Homework4

David Mathews

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Problem 1: Discrete Fourier Transform

Part i

$$\sum_{n=0}^{N-1} e^{i*\frac{2\pi nk}{N}} = N\delta_{k,0}$$

This will be solved for two different cases: When k = 0, and when $k \neq 0$

$$k = 0 : \sum_{n=0}^{N-1} e^{i*\frac{2\pi nk}{N}} = \sum_{n=0}^{N-1} e^{0} = N$$

$$k \neq 0 : \sum_{n=0}^{N-1} r^{n} = \frac{1 - r^{N}}{1 - r}$$

$$\sum_{n=0}^{N-1} e^{i*\frac{2\pi nk}{N}} = \frac{1 - e^{i*2\pi k}}{1 - e^{\frac{i*2\pi kn}{N}}}$$

As the denominator will never be 0 we can ignore it and focus on the numerator

$$1 - e^{i*2\pi k} = 1 - cos(2\pi * k) - i * sin(2\pi * k)$$

As k is always an integer: $cos(2\pi * k) = 1 : sin(2\pi * k) = 0$

$$\sum_{n=0}^{N-1} e^{i*\frac{2\pi nk}{N}} = 0 \text{ for } k \neq 0$$

Combining these results for different k values gives

$$\sum_{n=0}^{N-1} e^{i*\frac{2\pi nk}{N}} = N\delta_{k,0}$$

Part ii

We need to cancel out the summation over n and we can do this by summing over the complex conjugate of k. Doing this to both sides of the equation gives the following relationships.

$$G_k = \sum_{n=0}^{N-1} e^{\frac{-i*2\pi nk}{N}} g_n$$

$$e^{\frac{i*2\pi nk'}{N}} G_k = \sum_{n=0}^{N-1} e^{\frac{-i*2\pi n*(k-k')}{N}} g_n$$

$$e^{\frac{i*2\pi nk'}{N}} G_k = \delta_{k,k'} N g_n$$

Re-indexing our terms gives

$$e^{\frac{i*2\pi nk}{N}} G_k = \delta_{k,0} N g_n$$

$$\sum_{k=0}^{N-1} e^{\frac{i*2\pi nk}{N}} G_k = \sum_{k=0}^{N-1} \delta_{k,0} N g_n$$

$$\sum_{k=0}^{N-1} e^{\frac{i*2\pi nk}{N}} G_k = N g_n$$

$$g_n = \frac{1}{N} * \sum_{k=0}^{N-1} e^{\frac{i*2\pi nk}{N}} G_k$$

Part III

If g_n is real, then the values of G_k are imaginary.

Part IV

$$G_k = \sum_{n=0}^{N-1} e^{i*\frac{-2\pi nk}{N}} g_n$$

$$G_{k+N} = \sum_{n=0}^{N-1} e^{i*\frac{-2\pi(n+N)k}{N}} g_n$$

$$G_{k+N} = \sum_{n=0}^{N-1} e^{i*\frac{-2\pi nk-2\pi Nk}{N}} g_n$$

$$G_{k+N} = \sum_{n=0}^{N-1} e^{i*\frac{-2\pi nk}{N} - i2\pi k} g_n$$

$$G_{k+N} = \sum_{n=0}^{N-1} e^{i*\frac{-2\pi nk}{N}} g_n$$

The final step can be made as $e^{-2i\pi k}$ is always 1 for integer values of k.

Part V

$$G_k = \sum_{n=0}^{N-1} e^{\frac{-i*2\pi nk}{N}} g_n$$

$$G_k e^{\frac{i*2\pi nk}{N}} g_n{}^* = \sum_{n=0}^{N-1} |g_n|^2$$

$$g_n{}^* = \frac{1}{N} \sum_{k=0}^{N-1} e^{\frac{-i*2\pi nk}{N}} G_k{}^*$$

$$\frac{1}{N} \sum_{k=0}^{N-1} |G_k|^2 = \sum_{n=0}^{N-1} |g_n|^2$$

Problem 2

For the no force driving term, as in we are not driving the pendulum at all, I expect the graph of the Fourier transform to be a perfectly flat line as there should be no motion and therefore no periodicity.

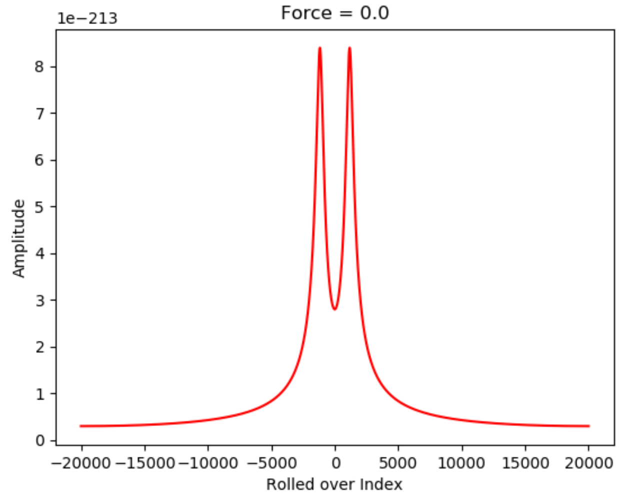
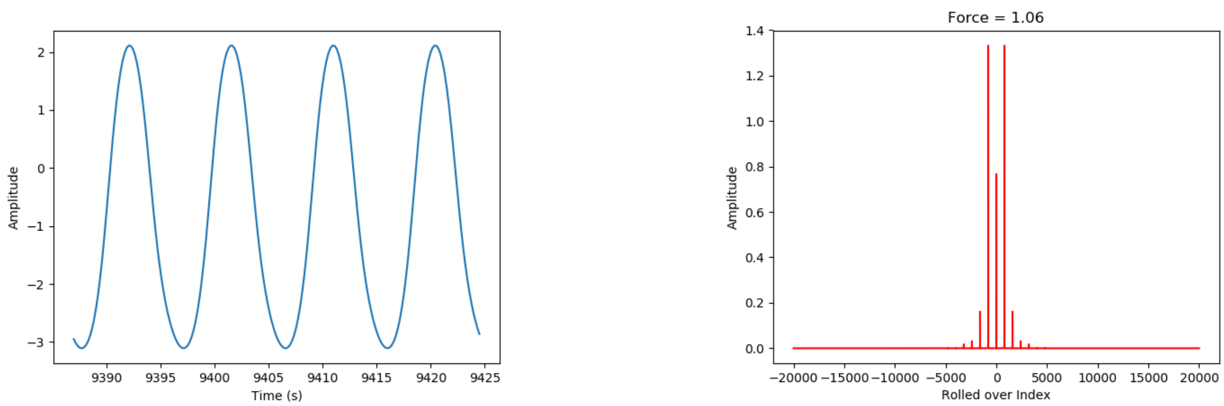


Figure 1: No driving Force term

Oddly enough, this graph is extremely curved. But due to how tiny the scale is, we can safely assume that this is due to the level of machine precision. If we were to look at this on a regular scale it would be a flat line.

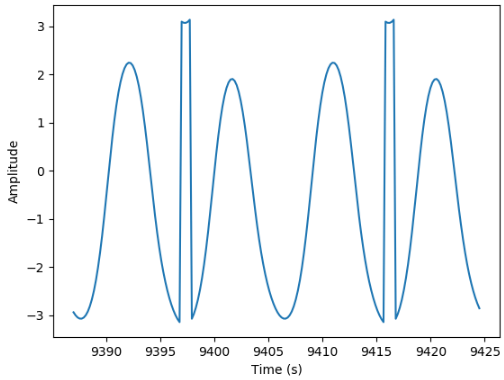


(a) Angular Position over Time

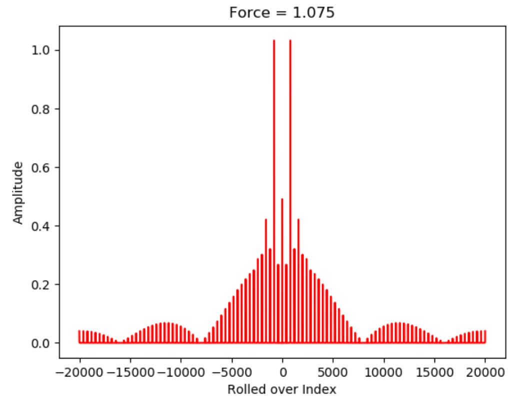
(b) Fourier Transform of the Position Function

Figure 2: Original Behavior and Fourier Transform of the pendulum's position over time for $F_D = 1.06$

As we noticed in the previous homework, this periodic behavior isn't an exact sine wave. Previously we saw that there were some non-linear components. These appear in the Fourier Transform as the other peaks. I expect two delta function peaks on either side of the 0 marker for a perfect sine wave, but for this case we also have several other small periodic terms adding to the behavior.



(a) Angular Position over Time

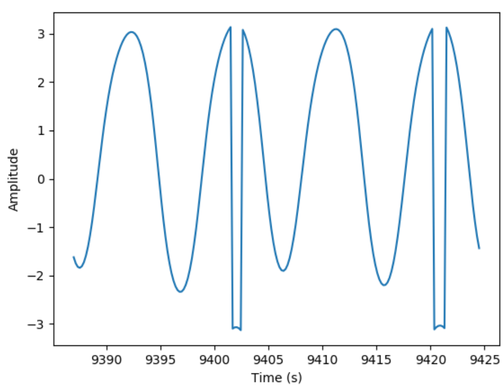


(b) Fourier Transform of the Position Function

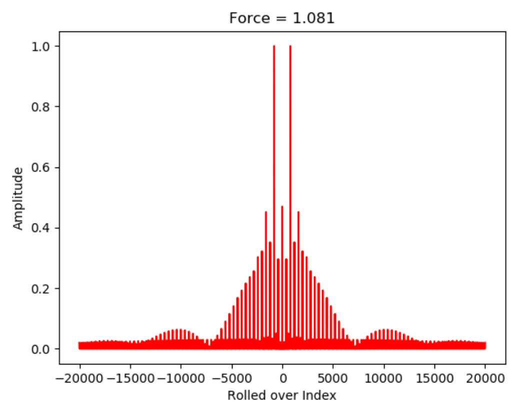
Figure 3: Original Behavior and Fourier Transform of the pendulum's position over time for $F_D = 1.075$

For the period doubling situation that we observed in the last assignment, the behavior of the Fourier Transform is about as expected. We can see that the number of peaks has substantially increased and the separation between each peak has also dramatically reduced.

From this, for the period tripling force, I expect there to be more peaks with smaller separations between each one.



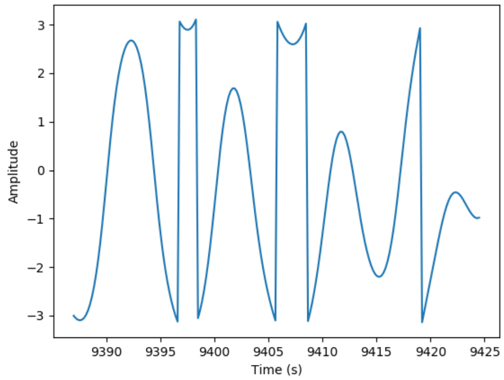
(a) Angular Position over Time



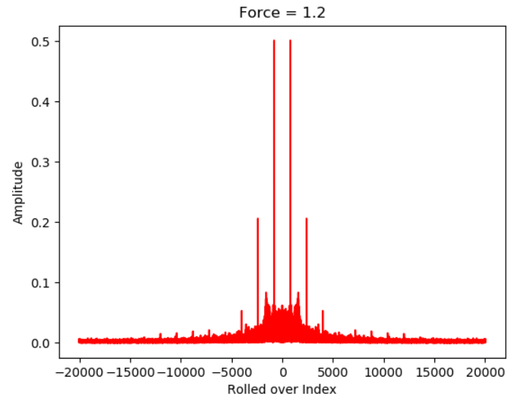
(b) Fourier Transform of the Position Function

Figure 4: Original Behavior and Fourier Transform of the pendulum's position over time for $F_D = 1.0815$

As expected, the number of peaks has dramatically increased and also the relative amplitudes of these extra peaks have decreased.



(a) Angular Position over Time



(b) Fourier Transform of the Position Function

Figure 5: Original Behavior and Fourier Transform of the pendulum's position over time for $F_D = 1.0815$

For the chaotic case, as expected besides the fact each side is identical, there is no real regularity in the data. Some higher spikes appear to happen at fairly regular intervals down each side, but for the most park they are buried in the noise of random peaks.