Math 344 Midterm 1

Sept. 30- Oct. 3, 2015

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Section: 1

Instructor: Evans

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Instructions

- Do not write on the barcode area at the top of each page, or near the four circles on each page.
- Completely fill in the correct boxes for your BYU ID and for the correct answer on the multiple choice.
- Mark the correct choice in the multiple choice section. Each multiple choice question is worth 3 points.
- You are allowed a single 4x6 or 5x8 notecard with whatever you wish written on it.
- You may cite theorems and examples from your book however you may not cite homework exercises.
- Do not talk to classmates about the exam until the exams have been returned in class.

Sign the following pledge below:
I pledge on my honor that I have not given or received any unauthorized assistance on this assignment/examination.

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Part 1: Multiple Choice Fill in the box for the correct response.

1 Let $S = \{x+3, x^2-x, x-3\} = \{s_1, s_2, s_3\}$ and $T = \{x^2, x, 1\} = \{t_1, t_2, t_3\}$ be bases for $\mathbb{R}[x]_2$. The transition matrix from S to T is:

☐ $\begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 2 \\ -2 & 1 & -2 \end{bmatrix}$.

☐ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.

☒ $\begin{bmatrix} 0 & 1 & 3 \\ 1 & -1 & 0 \\ 0 & 1 & -3 \end{bmatrix}$.

☐ $\begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 3 & 0 & -3 \end{bmatrix}$.

$t_1 = x^2 = \quad + 1s_2 +$

$s_1 = x+3 = 0t_1 + 1t_2 + 3t_3$

$s_2 = x^2 - x = 1t_1 + (-1)t_2 + 0t_3$

$s_3 = x-3 = 0t_1 + 1t_2 + (-3)t_3$

2 Which of the following subsets of $\mathbb{R}^3$ is complementary to $\operatorname{span}\{(1,0,2)\}$?

x, y, z

☐ ~~span($\mathbb{R}^3/\{(1,0,2)\}$)~~

☐ ~~span($\{(-2,1,1),(0,3,0)\}$)~~

☒ $\operatorname{span}\{(0,1,0)\}$

☐ ~~$\{(0,1,0),(-2,0,1)\}$~~

$x + \{(1,0,2)\}$

3 Which of the following subsets of $\mathbb{R}^3$ is not a subspace of $\mathbb{R}^3$?

- ☐ The vectors (b_1, b_2, b_3) that satisfy $b_3 - b_2 + 3b_1 = 0$.
- ☒ The vectors $\mathbf{b}$ with $b_1 b_2 = 0$. $(1,0,2) + (0,1,2) = (1,1,4) \notin W$
- ☐ ~~The plane of all vectors $\mathbf{b}$ with first component $b_1 = 0$.~~ $(0,1,2) + (0,2,3) = (0,3,5) \in W$
- ☐ ~~The solitary vector $\mathbf{b} = (0,0,0)$.~~
- ☐ All combinations of two given vectors $\mathbf{x} = (1,1,0)$ and $\mathbf{y} = (2,0,1)$.

4 Which of the following is NOT a property of the Bernstein basis?

- ☐ $B_j^n(x)$ has a single extremum, at $x = j/n$.
- missing x @ B_{j-1}^{n-1} ? ☐ $\dfrac{d}{dx} B_j^n(x) = n\left[B_{j-1}^{n-1}(x) - B_j^{n-1}(x)\right]$.
- ☒ $B_j^n(x) = \dfrac{j+1}{n+1} B_{j+1}^{n+1}(x) + \dfrac{j-1}{n+1} B_j^{n+1}(x), \quad j = 0, 1, \ldots, n.$
- ☐ ~~$B_{n-j}^n(1-x) = B_j^n(x)$.~~ $\frac{n!}{(n-j)!(j)!}(1-x)^{n-j}(x)^j$
- ☐ ~~$B_j^{n+1}(x) = xB_{j-1}^n(x) + (1-x)B_j^n(x)$.~~

$$\frac{j+1}{n+1}\cdot\frac{(n+1)!}{(j+1)!(n-j)!}x^{j+1}(1-x)^{n-j} + \frac{j-1}{n+1}\cdot\frac{(n+1)!}{j!(n+1-j)!}x^j(1-x)^{n+1-j}$$
$$= \frac{n!}{j!(n-j)!}x^{j+1}(1-x)^{n-j} + \frac{n!}{j!(n-j)!}\cdot\frac{j-1}{n-j+1}x^j(1-x)^{n-j+1}$$
$$= \frac{n!}{j!(n-j)!}x^j(1-x)^{n-j}\left(x + (1-x)\cdot\frac{j-1}{n-j+1}\right)$$
$$= B_j^n(x)\cdot\left(x + (1-x)\cdot\frac{j-1}{n-j+1}\right)$$
$$\neq B_j^n(x)$$

5 Suppose V is a vector space of dimension 7 and W is subspace of dimension 4. Consider the following statements:

i Every basis for W can be extended to a basis for V by adding three more vectors.

ii Every basis for V can be reduced to a basis for W by removing three vectors.

$V = \{1, x, x^2, x^3, x^4, x^5, x^6\}$

$W = \{1, x, x^2, x^3\}$

- ☐ ~~(i) is false and (ii) is true~~
- ☐ (i) and (ii) are both false
- ☐ (i) and (ii) are both true
- ☒ (i) is true and (ii) is false by Extension Theorem

$V \setminus \{x^2, x^3, x^4\}$

but this is not a basis for W.

6 Assume that A is n by n and B is n by n. What is $\det \begin{bmatrix} 0 & A \\ -B & I \end{bmatrix}$?

- [ ] $\det(AB)$ $0 - (\det(-B) \cdot \det(A))$
- [x] $-\det(A)\det(B)$
- [ ] Not enough information to determine.
- [ ] 1
- [ ] 0

7 ♣ Select all of the following which are the result of the **First Isomorphism Theorem**?

- [ ] Let V and W be finite-dimensional vector spaces and $L : V \to W$ a linear transformation then $\dim V = \text{rank} L + \text{nullity} L$.
- [ ] If V is a vector space and W is a subspace of V, the mapping $\pi : V \to V/W$ defined by $\pi(\mathbf{v}) = \mathbf{v} + W$ is a surjective linear transformation.
- [ ] If V is a vector space and W is a subspace of V then $\dim V = \dim W + \dim V/W$.
- [ ] If a vector space over $\mathbb{F}$ is n-dimensional, then it is isomorphic to $\mathbb{F}^n$. A corollary
- [ ] A linear map is injective if and only if $\mathscr{N}(L) = \{\mathbf{0}\}$. Lemma to F.I.T.
- [x] *None of these answers are correct.*

The direct result? or follows from?

Assuming direct result

in other words None of these are the First Isomorphism Theorem

8 The general solution to

$$\begin{bmatrix}1 & 2 & 2\\ 2 & 4 & 5\end{bmatrix}\begin{bmatrix}u\\ v\\ w\end{bmatrix} = \begin{bmatrix}1\\ 4\end{bmatrix}$$

is?

- [x] $\begin{bmatrix}u\\ v\\ w\end{bmatrix} = v\begin{bmatrix}-2\\ 1\\ 0\end{bmatrix} + \begin{bmatrix}-3\\ 0\\ 2\end{bmatrix}.$
- [ ] $\begin{bmatrix}u\\ v\\ w\end{bmatrix} = u\begin{bmatrix}1\\ -1\\ 0\end{bmatrix} + w\begin{bmatrix}0\\ -2\\ 1\end{bmatrix} + \begin{bmatrix}0\\ 4\\ 0\end{bmatrix}.$
- [ ] $\begin{bmatrix}u\\ v\\ w\end{bmatrix} = v\begin{bmatrix}-2\\ 1\\ 3\end{bmatrix} + \begin{bmatrix}4\\ 0\\ -1\end{bmatrix}.$
- [ ] $\begin{bmatrix}u\\ v\\ w\end{bmatrix} = v\begin{bmatrix}-3\\ 0\\ 2\end{bmatrix} + \begin{bmatrix}-2\\ 1\\ 0\end{bmatrix}.$
- [ ] $\begin{bmatrix}u\\ v\\ w\end{bmatrix} = \begin{bmatrix}1\\ 2\\ 2\end{bmatrix}.$

$\begin{matrix}1 & 2 & 2 & | & 0\\ 2 & 4 & 5 & | & 0\end{matrix}$

$\begin{matrix}1 & 2 & 2 & | & 0\\ 0 & 0 & 1 & | & 0\end{matrix}$

$u + 2v + 2w = 0$

$w = 0$

$u = -2v - 2w$

$v = 1v$

$w = 0$

$N(L) = v\begin{pmatrix}-2\\ 1\\ 0\end{pmatrix}$

$S + N(L)$

9 The set $S = \{e^{3t}, te^{3t}, t^2e^{3t}\}$ is a basis of a vector space V of functions $f : \mathbb{R} \to \mathbb{R}$. Let D be the differential operator on V; that is, $D(f) = df/dt$. Find the matrix representation of D relative to the basis S.

$DS = \{3e^{3t}, 3te^{3t} + e^{3t}, 3t^2e^{3t} + 2te^{3t}\}$

☐ $\begin{bmatrix} 9 & -3 & 2 \\ 0 & 9 & -6 \\ 0 & 1 & 9 \end{bmatrix}$.

☒ $\begin{bmatrix} 3 & 0 & 0 \\ 1 & 3 & 0 \\ 0 & 2 & 3 \end{bmatrix}$.

☐ $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.

☐ $\begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 2 \\ 0 & 0 & 3 \end{bmatrix}$.

☐ $\begin{bmatrix} 9 & 3 & 0 \\ 0 & 9 & 6 \\ 0 & 0 & 0 \end{bmatrix}$.

$d_1 = 3s_1 + 0s_2 + 0s_3$

$d_2 = 1s_1 + 3s_2 + 0s_3$

$d_3 = 0s_1 + 2s_2 + 3s_3$

$$\begin{bmatrix} 3 & 0 & 0 \\ 1 & 3 & 0 \\ 0 & 2 & 3 \end{bmatrix}$$

10 In each of the choices below a transformation $L : \mathbb{F}^3 \to \mathbb{F}^3$ is defined by the formula given for $L(x, y, z)$ where (x, y, z) is an arbitrary point in $\mathbb{F}^3$. Identify the linear transformation.

☐ $\cancel{L(x, y, z) = (x, y^2, z^3)}$ $L(2(1,1,2) + (1,1,1)) = L(3,3,5) = (3,9,125)$

$2L(1,1,2) + L(1,1,1) = 2(1,1,8) + (1,1,1) = (3,3,17)$

☐ $\cancel{L(x, y, z) = (x, y, 1)}$ $L(0) \neq 0$

☒ $L(x, y, z) = (x, y, 0)$

☐ $\cancel{L(x, y, z) = (x + 1, y + 1, z + 1)}$ $L(0) \neq 0$

☐ $\cancel{L(x, y, z) = (x + 1, y + 2, z + 3)}$ $L(0) \neq 0$

$L(2(1,2,1) + (2,1,1)) = L((2,4,2) + (2,1,1)) = L(4,5,3) = (4,5,0)$

$2L(1,2,1) + L(2,1,1) = 2(1,2,0) + (2,1,0) = (2,4,0) + (2,1,0) = (4,5,0)$

Part II: Long Answer. Before attempting any problems be sure to read all four problems. The easiest problems are not necessarily first. Start with the problem you feel most confident about. Only three problems will be graded, be sure to indicate which problems you wish graded. Show all work for each question in the space provided. If necessary, write "work at end" at the bottom of the space, and continue working on the extra pages provided at the back of the exam. **DO NOT** write in the administrative areas.

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Let W_1 and W_2 be nontrivial subspaces of a finite dimensional vector space V; that is, W_1 and W_2 are neither $\{\mathbf{0}\}$ nor V. Moreover assume $W_1 \neq W_2$. Show that there exists an element $\alpha \in V$ such that $\alpha \notin W_1$ and $\alpha \notin W_2$. Show further that there exists a basis of V such that none of the vectors in the basis is contained in either W_1 or W_2.

Assume the hypothesis described above, but assume there is an $\alpha \in V$ such that $\alpha \in W_1$ or $\alpha \in W_2$. Then $\alpha \in W_1 \oplus W_2$ and it follows that $\alpha \notin V/W_1 \cap W_2$ but this implies that $\alpha \notin V$ which is a contradiction. Therefore there must exist $\alpha \in V$ such that $\alpha \notin W_1$ and $\alpha \notin W_2$.

Since there exists $\alpha \in V$ such that $\alpha \notin W_1$ and $\alpha \notin W_2$, then for each basis for W_1 and W_2, we can add α to each element in a basis for V such that it is still linearly independent. Furthermore, since α is added to each vector in the basis, it cannot be that any vector is in W_1 or W_2 but we can still span V since our basis with α does not change the dimension of the basis. Therefore, the basis for V with no vectors of the basis contained in W_1 or W_2 exists. □

+32/10/37+

~~12~~

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Suppose W is a subspace of a vector space V such that V/W is finite-dimensional. Prove that V is isomorphic to $W \times (V/W)$.

13

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11 12 13 14 15 16 17 18 19 20

Let V be an n-dimensional vector space. We say a linear operator on V is *idempotent* if $A^2 = A$ (that is, if $v \in V$, then $A^2v = Av$). Let A be an idempotent operator on V, and let I be the identity operator on V Show the following (hint: since V is finite dimensional, A and I can be represented as matrices):

1. $I - A$ is idempotent.
2. $(I - A)(I - tA) = I - A$ for any scalar t.
3. $(2A - I)^2 = I$.
4. $A + I$ is invertible.
5. Find (not show) $\mathscr{N}(A)$.
6. Find (not show) $\mathscr{R}(A)$.
7. $V = \mathscr{N}(A) \oplus \mathscr{R}(A)$.
8. $Ax = x$ for every $x \in \mathscr{N}(A)$.
9. (Bonus worth 4 extra points) Each eigenvalue of A is either 1 or 0.

1. $(I-A)^2 = (I-A)(I-A) = I^2 - AI - IA + A^2 = I - A - A + A = I - A$.

2. $(I-A)(I-tA) = I^2 - AI - tIA + tA^2 = I - A - tA + tA = I - A$.

3. $(2A-I)^2 = (2A-I)(2A-I) = 4A^2 - 2IA - 2AI + I^2 = 4A - 2A - 2A + I = 4A - 4A + I = I$.

4. Note that $(I - \frac{1}{2}A)$ is the inverse of $(A+I)$ since
$(I - \frac{1}{2}A)(A+I) = IA - \frac{1}{2}A^2 + I^2 - \frac{1}{2}AI = A - \frac{1}{2}A + I - \frac{1}{2}A = A - A + I = I$.

5. Since A is a linear operator from V to V, it is injective so by Lemma 2.2.3 we have that $N(A) = \{\vec{0}\}$. Thus $Av = 0$ for $v \in V$ implies $v = \vec{0} \in V$.

6. Since A is bijective, by the First Isomorphism Theorem we have that $V/N(A) \cong R(A)$ therefore $R(A) = \{v + \{\vec{0}\} \mid v \in V, v \neq \vec{0}\}$.

7. $\Rightarrow$ Let $v \in V$ be arbitrary. If $v = \vec{0}$, then $v \in N(A) \subset N(A) \oplus R(A)$. If $v \neq \vec{0}$, then $v \in V$ implies that $v \in R(A) \subset N(A) \oplus R(A)$. Therefore, $V \subset N(A) \oplus R(A)$.
$\Leftarrow$ Let $x \in N(A) \oplus R(A)$ be arbitrary. If $x \in N(A) \cap R(A) = \{\vec{0}\}$ then $x = \vec{0}$ and $\vec{0} \in V$ so $x \in V$. If $x \in N(A)$ then $x = \vec{0} \in V$. If $x \in R(A)$, then $x = v + \{\vec{0}\}$ and $v \in V$ implies that $x \in V$. Therefore, $V = N(A) \oplus R(A)$. □

8. Let $x \in N(A)$ be given. Then $x = \vec{0}$ since $N(A) = \{\vec{0}\}$ and because A is a linear operator, we have that $Ax = \vec{0}$ for every $x \in N(A)$. Thus $Ax = x$ for every $x \in N(A)$.

9. See Blank Pages

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I take this to mean each $a_{ij} \in A$ could be either 1 or -1; A is not necessarily a matrix of all 1's or all -1's.

If A is an $n \times n$ matrix all of whose entries are either 1 or -1, prove that $\det(A)$ is evenly ? divisible by 2^{n-1}.

We prove by Induction. Let $n=1$. Then $\det(A) = \det([1]) = 1$ or $\det([-1]) = -1$ which is evenly divisible by $2^0 = 1$.

Now let $n=2$. Then $\det(A) = \sum_{\sigma \in S_2} \operatorname{sign}(\sigma) a_{1\sigma(1)} a_{2\sigma(2)} = (a_{11}a_{22}) - (a_{12}a_{21})$ which must be ± 0 or ± 2 since $a_{11}a_{22}$ is either 1 or -1 and $a_{12}a_{21}$ is either 1 or -1. Then $2^{n-1} = 2^{2-1} = 2$ and thus $\det(A)$ is divisible by 2. Let $n=3$. Then $\det(A) = \sum_{\sigma \in S_3} \operatorname{sign}(\sigma) a_{1\sigma(1)} a_{2\sigma(2)} a_{3\sigma(3)}$
$= a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} - a_{13}a_{22}a_{31} + a_{13}a_{21}a_{32}$ which must be ± 0 or ± 4 and $2^{n-1} = 2^2 = 4$ so $\det(A)$ is divisible by 2^2.

Now assume $\det(A)$ is divisible by 2^{n-1} for all $n \leq 3$, $n \in \mathbb{N}$. We show $A_{n+1 \times n+1}$ is divisible by 2^n. Suppose $A_{n+1 \times n+1}$ is a matrix whose entries are either 1 or -1. Then

$$\det(A_{n+1\times n+1}) = \sum_{\sigma \in S_{n+1}} \operatorname{sign}(\sigma) a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)} a_{n+1\,\sigma(n+1)}$$
$$= \left(\sum_{\sigma \in S_n} \operatorname{sign}(\sigma) a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)}\right)\left(\sum_{\substack{\sigma \in S_{n+1} \\ \sigma \in S_n}} \operatorname{sign}(\sigma) a_{n+1\,\sigma(n+1)}\right)$$

Thus $\sum_{\substack{\sigma \in S_{n+1} \\ \sigma \in S_n}} \operatorname{sign}(\sigma) a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)}$ is divisible by 2^{n-1}.
$\sum_{\substack{\sigma \in S_{n+1} \\ \sigma \in S_n}} \operatorname{sign}(\sigma) a_{n+1\,\sigma(n+1)}$ is either ± 0 or ± 2 which is divisible by 2^1.
Therefore, $\det(A_{n+1\times n+1})$ is divisible by $2^{n-1} \cdot 2^1 = 2^n$ by induction. $\square$

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9. Note that $\det(A - \lambda I)$ gives the characteristic polynomial. Then for $A_{n \times n}$ we obtain $(1-\lambda)$ or $(-1-\lambda)$ for the characteristic polynomial. ~~If~~ Observe

$$(1-\lambda)(-1-\lambda) = \lambda^2 + \lambda - \lambda - 1 = \lambda^2 - 1$$

So $\lambda = 1$ or

$$(-1-\lambda)(-1-\lambda) = \lambda^2 + 2\lambda + 1 = (1-\lambda)^2$$

$$\lambda(1-\lambda) \text{ so } \lambda = 0$$

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