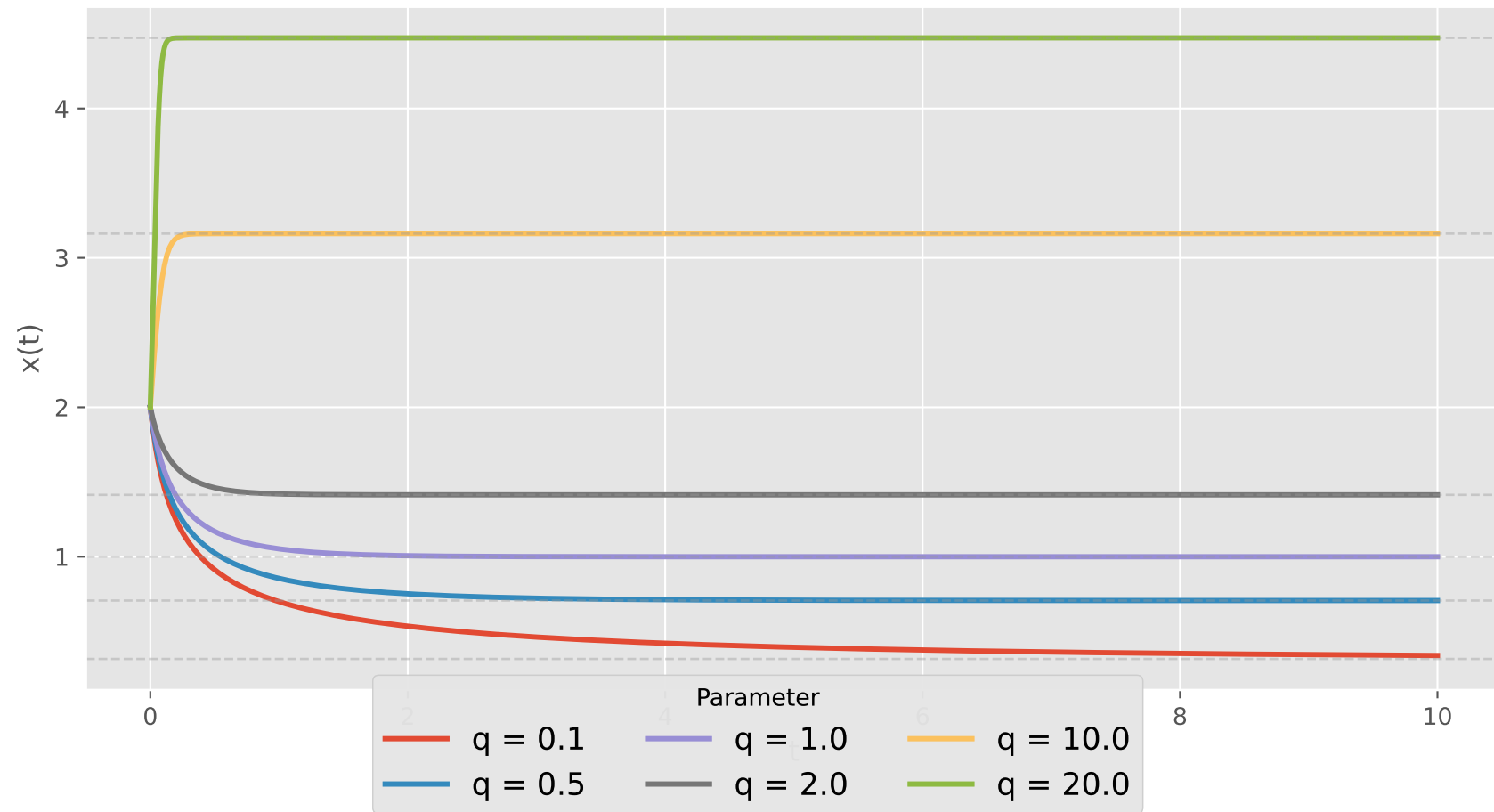
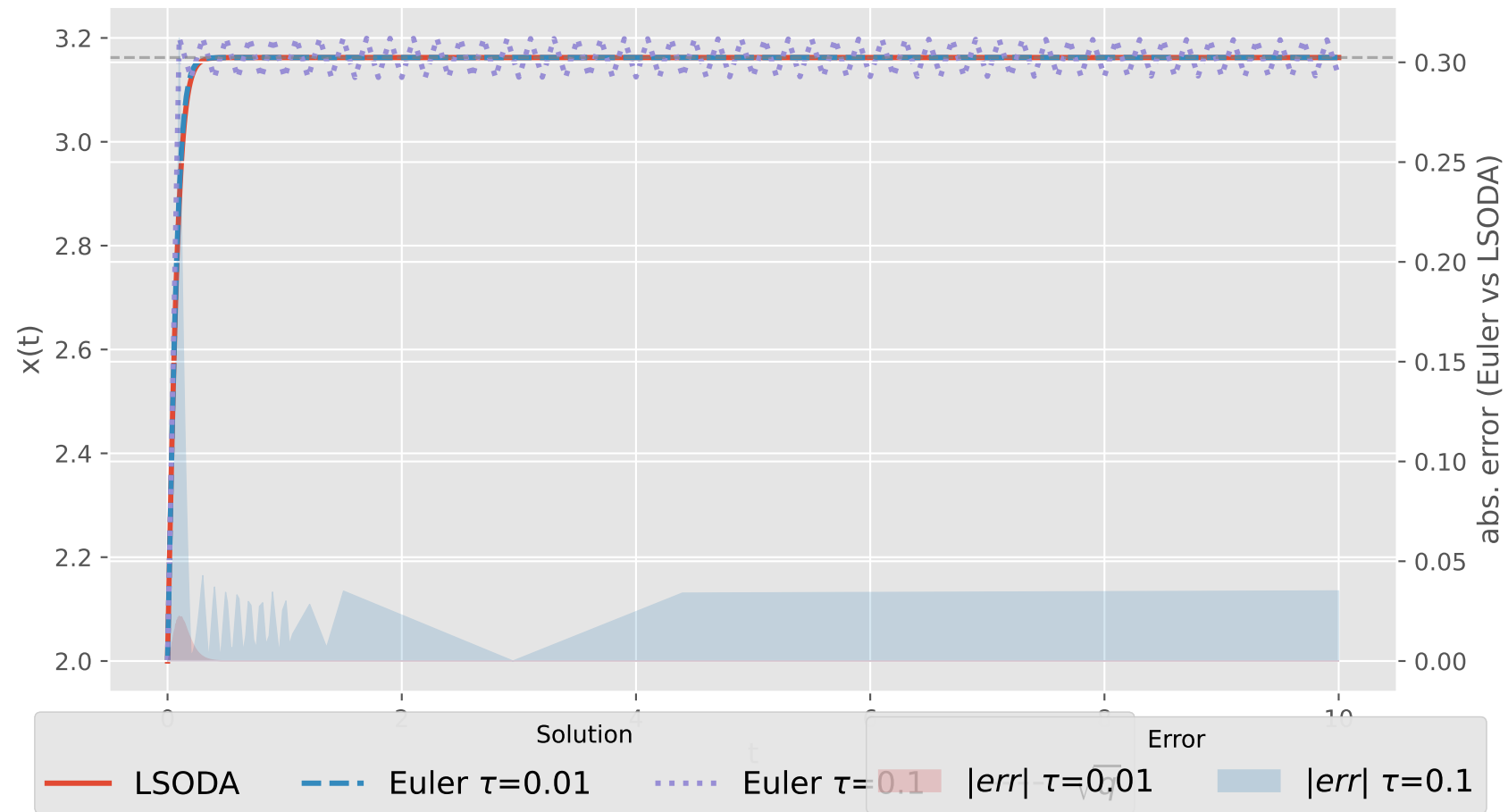


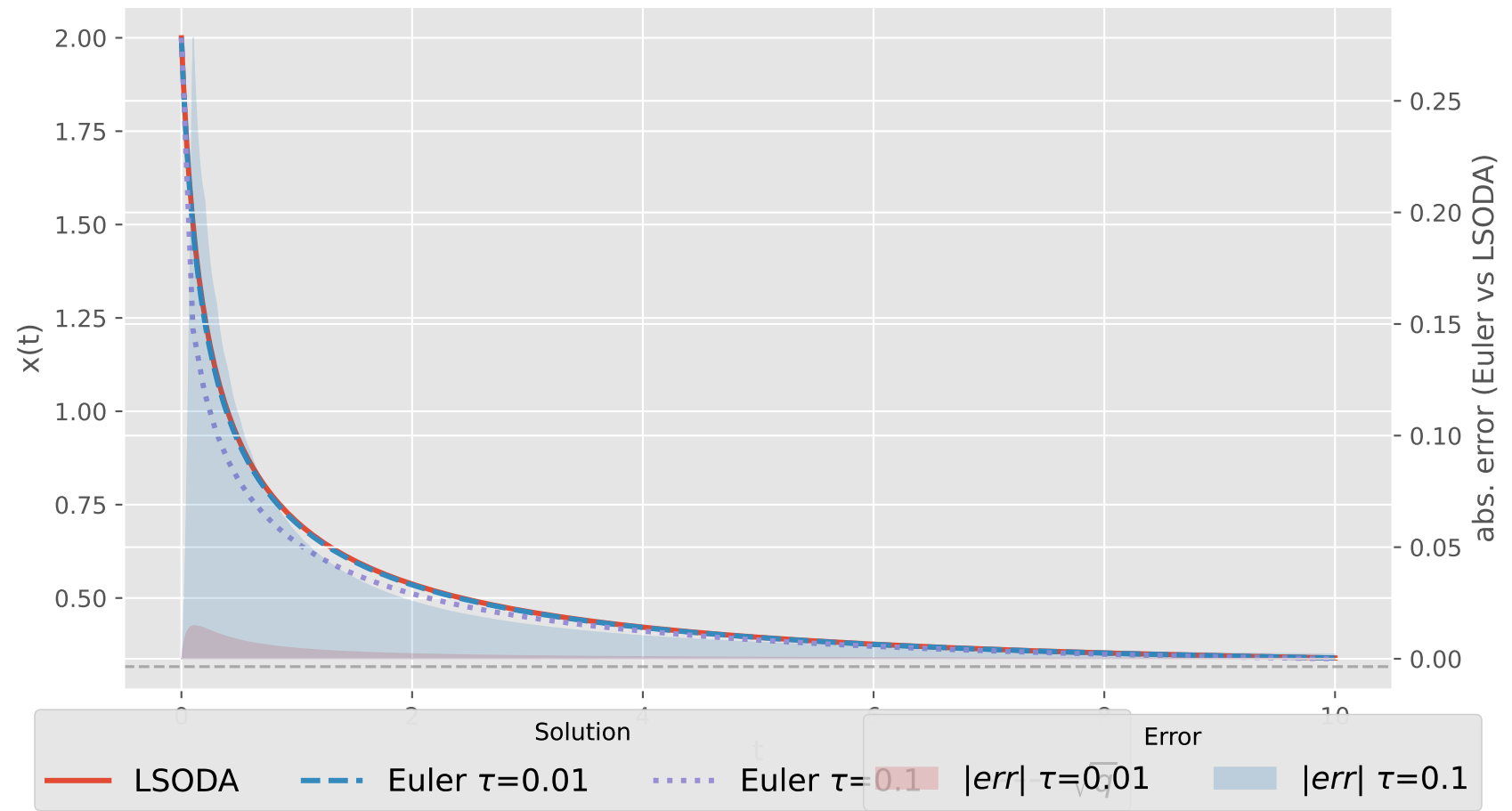
Cubic ODE parameter sweep (equilibria shown)



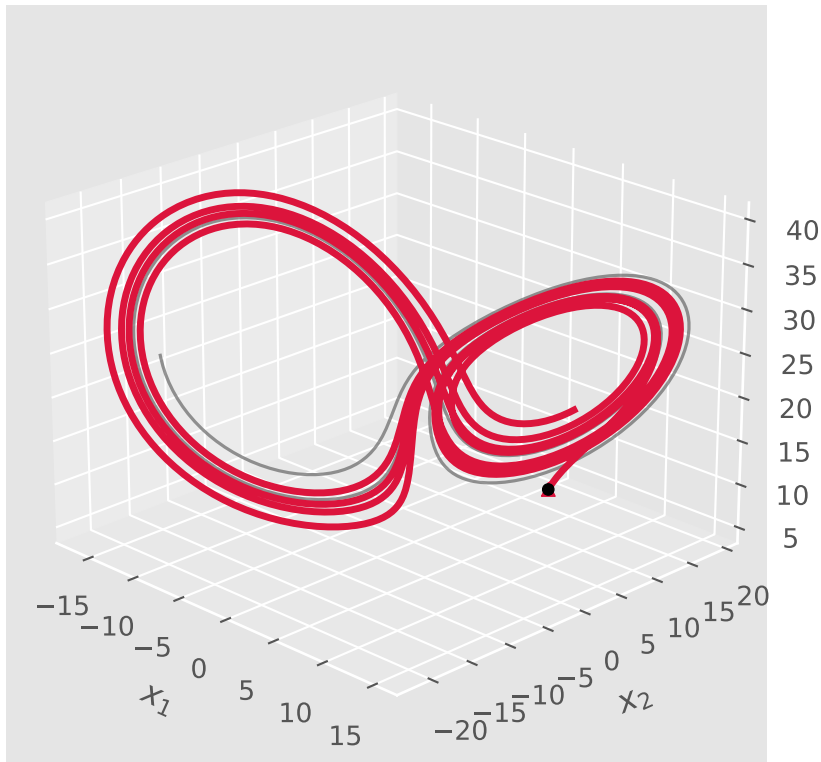
Method comparison for $q = 10.0$



Method comparison for $q = 0.1$

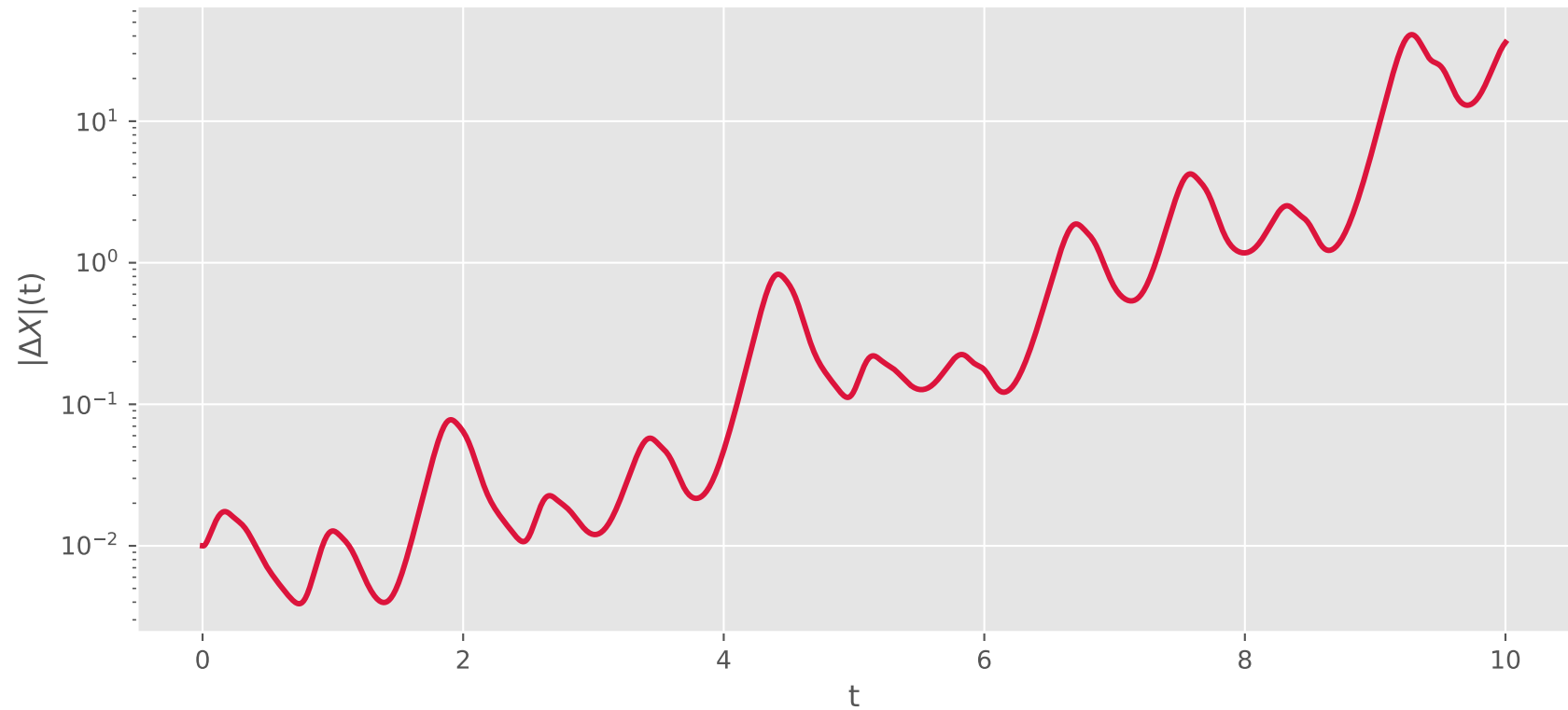


Lorenz sensitivity to perturbing $x_2(0)$

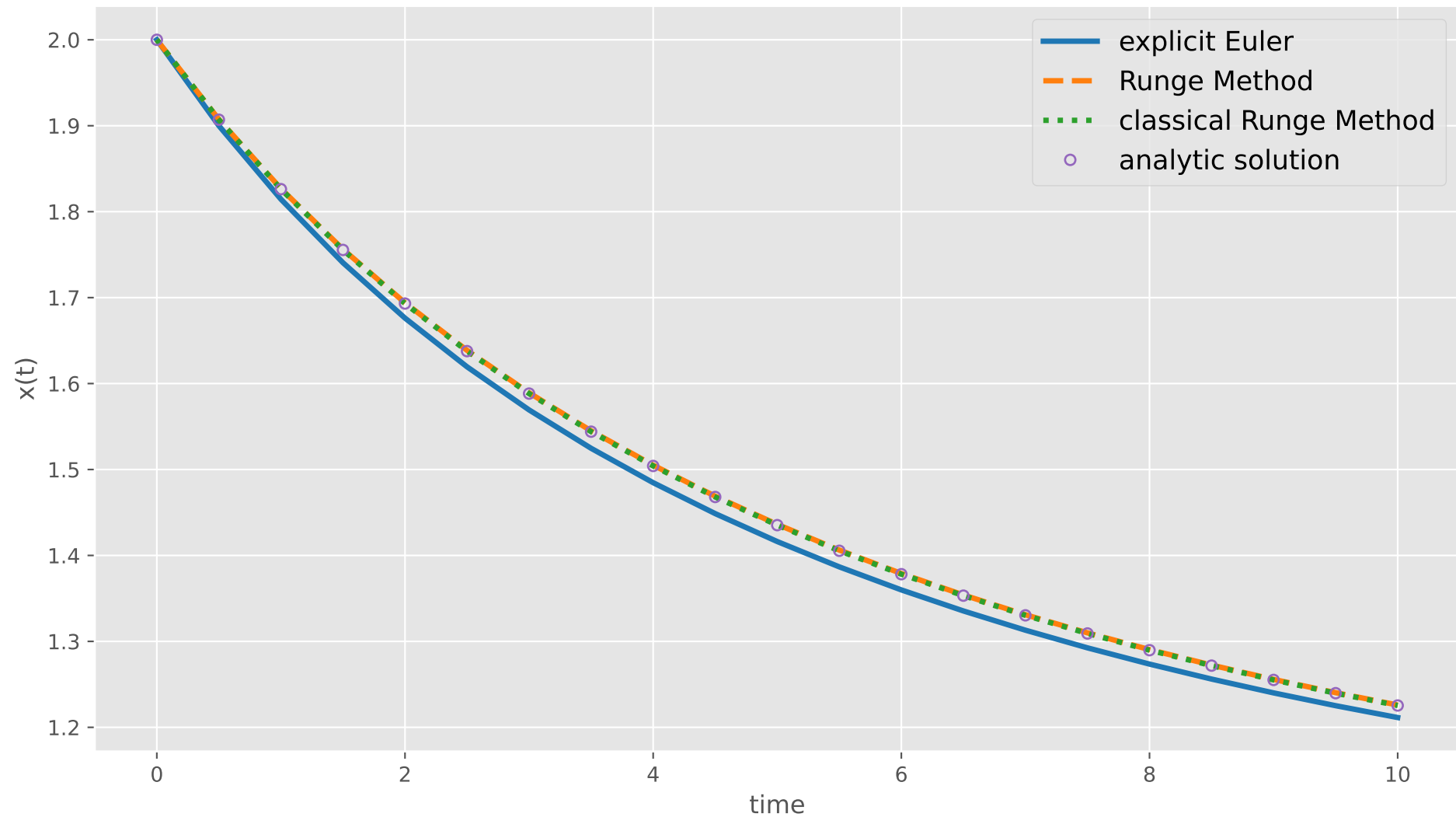


— Baseline — $x_2(0) = 5.01$ perturbed

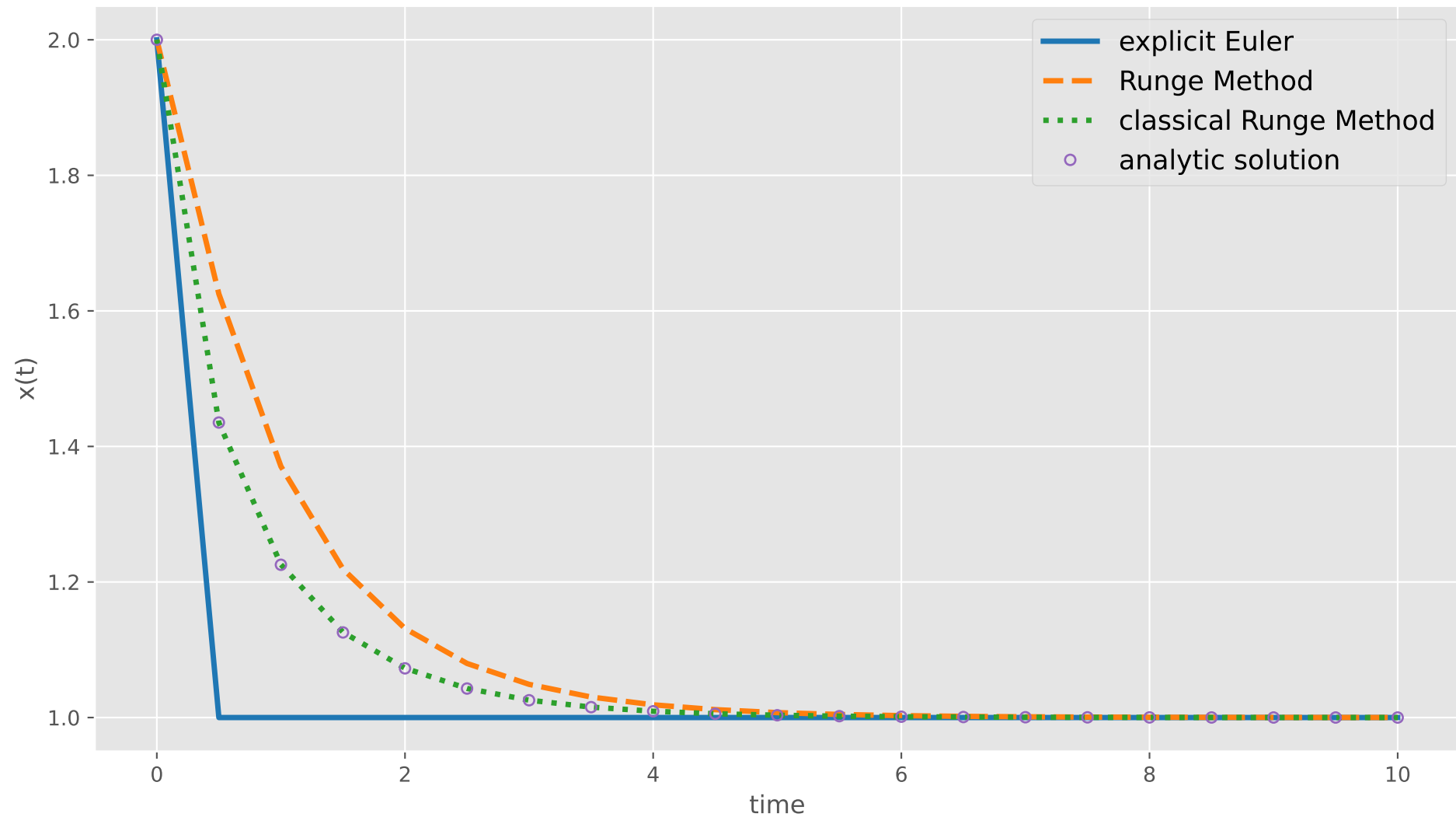
Lorenz trajectory separation



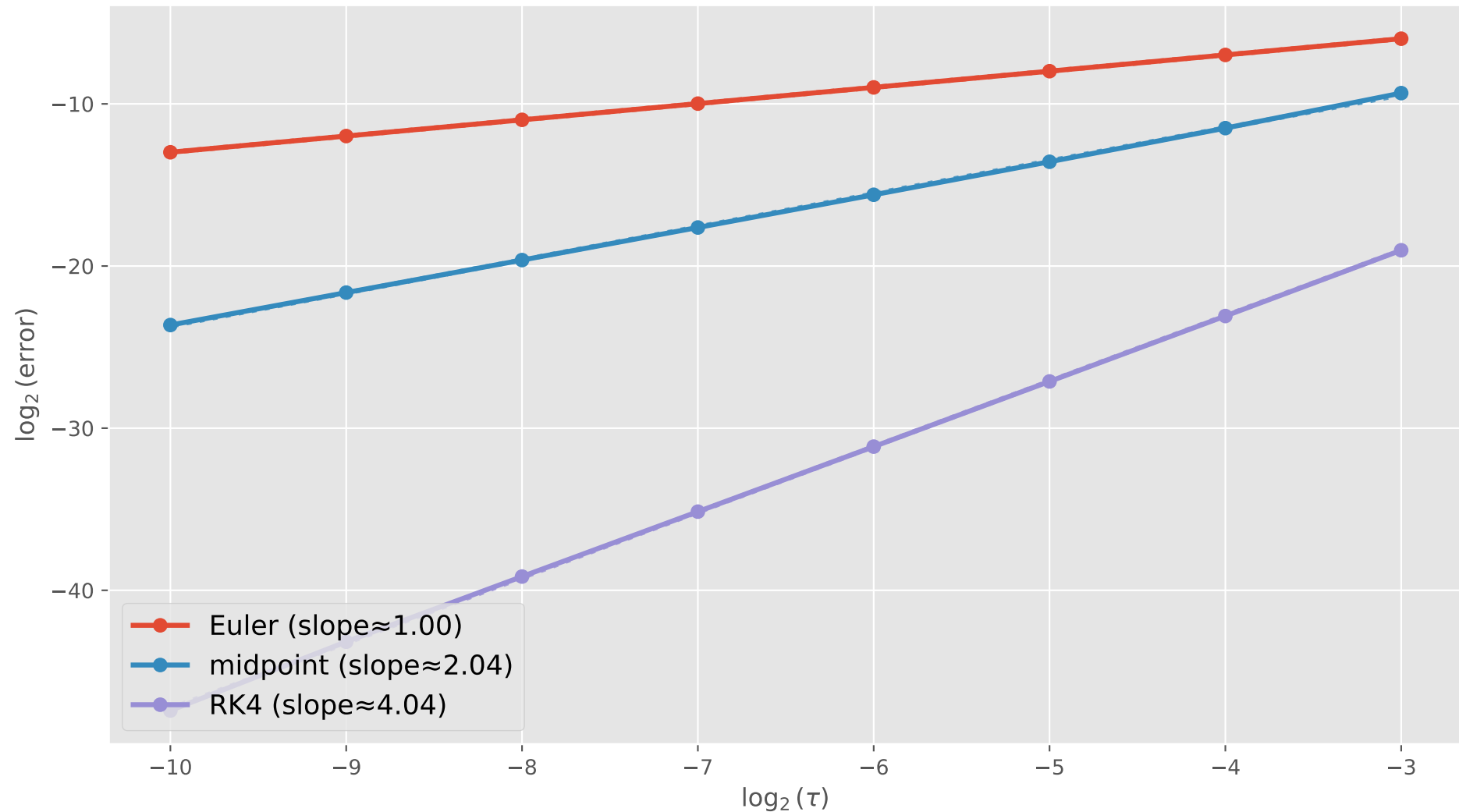
Approximation of the solution for different RK methods



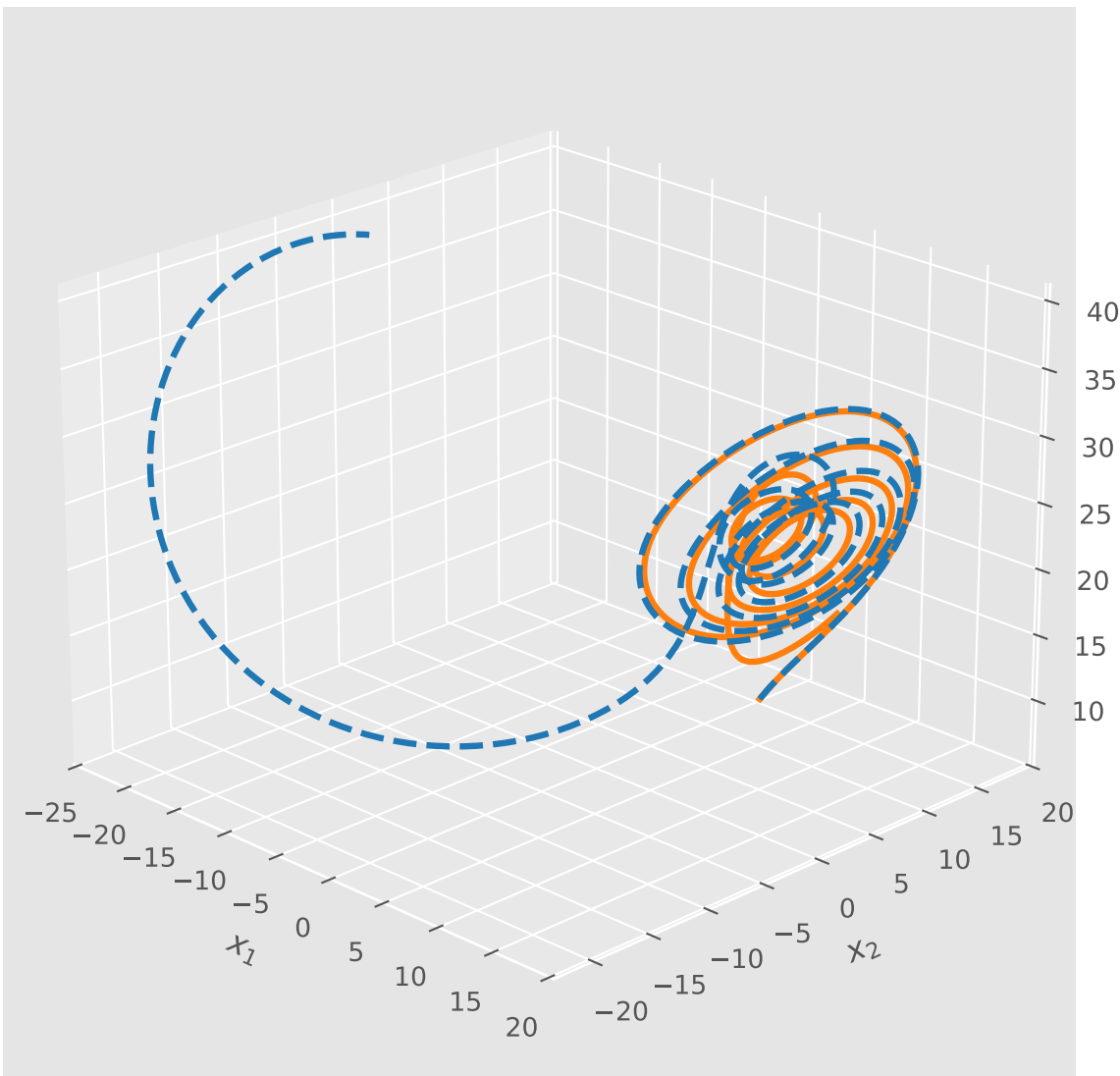
Approximation of the solution for different RK methods



\log_2, \log_2 plot of the error for the three different RK methods

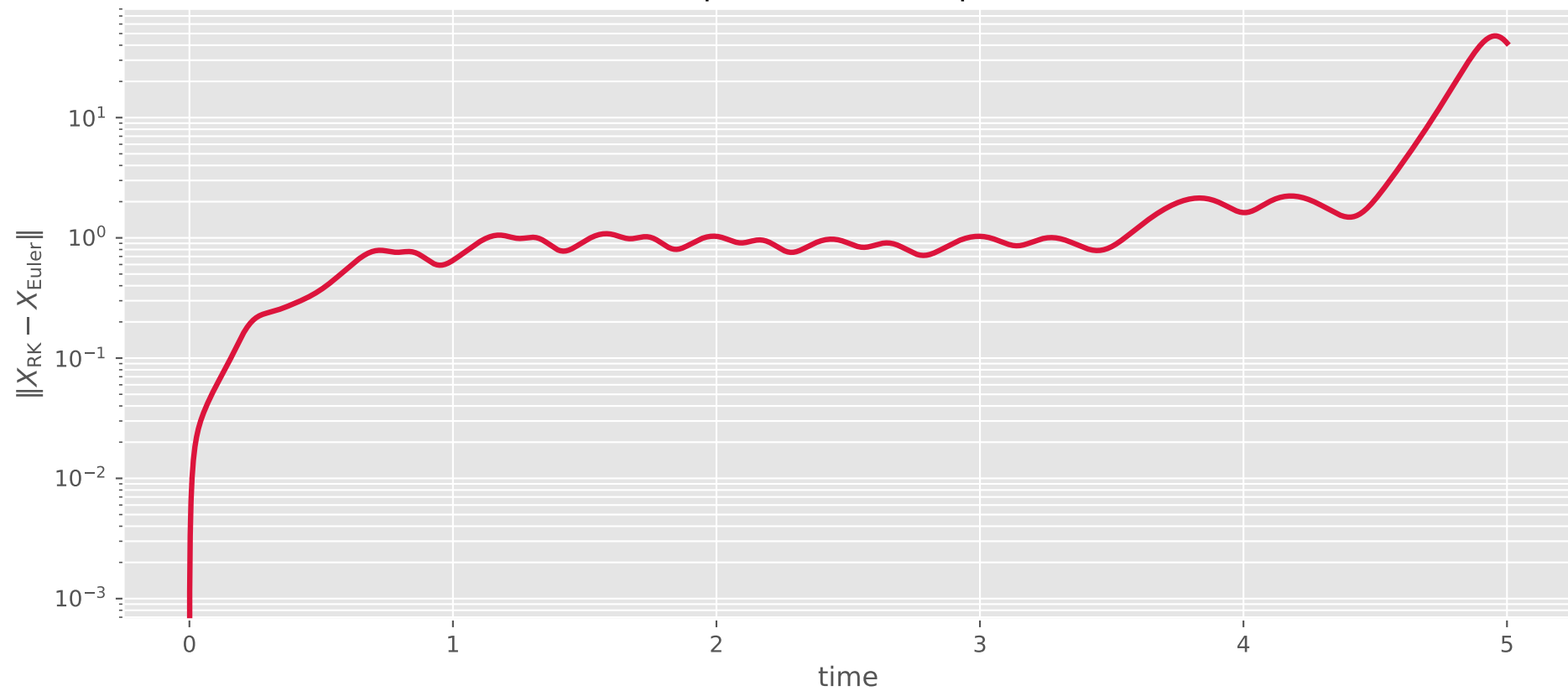


Forced Lorenz trajectories



midpoint RK explicit Euler

Difference between midpoint RK and explicit Euler (forced Lorenz)



Answers

(b) Long-term behaviour as a function of q

\textbf{(b) Long-term behaviour as a function of q .}

We solve $x'(t)=qx-x^3$ with $x(0)=2$ and $q>0$. The equilibria are 0 and $\pm\sqrt{q}$. For $q>0$, $x=0$ is unstable and $x=\pm\sqrt{q}$ are asymptotically stable since $f'(x)=q-3x^2$ implies $f'(\pm\sqrt{q})=-2q<0$.

With $x(0)=2>0$, the trajectory approaches the stable equilibrium $+\sqrt{q}$:

```
\[
\begin{cases}
q<4\!:\ & \sqrt{q}<2 \ \Rightarrow x(t)\ \text{decreases monotonically to}\ \sqrt{q},\ \ \ [2pt]
q=4\!:\ & x(t)\equiv 2\ \text{(equilibrium; adding this case gives a flat line at }2\text{)},\ \ \ [2pt]
q>4\!:\ & \sqrt{q}>2 \ \Rightarrow x(t)\ \text{increases monotonically to}\ \sqrt{q}.
\end{cases}
\]
```

This matches the parameter-sweep plot: for small q the approach to \sqrt{q} is slower, so at $T=10$ the solution can still be slightly above the limiting value.

(c) Method comparison (Euler vs. LSODA) and effect of q

\textbf{(c) Method comparison (Euler vs. LSODA) and effect of q .}

We compare explicit Euler with step sizes $\tau=0.1$ and $\tau=0.01$ against an LSODA reference on $[0,10]$.

\emph{Accuracy order.} Explicit Euler is first order: the global error scales as $\mathcal{O}(\tau)$ for smooth problems on a fixed time horizon. Hence, reducing τ from 0.1 to 0.01 should reduce the error by about a factor of 10 (modulo transients).

\emph{Linear stability near the attractor.} Linearizing at the stable equilibrium $x^*=\sqrt{q}$ gives $y'=f'(x^*)y=-2qy$. For the test equation $y'=\lambda y$ with $\lambda=-2q$, explicit Euler is stable iff

```
\[
|1-\tau\lambda|<1 \quad \Longleftrightarrow \quad 0<\tau<\frac{1}{q}.
\]
```

\emph{Case $q=10$.} The stability bound is $\tau<0.1$, so $\tau=0.1$ lies on the boundary and yields visible phase/amplitude error and mild oscillation around the equilibrium; $\tau=0.01$ is well inside the stable region and closely tracks LSODA. Empirically, the absolute error curve for $\tau=0.1$ sits roughly an order of magnitude above that for $\tau=0.01$ over most of $[0,10]$, consistent with first-order convergence \emph{and} the stability-edge effect at $\tau=0.1$.

\emph{Case $q=0.1$.} The bound is $\tau<10$, so both $\tau=0.1$ and 0.01 are deep inside the stability region and the dynamics are slow. Both Euler solutions lie very close to LSODA; the $\tau=0.01$ error is still smaller (by about the expected $\sim 10\times$ factor), but the difference is barely visible in the solution plot because all errors are small.

(d) Sensitivity for the Lorenz system

\textbf{(d) Sensitivity for the Lorenz system.}

With standard parameters $(a,b,c)=(10,25,8/3)$ the Lorenz system exhibits sensitive dependence on initial conditions (positive largest Lyapunov exponent). We integrate on $[0,10]$ with explicit Euler ($\tau=0.001$) from $(x_1(0),x_2(0),x_3(0))=(10,5,12)$ and from the perturbed $(10,5.01,12)$.

The two trajectories coincide initially but separate clearly after a short time, ultimately exploring different parts of the attractor. This is the expected behaviour for a chaotic system: for a small perturbation $\|\delta x(0)\|$ the separation typically grows like $\|\delta x(t)\| \approx \|\delta x(0)\| e^{\lambda t}$ with $\lambda>0$.

\emph{Conclusion.} Yes, the solution changes significantly when $x_2(0)$ is perturbed to 5.01; the 3D plot makes this divergence clearly visible.