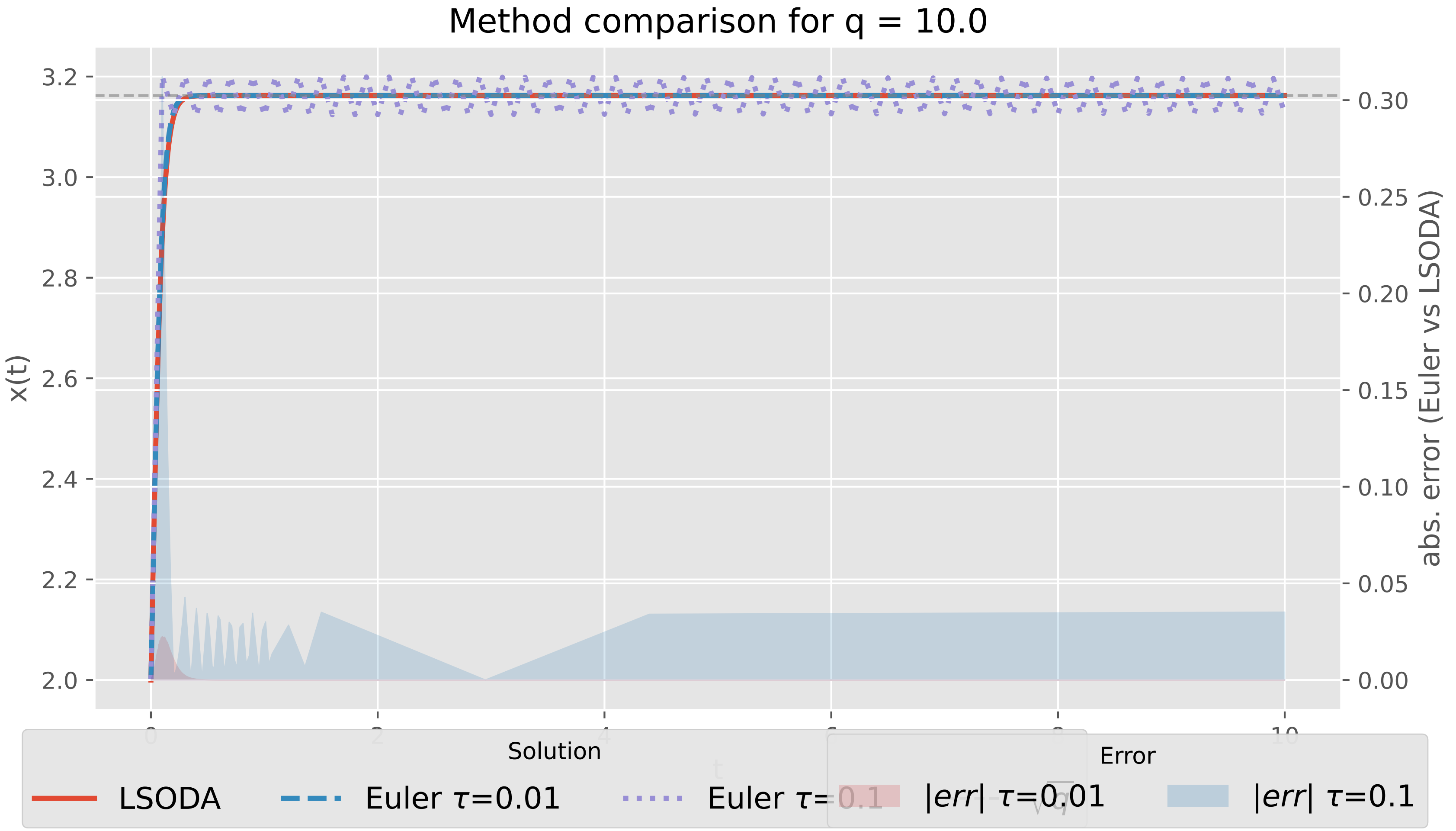
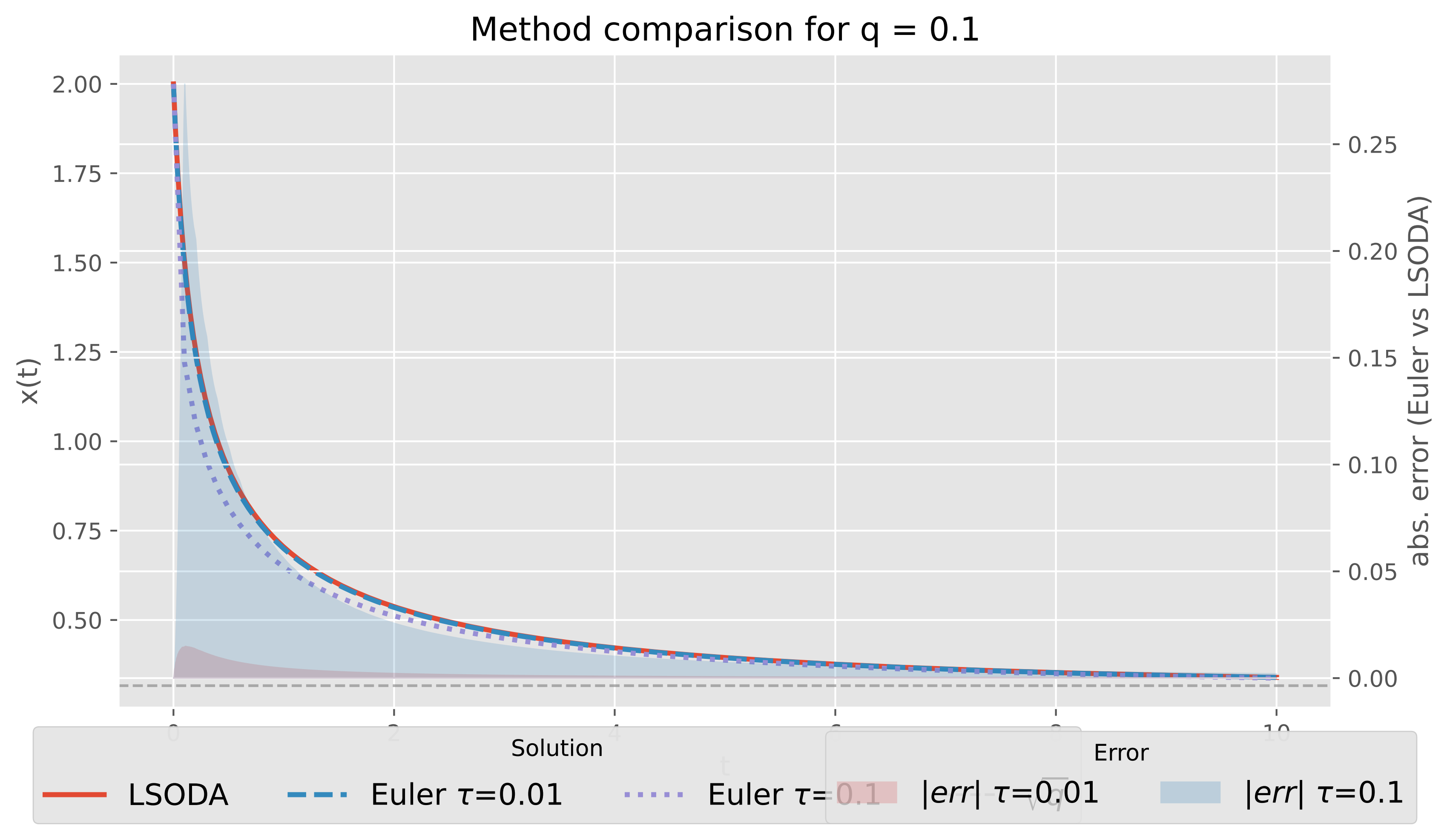


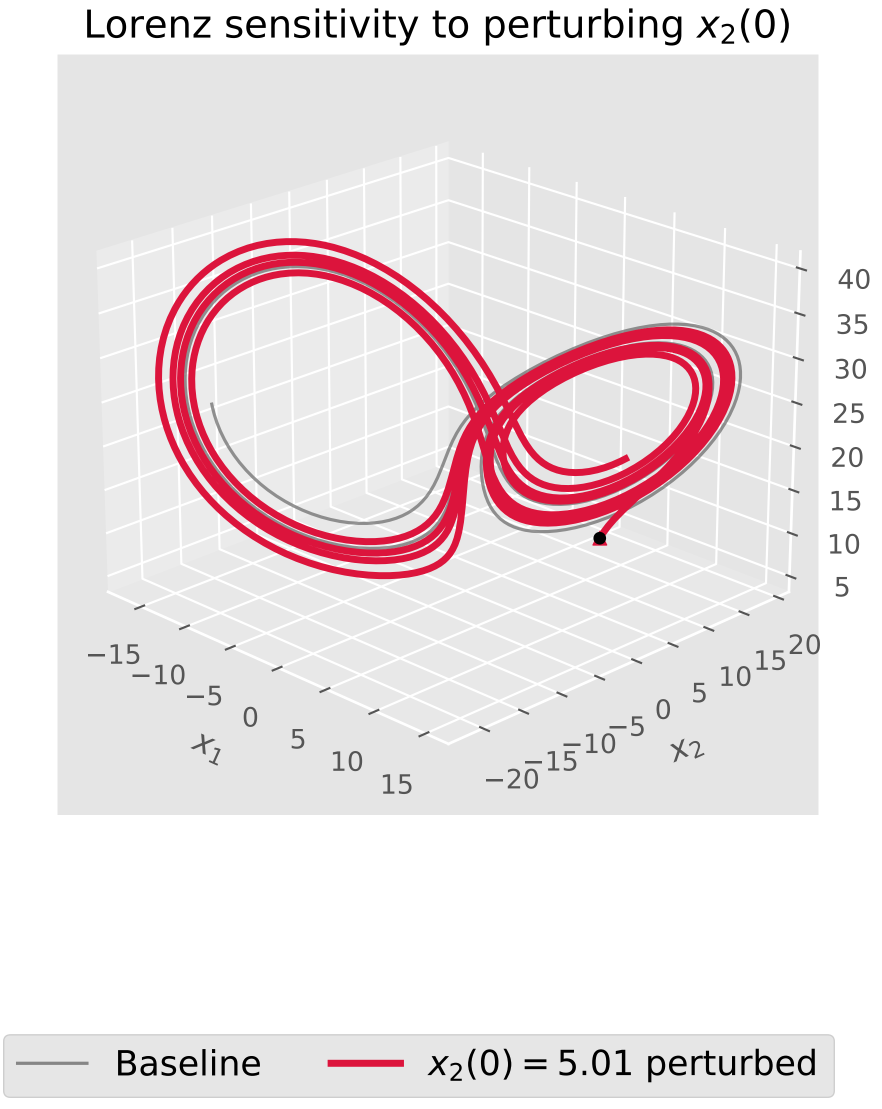
Cubic ODE parameter sweep (equilibria shown)
x(t)
t
0
2
4
6
8
10
1
2
3
4
Parameter
q = 0.1
q = 0.5
q = 1.0
q = 2.0
q = 10.0
q = 20.0



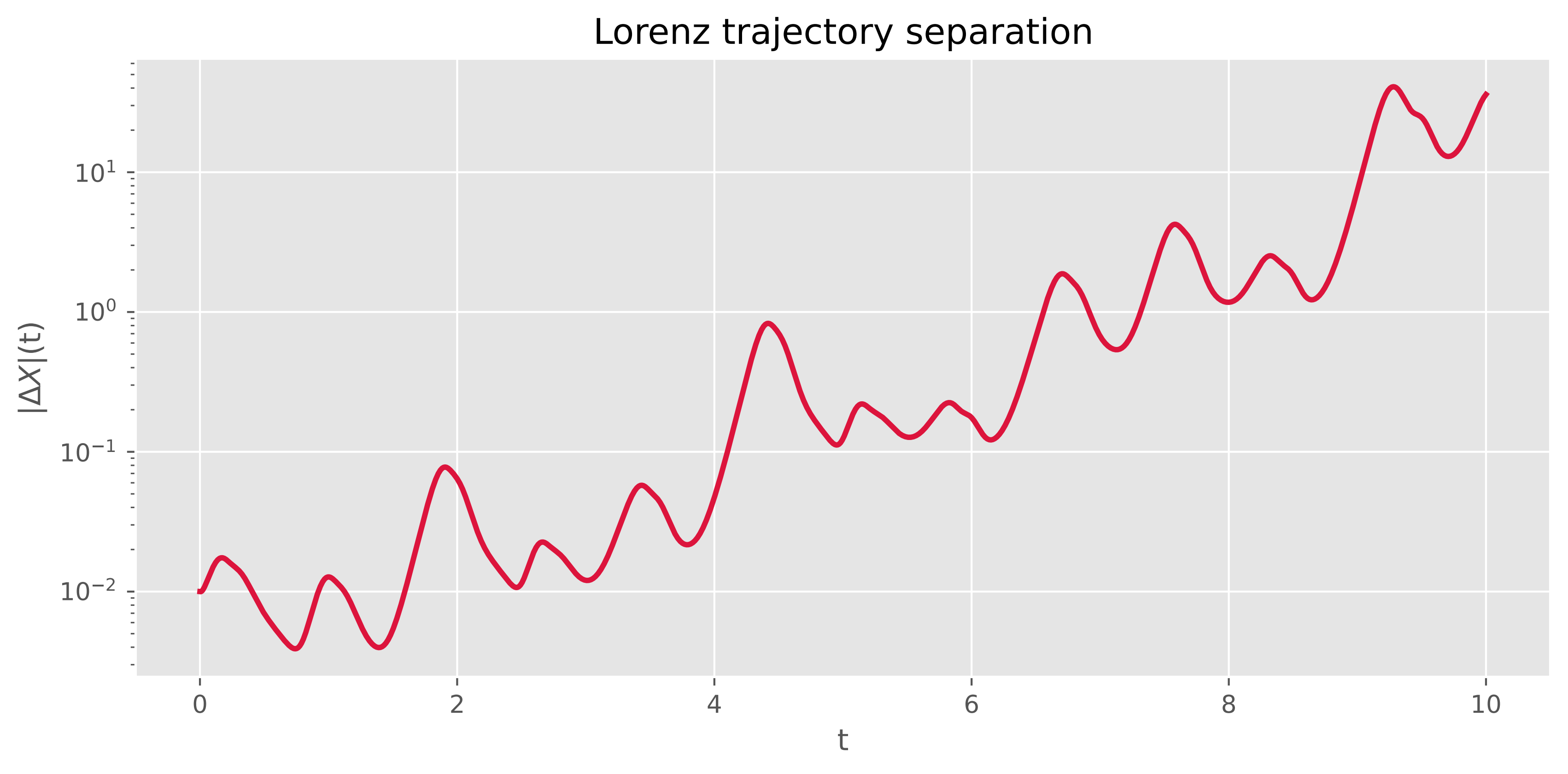
Method comparison for q = 10.0
x(t)
3.2
3.0
2.8
2.6
2.4
2.2
2.0
abs. error (Euler vs LSODA)
0.30
0.25
0.20
0.15
0.10
0.05
0.00
0
2
4
6
8
10
t
Solution
LSODA
Euler τ=0.01
Euler τ=0.1
Error
|err| τ=0.01
|err| τ=0.1

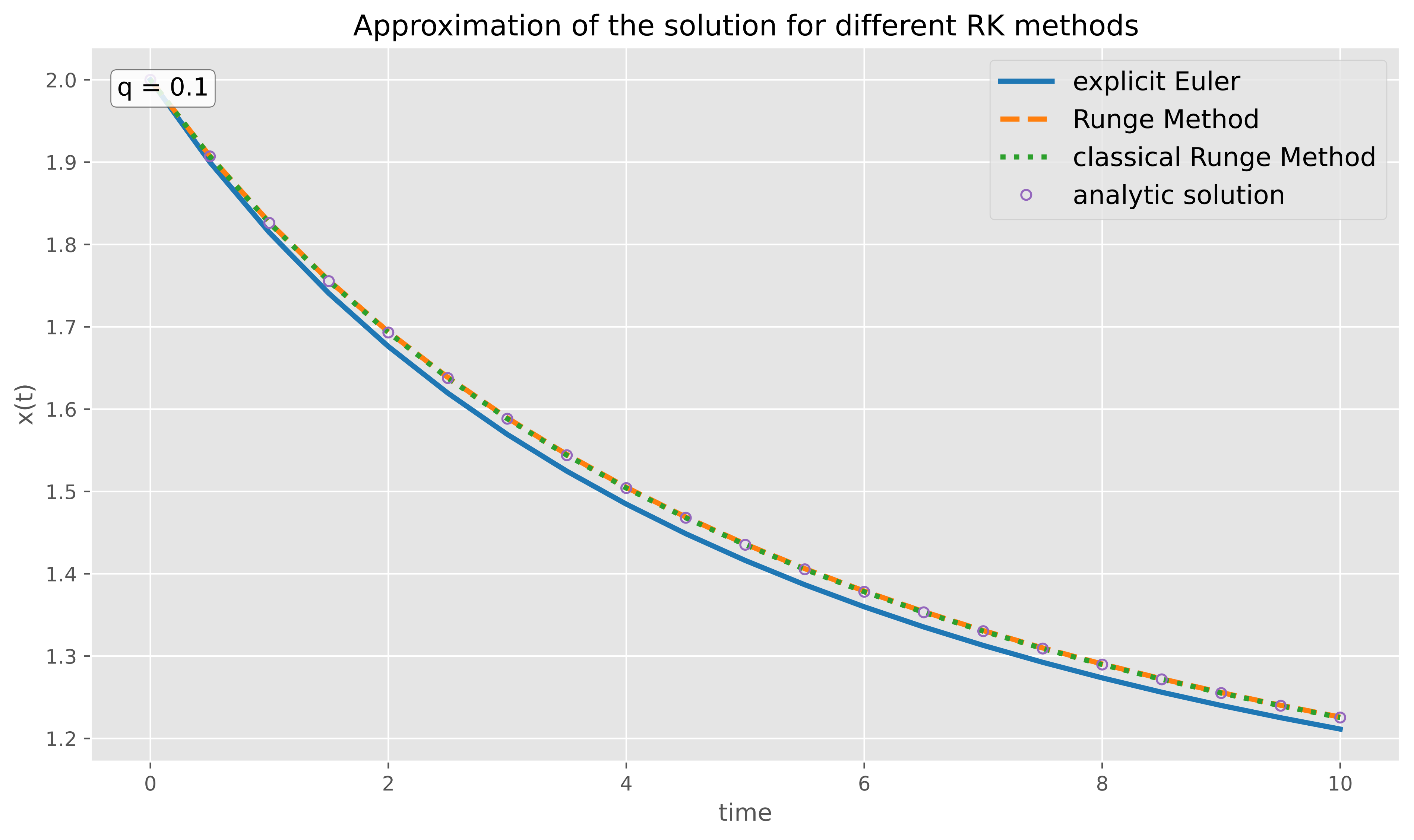


Method comparison for q = 0.1
x(t)
abs. error (Euler vs LSODA)
t
Solution
LSODA
Euler τ=0.01
Euler τ=0.1
Error
|err| τ=0.01
|err| τ=0.1

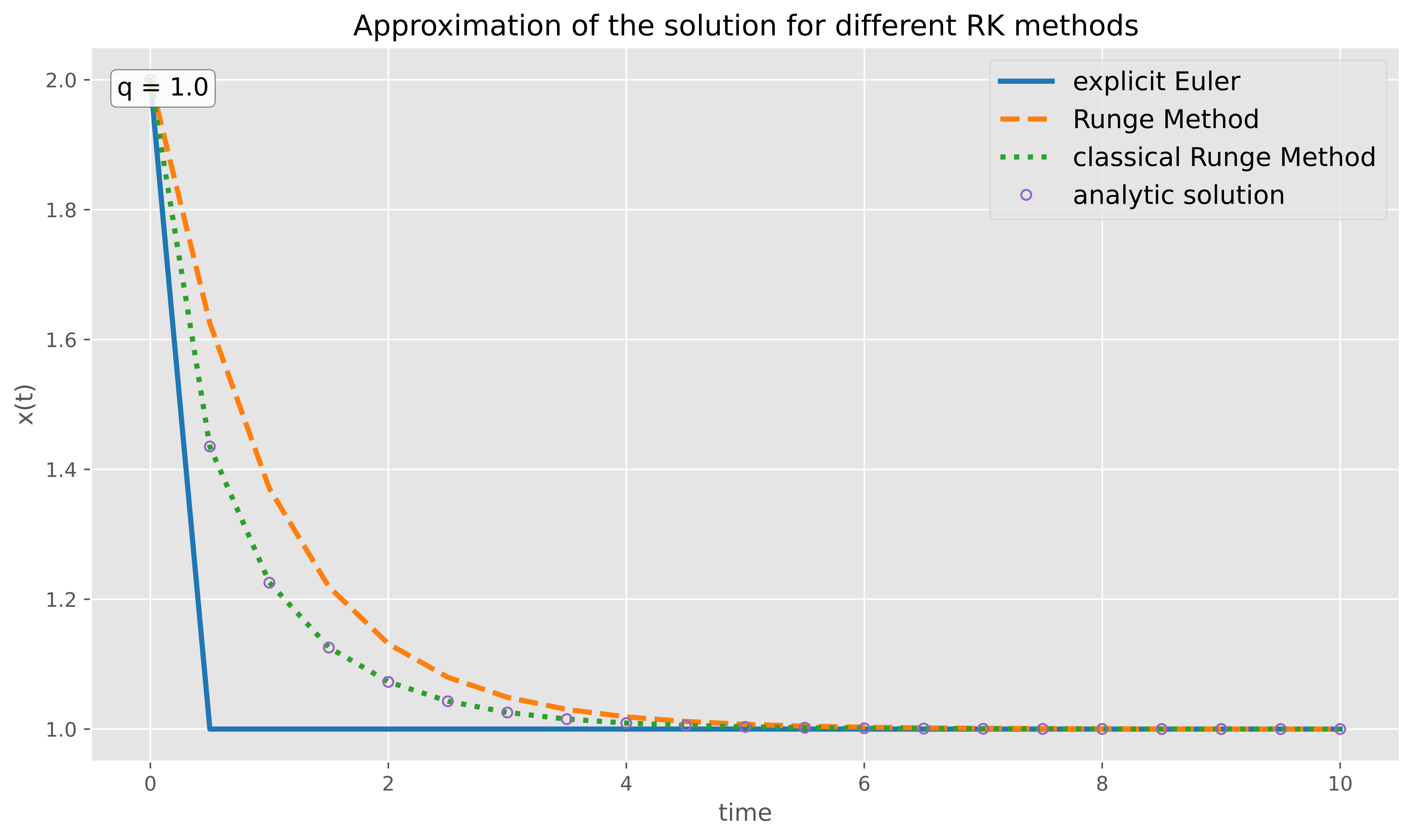


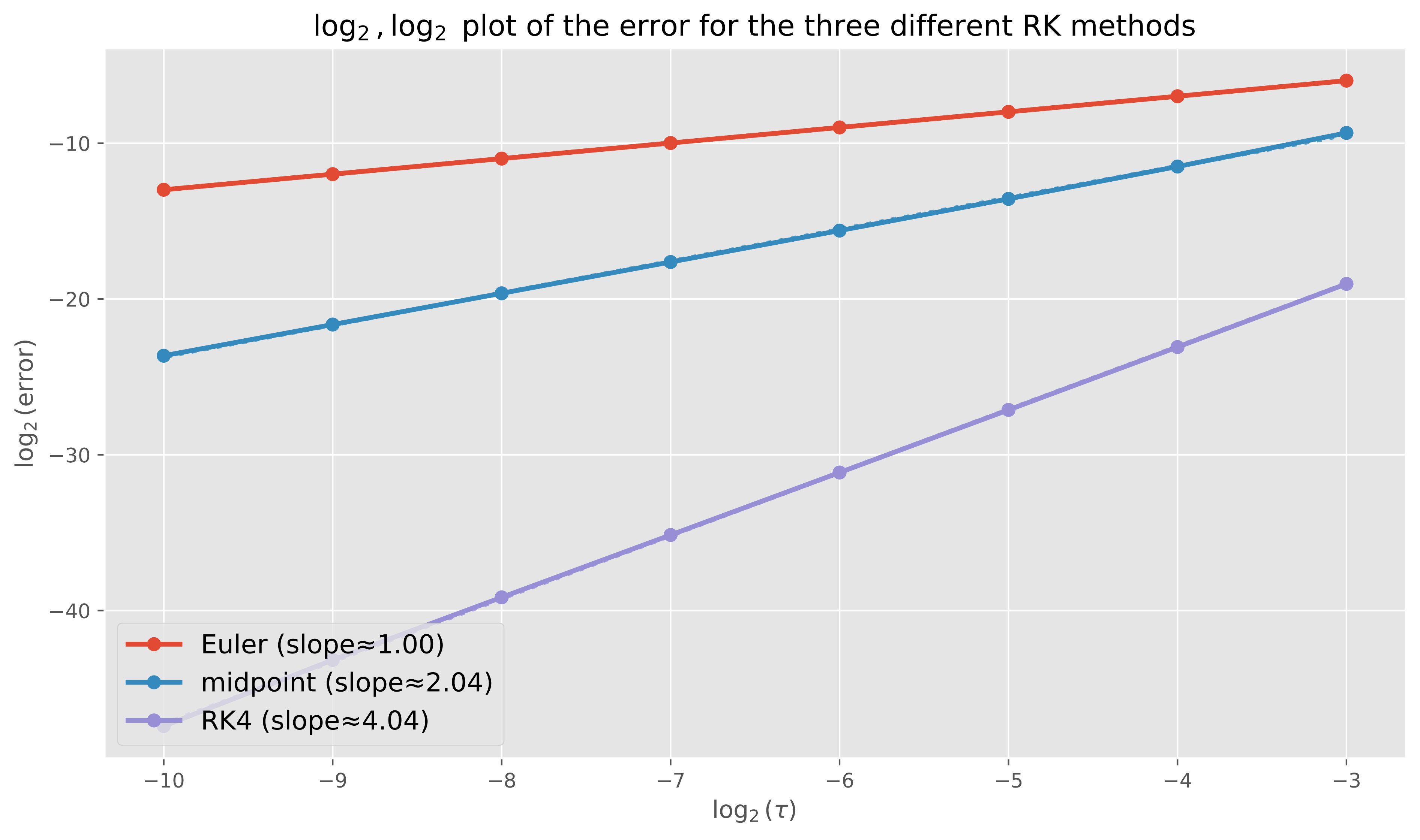
Lorenz sensitivity to perturbing $x_2(0)$
x_1
x_2
−15
−10
−5
0
5
10
15
−20
−15
−10
−5
0
5
10
15
20
5
10
15
20
25
30
35
40
Baseline
$x_2(0) = 5.01$ perturbed


Lorenz trajectory separation
10^{1}
10^{0}
10^{-1}
10^{-2}
$|\Delta X|(t)$
0
2
4
6
8
10
t

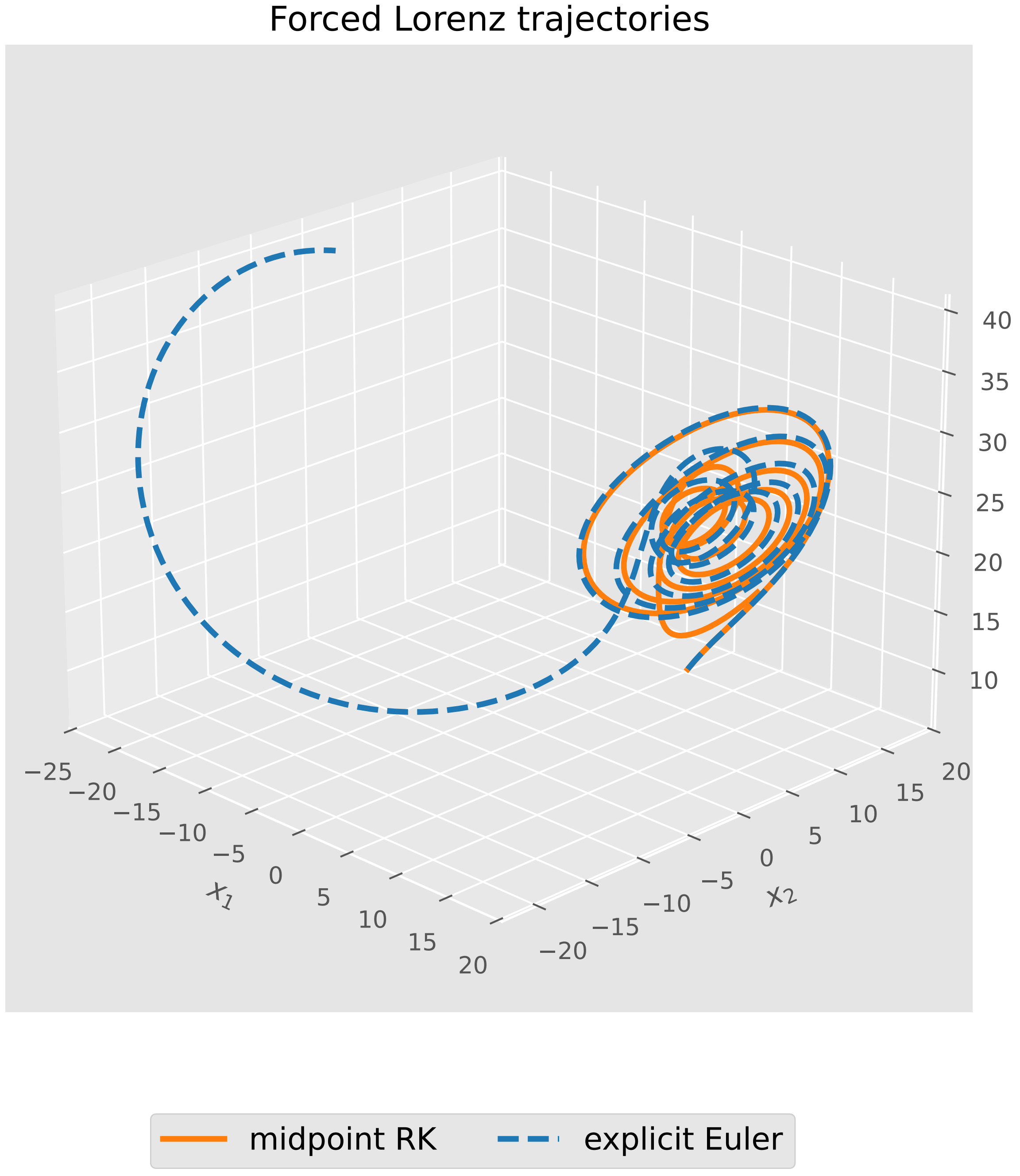


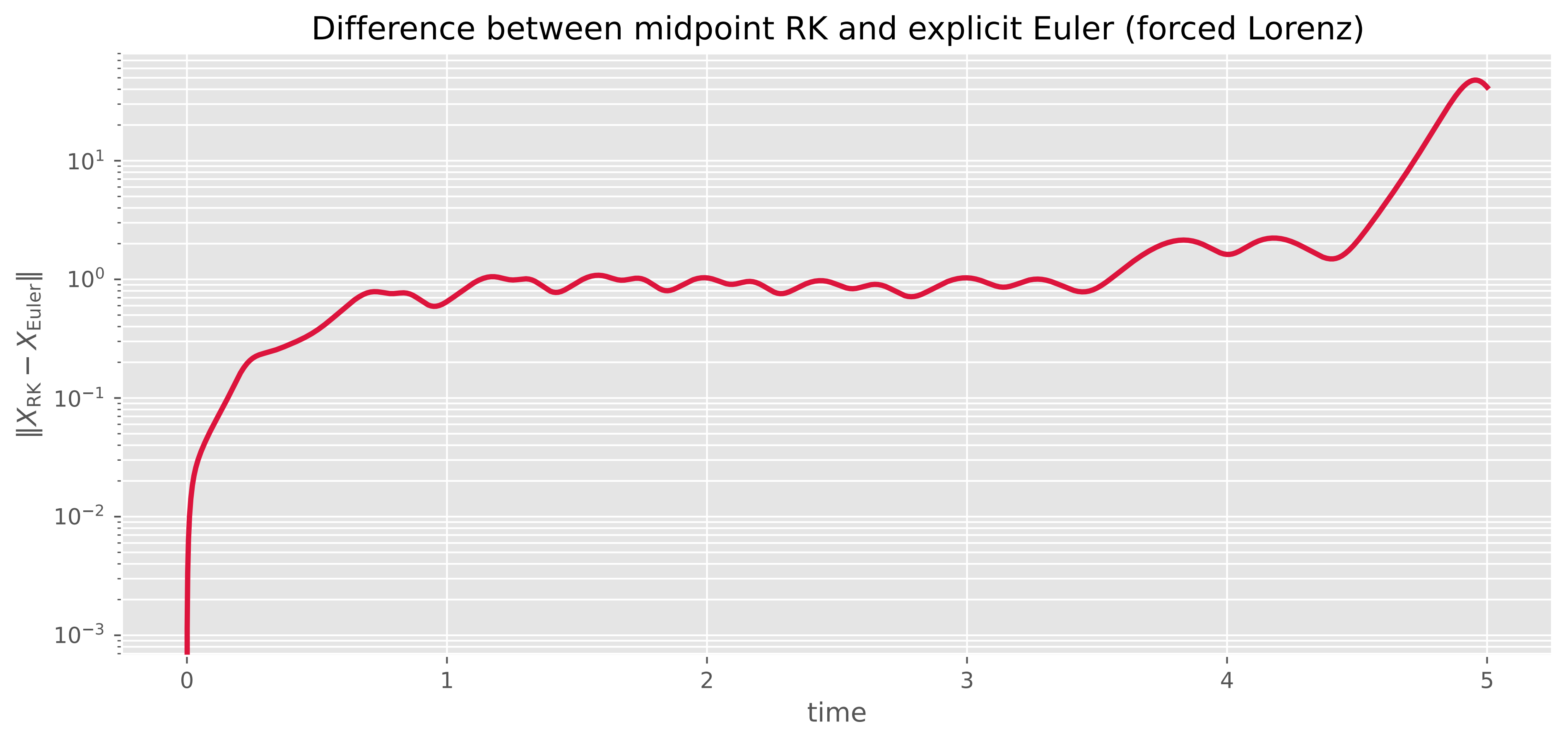
Approximation of the solution for different RK methods
q = 0.1
explicit Euler
Runge Method
classical Runge Method
analytic solution
x(t)
time
2.0
1.9
1.8
1.7
1.6
1.5
1.4
1.3
1.2
0
2
4
6
8
10


Approximation of the solution for different RK methods
q = 1.0
explicit Euler
Runge Method
classical Runge Method
analytic solution
x(t)
time
2.0
1.8
1.6
1.4
1.2
1.0
0
2
4
6
8
10

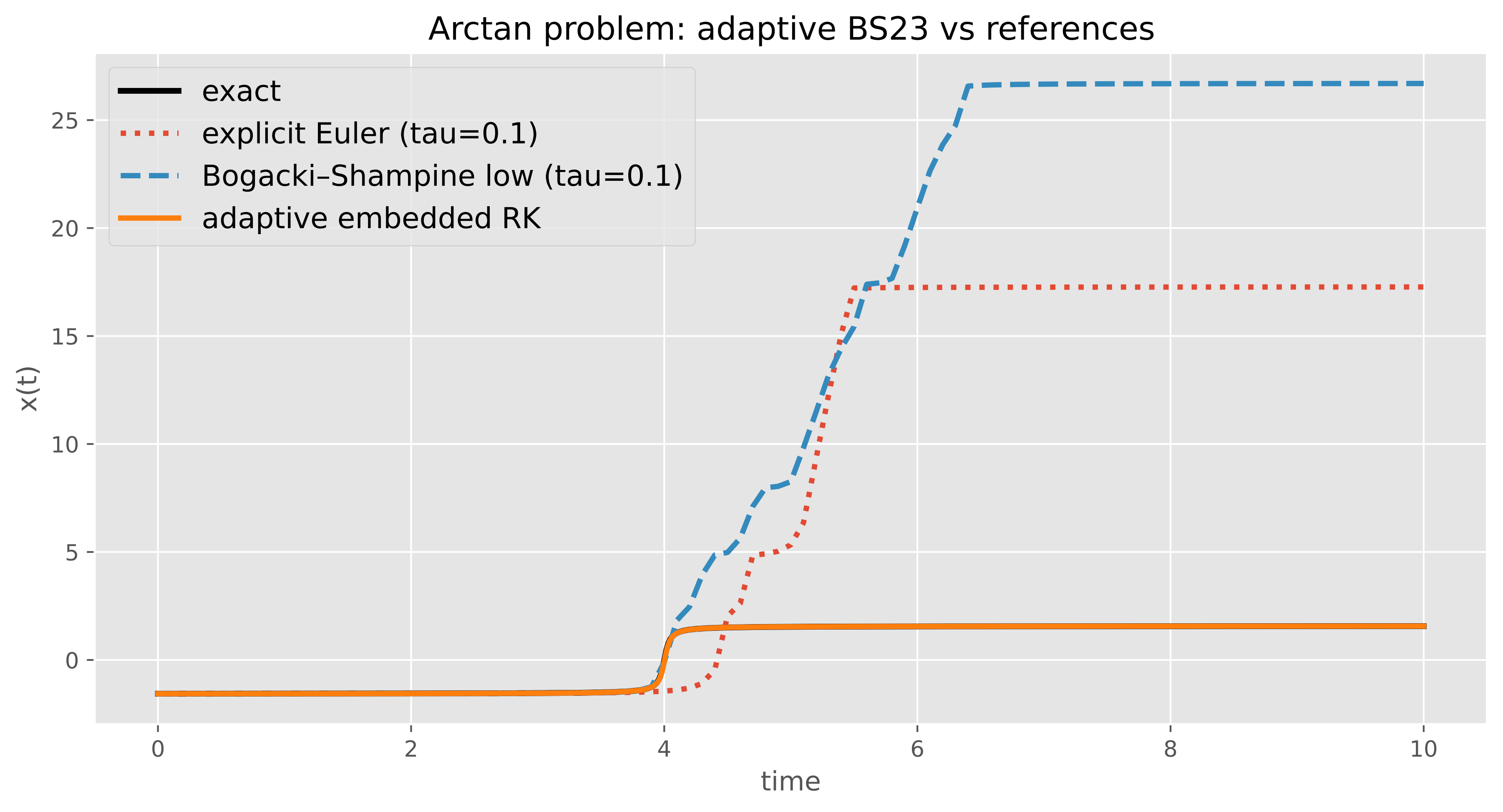


$\log_2, \log_2$ plot of the error for the three different RK methods
−10
−20
−30
−40
$\log_2(\text{error})$
−10
−9
−8
−7
−6
−5
−4
−3
$\log_2(\tau)$
Euler (slope≈1.00)
midpoint (slope≈2.04)
RK4 (slope≈4.04)

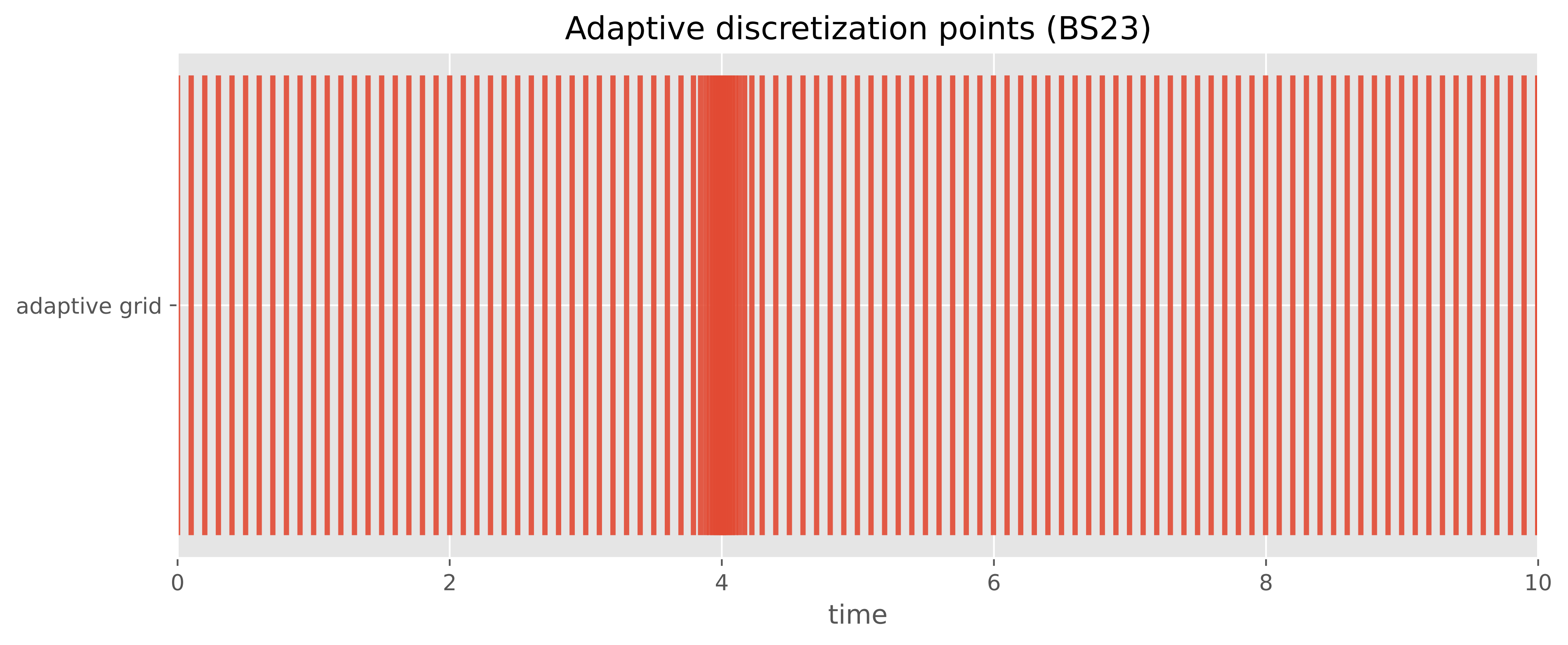

Forced Lorenz trajectories
40
35
30
25
20
15
10
−25
−20
−15
−10
−5
0
5
10
15
20
x_1
20
15
10
5
0
−5
−10
−15
−20
x_2
midpoint RK
explicit Euler



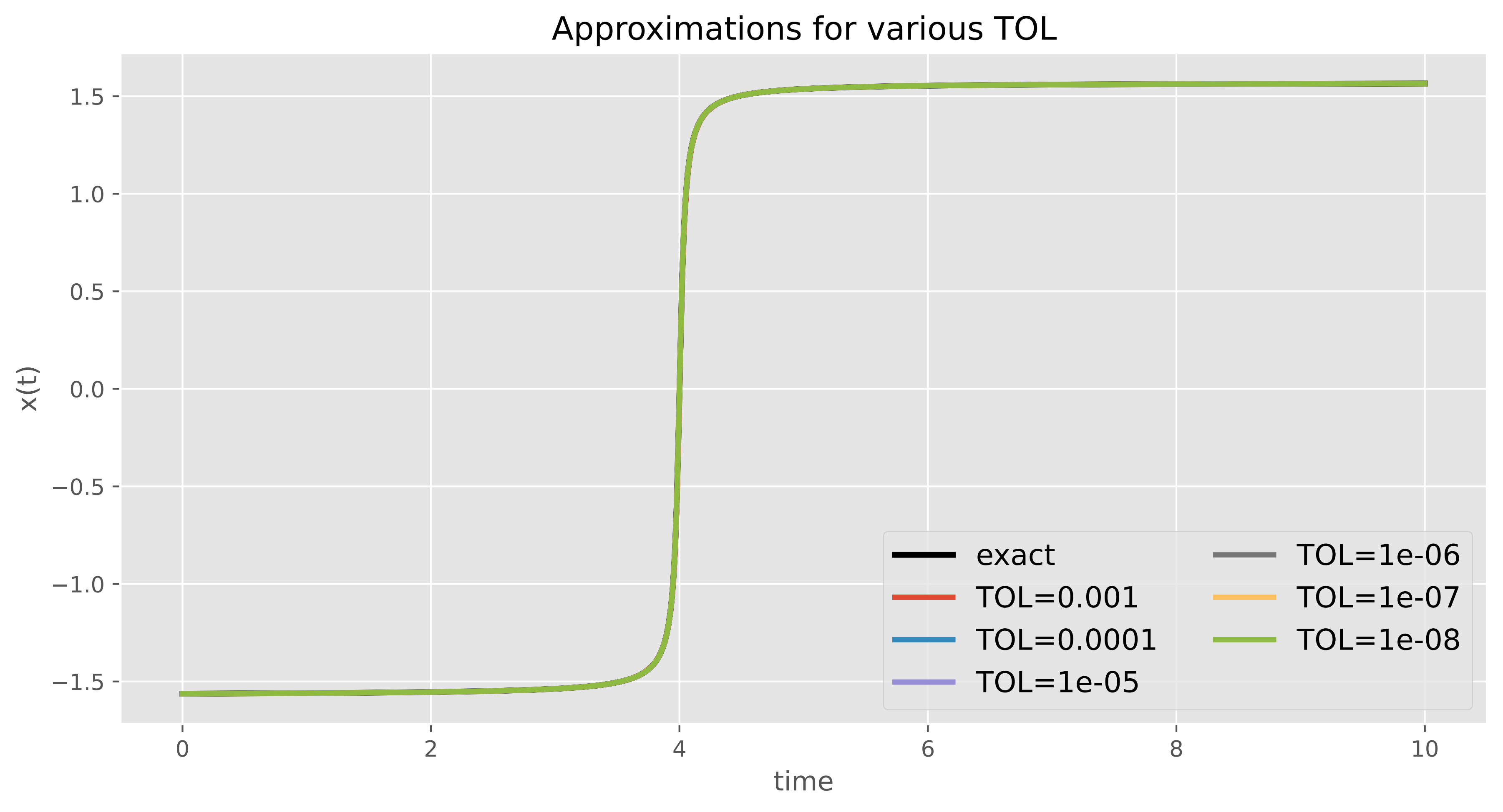
Difference between midpoint RK and explicit Euler (forced Lorenz)
$\|X_{\mathrm{RK}} - X_{\mathrm{Euler}}\|$
10^{1}
10^{0}
10^{-1}
10^{-2}
10^{-3}
0
1
2
3
4
5
time

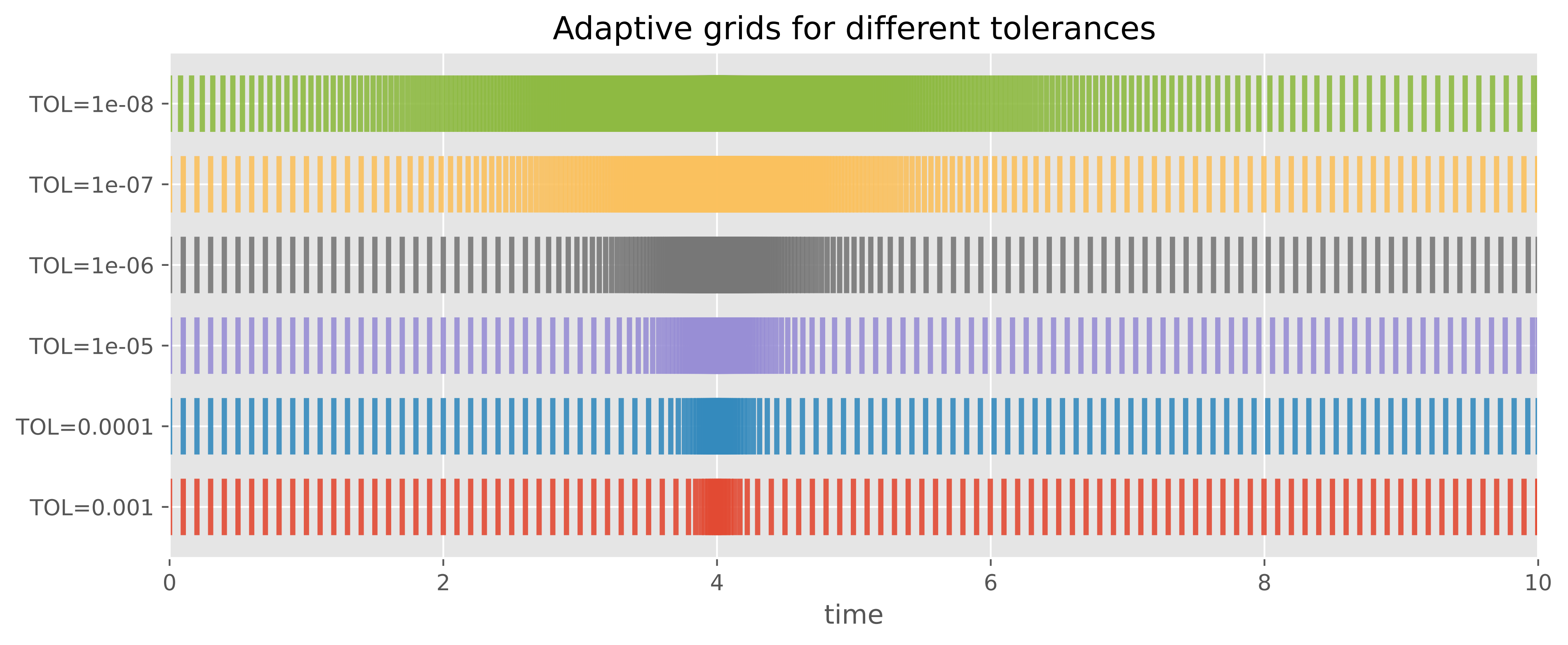


Arctan problem: adaptive BS23 vs references
exact
explicit Euler (tau=0.1)
Bogacki–Shampine low (tau=0.1)
adaptive embedded RK
x(t)
time
0
5
10
15
20
25
0
2
4
6
8
10

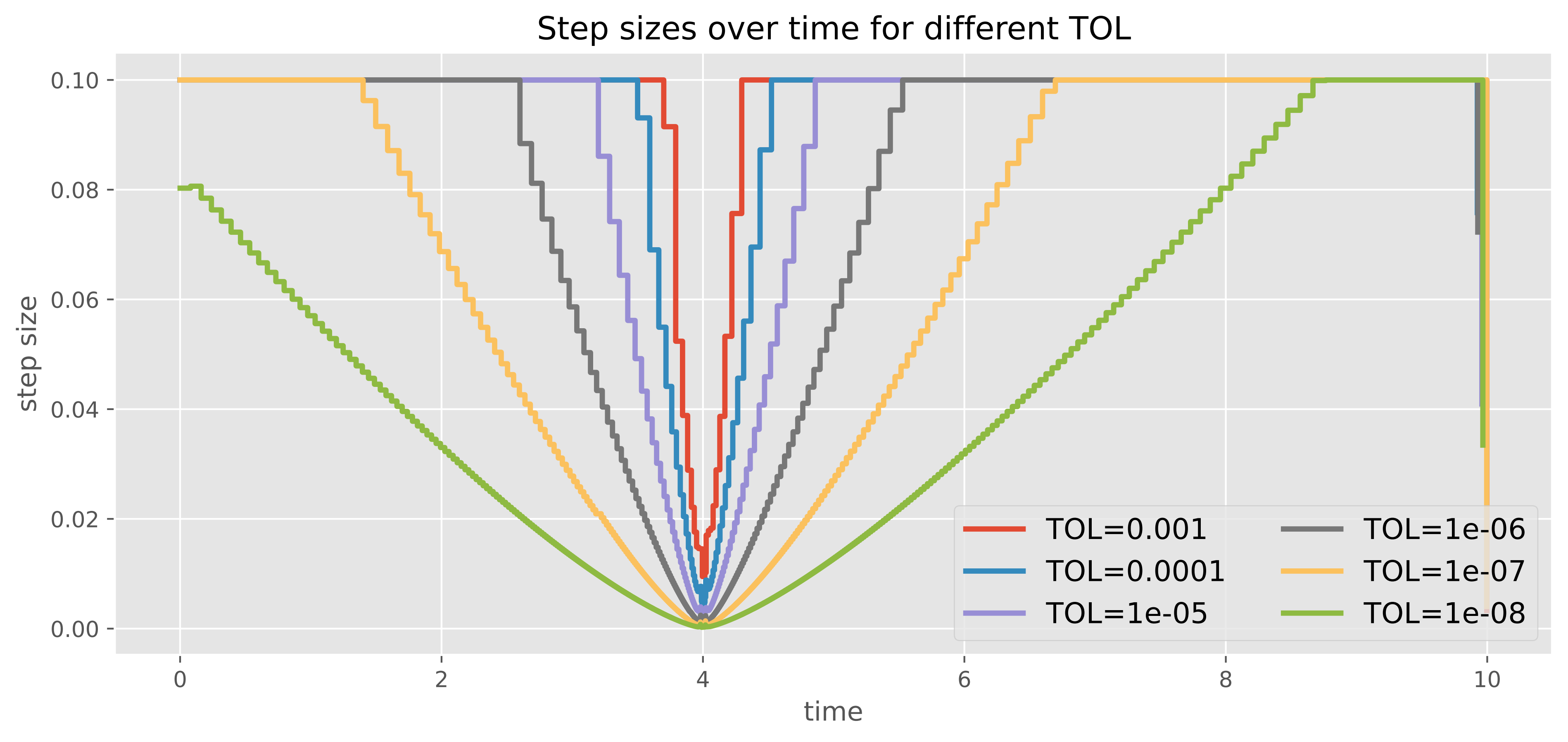


Adaptive discretization points (BS23)
adaptive grid
0
2
4
6
8
10
time

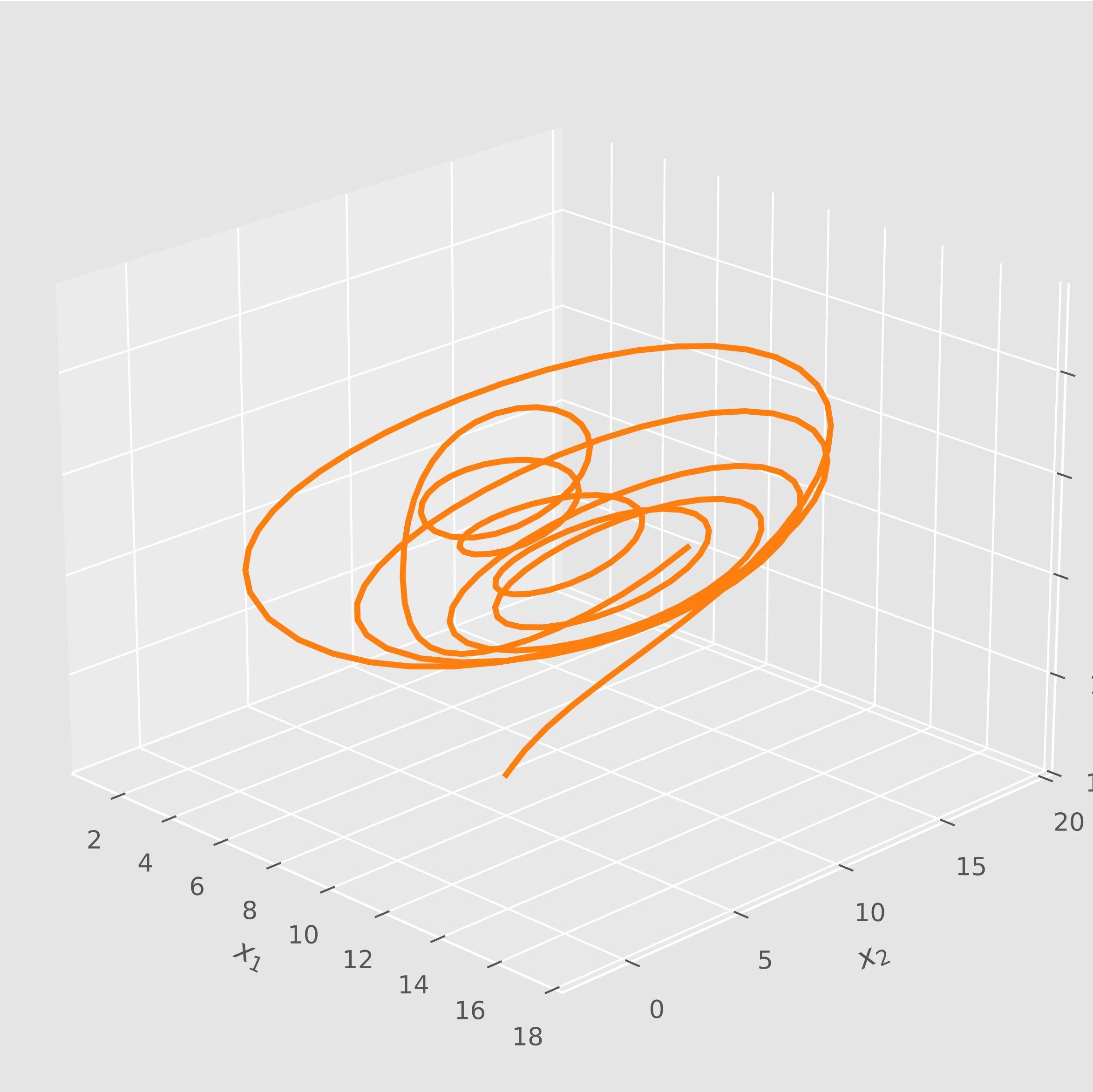

Approximations for various TOL
x(t)
time
exact
TOL=0.001
TOL=0.0001
TOL=1e-05
TOL=1e-06
TOL=1e-07
TOL=1e-08
1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
0
2
4
6
8
10

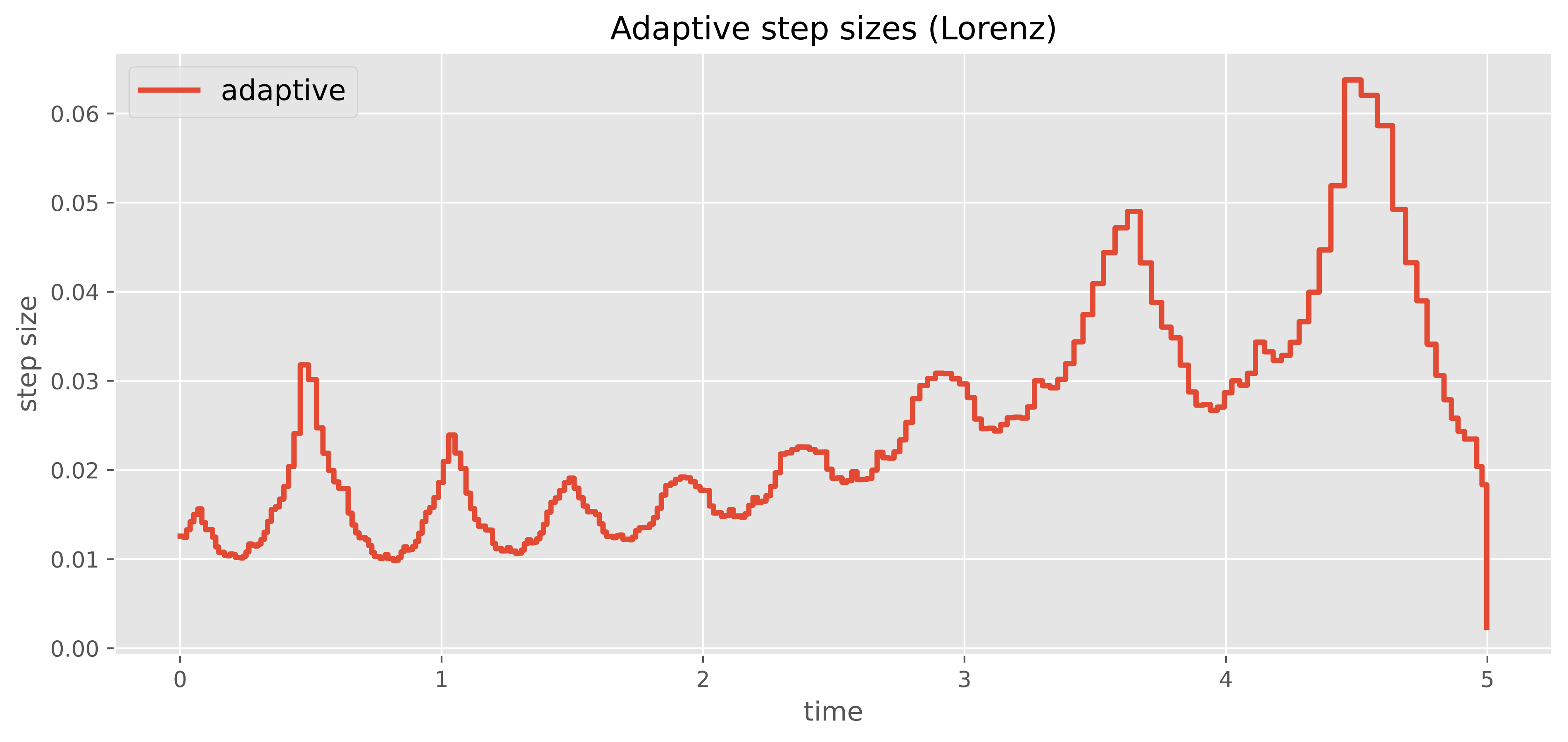


Adaptive grids for different tolerances
TOL=1e-08
TOL=1e-07
TOL=1e-06
TOL=1e-05
TOL=0.0001
TOL=0.001
0
2
4
6
8
10
time


Step sizes over time for different TOL
step size
time
0.10
0.08
0.06
0.04
0.02
0.00
0
2
4
6
8
10
TOL=0.001
TOL=0.0001
TOL=1e-05
TOL=1e-06
TOL=1e-07
TOL=1e-08

Forced Lorenz (adaptive BS23)





Adaptive step sizes (Lorenz)
adaptive
0.06
0.05
0.04
0.03
0.02
0.01
0.00
step size
0
1
2
3
4
5
time

Answers

(b) Long-term behaviour as a function of q

\textbf{(b) Long-term behaviour as a function of q.}

We solve $x'(t) = qx - x^3$ with $x(0) = 2$ and $q > 0$. The equilibria are
0 and $\pm\sqrt{q}$. For $q > 0$, $x = 0$ is unstable and $x = \pm\sqrt{q}$ are
asymptotically stable since $f'(x) = q - 3x^2$ implies $f'(\pm\sqrt{q}) = -2q < 0$.

With $x(0) = 2 > 0$, the trajectory approaches the stable equilibrium $+\sqrt{q}$:
\[
\begin{cases}
q<4\!: & \sqrt{q}<2 \Rightarrow x(t)\ \text{decreases monotonically to}\ \sqrt{q},\\[2pt]
q=4\!: & x(t)\equiv 2\ \text{(equilibrium; adding this case gives a flat line at
2)},\\[2pt]
q>4\!: & \sqrt{q}>2 \Rightarrow x(t)\ \text{increases monotonically to}\ \sqrt{q}.
\end{cases}
\]
This matches the parameter-sweep plot: for small q the approach to $\sqrt{q}$ is slower,
so at $T = 10$ the solution can still be slightly above the limiting value.

(c) Method comparison (Euler vs. LSODA) and effect of q

\textbf{(c) Method comparison (Euler vs.\ LSODA) and effect of q.}

We compare explicit Euler with step sizes $\tau = 0.1$ and $\tau = 0.01$ against an
LSODA reference on $[0, 10]$.

\emph{Accuracy order.} Explicit Euler is first order: the global error scales
as $\mathcal{O}(\tau)$ for smooth problems on a fixed time horizon. Hence,
reducing τ from 0.1 to 0.01 should reduce the error by about a factor
of 10 (modulo transients).

\emph{Linear stability near the attractor.} Linearizing at the stable equilibrium
$x^* = \sqrt{q}$ gives $y' = f'(x^*)\,y = -2q\,y$. For the test equation
$y' = \lambda y$ with $\lambda = -2q$, explicit Euler is stable iff
\[
|1+\tau\lambda|<1 \quad\Longleftrightarrow\quad 0<\tau<\frac{1}{q}.
\]

\emph{Case $q = 10$.} The stability bound is $\tau < 0.1$, so $\tau = 0.1$ lies
\emph{on the boundary} and yields visible phase/amplitude error and mild
oscillation around the equilibrium; $\tau = 0.01$ is well inside the stable
region and closely tracks LSODA. Empirically, the absolute error curve for
$\tau = 0.1$ sits roughly an order of magnitude above that for $\tau = 0.01$ over
most of $[0, 10]$, consistent with first-order convergence \emph{and} the
stability-edge effect at $\tau = 0.1$.

\emph{Case $q = 0.1$.} The bound is $\tau < 10$, so both $\tau = 0.1$ and 0.01
are deep inside the stability region and the dynamics are slow. Both Euler
solutions lie very close to LSODA; the $\tau = 0.01$ error is still smaller
(by about the expected $\sim 10\times$ factor), but the difference is barely
visible in the solution plot because all errors are small.

(d) Sensitivity for the Lorenz system

\textbf{(d) Sensitivity for the Lorenz system.}

With standard parameters $(a, b, c) = (10, 25, 8/3)$ the Lorenz system exhibits
sensitive dependence on initial conditions (positive largest Lyapunov exponent).
We integrate on $[0, 10]$ with explicit Euler ($\tau = 0.001$) from
$(x_1(0), x_2(0), x_3(0)) = (10, 5, 12)$ and from the perturbed
$(10, 5.01, 12)$.

The two trajectories coincide initially but separate clearly after a short time,
ultimately exploring different parts of the attractor. This is the expected
behaviour for a chaotic system: for a small perturbation $\|\delta x(0)\|$ the
separation typically grows like $\|\delta x(t)\| \approx\ \|\delta x(0)\|\, e^{\lambda t}$
with $\lambda > 0$.

\emph{Conclusion.} Yes, the solution changes significantly when $x_2(0)$ is
perturbed to 5.01; the 3D plot makes this divergence clearly visible.