ODEs and Dynamical Systems — Answers (b)–(d)

Compiled from the original Python content

(b) Long-term behaviour as a function of q

We solve $x'(t) = qx - x^3$ with $x(0) = 2$ and $q > 0$. The equilibria are 0 and $\pm\sqrt{q}$. For $q > 0$, $x = 0$ is unstable and $x = \pm\sqrt{q}$ are asymptotically stable since $f'(x) = q - 3x^2$ implies $f'(\pm\sqrt{q}) = -2q < 0$.

With $x(0) = 2 > 0$, the trajectory approaches the stable equilibrium $+\sqrt{q}$:

$$\begin{cases} q < 4: & \sqrt{q} < 2 \Rightarrow x(t) \text{ decreases monotonically to } \sqrt{q}, \\ q = 4: & x(t) \equiv 2 \text{ (equilibrium; adding this case gives a flat line at 2),} \\ q > 4: & \sqrt{q} > 2 \Rightarrow x(t) \text{ increases monotonically to } \sqrt{q}. \end{cases}$$

This matches the parameter-sweep plot: for small q the approach to $\sqrt{q}$ is slower, so at $T = 10$ the solution can still be slightly above the limiting value.

(c) Method comparison (Euler vs. LSODA) and effect of q

We compare explicit Euler with step sizes $\tau = 0.1$ and $\tau = 0.01$ against an LSODA reference on $[0, 10]$.

Accuracy order. Explicit Euler is first order: the global error scales as $\mathcal{O}(\tau)$ for smooth problems on a fixed time horizon. Hence, reducing τ from 0.1 to 0.01 should reduce the error by about a factor of 10 (modulo transients).

Linear stability near the attractor. Linearizing at the stable equilibrium $x^* = \sqrt{q}$ gives $y' = f'(x^*) y = -2q y$. For the test equation $y' = \lambda y$ with $\lambda = -2q$, explicit Euler is stable iff

$$|1 + \tau\lambda| < 1 \iff 0 < \tau < \frac{1}{q}.$$

Case $q = 10$. The stability bound is $\tau < 0.1$, so $\tau = 0.1$ lies *on the boundary* and yields visible phase/amplitude error and mild oscillation around the equilibrium; $\tau = 0.01$ is well inside the stable region and closely tracks LSODA. Empirically, the absolute error curve for $\tau = 0.1$ sits roughly an order of magnitude above that for $\tau = 0.01$ over most of $[0, 10]$, consistent with first-order convergence *and* the stability-edge effect at $\tau = 0.1$.

Case $q = 0.1$. The bound is $\tau < 10$, so both $\tau = 0.1$ and 0.01 are deep inside the stability region and the dynamics are slow. Both Euler solutions lie very close to LSODA; the $\tau = 0.01$ error is still smaller (by about the expected $\sim 10\times$ factor), but the difference is barely visible in the solution plot because all errors are small.

(d) Sensitivity for the Lorenz system

With standard parameters $(a, b, c) = (10, 25, 8/3)$ the Lorenz system exhibits sensitive dependence on initial conditions (positive largest Lyapunov exponent). We integrate on $[0, 10]$ with explicit Euler ($\tau = 0.001$) from $(x_1(0), x_2(0), x_3(0)) = (10, 5, 12)$ and from the perturbed $(10, 5.01, 12)$.

The two trajectories coincide initially but separate clearly after a short time, ultimately exploring different parts of the attractor. This is the expected behaviour for a chaotic system: for a small perturbation $\|\delta x(0)\|$ the separation typically grows like $\|\delta x(t)\| \approx \|\delta x(0)\| \, e^{\lambda t}$ with $\lambda > 0$.

Conclusion. Yes, the solution changes significantly when $x_2(0)$ is perturbed to 5.01; the 3D plot makes this divergence clearly visible.

Notes. Typeset with LaTeX; boxes via `tcolorbox`. All mathematics reproduced verbatim from the original Python strings.