Math 344-01 Evans **Midterm Exam 2** **Fall 2014**

1. (20 points) Let A, B and C be real matrices with dimensions such that the product $A^T C B^T$ is well defined. Let $\mathcal{X}$ be the set of matrices X minimizing $||AXB - C||_F$, and let X_0 be the unique member of $\mathcal{X}$ minimizing $||X||_F$. Show that $X_0 = A^\dagger C B^\dagger$. Hint: $||\cdot||_F$ refers to the norm using the Frobenius inner product, the singular value decompositions of A and B are helpful in solving this problem.

2. (20 points) Given an $n \times n$ matrix A with real entries such that $A^2 = -I$. Prove the following statements about A.

   (a) A is nonsingular.

   (b) n is even.

   (c) A has no real eigenvalues.

   (d) $\det A = 1$.

3. (20 points) Occasionally it is convenient to relax the definition of the inner product somewhat. A function $f : V \times V \to \mathbb{F}$ is called a semiscalar product if f satisfies the following conditions for any $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z} \in V$ and any $a, b \in \mathbb{F}$:

   (a) $f(\mathbf{x}, \mathbf{x}) \geq 0$ for all $\mathbf{x} \in V$.

   (b) $f(\mathbf{x}, a\mathbf{y} + b\mathbf{z}) = af(\mathbf{x}, \mathbf{y}) + bf(\mathbf{x}, \mathbf{z})$.

   (c) $f(\mathbf{x}, \mathbf{y}) = \overline{f(\mathbf{y}, \mathbf{x})}$.

   Give an example of a semiscalar product that is not an inner product on V You may not use the trivial semiscalar product (i.e., $f(\mathbf{x}, \mathbf{x}) = 0$ for all $\mathbf{x} \in V$.). Show that the Cauchy-Schwartz inequality remains valid for any semiscalar product f.

4. (20 points) Let V and W be finite dimensional vector spaces. Prove that

$$\dim \mathscr{N}(T^*) = \dim \mathscr{N}(T) + \dim W - \dim V$$

   and

$$\dim \mathscr{R}(T) = \dim \mathscr{R}(T^*)$$

   for every $T \in \mathscr{L}(V, W)$.

5. (20 points) Consider the space $M_2(\mathbb{R})$ with inner product $\langle A, B \rangle = \operatorname{tr}(A^T B)$.

   (a) Show the following is an orthonormal basis for $M_2(\mathbb{R})$:

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}.$$

   (b) Find an orthogonal basis for the orthogonal complement of the set of diagonal matrices.

   (c) Find an orthogonal basis for the orthogonal complement of the set of symmetric matrices.