Math 344-01 Evans **Practice Questions** **Fall 2014**

1. (Adapted from Atkinson) Let A be a symmetric matrix, and let λ and x be an eigenvalue-eigenvector pair for A with $\|x\|_2 = 1$. Let P be an orthogonal matrix for which

$$Px = e_1 \equiv [1, 0, \ldots, 0]^T$$

Consider the similar matrix $B = PAP^T$, and show that the first row and column are zero except for the diagonal element, which equals λ.

2. For the matrix

$$A = \begin{bmatrix} 2 & 10 & 2 \\ 10 & 5 & -8 \\ 2 & -8 & 11 \end{bmatrix}$$

$\lambda = 9$ is an eigenvalue with associated eigenvector $x = \left[\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right]^T$. Produce a Householder matrix P for which $Px = e_1$, and then produce $B = PAP^T$. The matrix eigenvalue problem for B can then easily be reduced to a problem for a 2×2 matrix. Use this procedure to calculate the remaining eigenvalues and eigenvectors of A.

Note: The process of changing A to B and of then solving a matrix eigenvalue problem of order one less than for A, is known as *deflation*. It can be used to extend the applicability of the power method to other than the dominant eigenvalue.

3. (Adapted from Demmel) Matrix A is called *strictly column diagonally dominant*, if

$$|a_{ii}| > \sum_{j=1,\ j\neq i}^{n} |a_{ji}|.$$

   (a) Show that A is nonsingular.

   (b) Show that Gaussian elimination with partial pivoting does not actually permute any rows, i.e. that it is identical to Gaussian elimination without pivoting.

4. (Adapted from Strang) Describe all 3 by 3 matrices that are simultaneously Hermitian, orthonormal and diagonal. How many of these are there?

5. (Adapted from Strang) Suppose Fibonacci had started his sequence with $F_0 = 1$ and $f_1 = 3$, and then followed the same rule $F_{k+1} = F_{k+1} + F_k$. Find the new initial vector u_0, the 10th Fibonacci number, the 50th Fibonacci number and show that F_{k+1}/F_k approaches the golden mean.

6. (Adapted from Strang) Describe the four fundamental subspace associated with

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

7. (Adapted from Strang) If A has the same four fundamental subspaces as B, does $A = B$?

8. (Adapted from Apostol) In $\mathbb{R}[x]_n$ define

$$\langle f, g\rangle = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) g\left(\frac{k}{n}\right).$$

   (a) Prove that $\langle f, g\rangle$ is an inner product for $\mathbb{R}[x]_n$.

(b) Compute $\langle f, g\rangle$ when $f(t) = t$ and $g(t) = at + b$.

(c) If $f(t) = t$, find all linear polynomials g orthogonal to f.

9. (Adapted from Apostol) In $C((0, \pi); \mathbb{R})$ with inner product $\langle f, g\rangle = \int_0^\pi f(t)g(t)\, dt$, let $f_n(t) = \cos nt$ for $n = 0, 1, 2, \ldots$. Prove the functions $g_0, g_1, g_2, \ldots,$ given by

$$g_0 = \frac{1}{\pi} \quad \text{and} \quad g_n = \sqrt{\frac{2}{\pi}} \cos nt \qquad \text{for } n \geq 1,$$

form an orthonormal set spanning the same subspace as $f_0,\ f_1,\ f_2, \ldots$.

10. (Adapted from Trefthen) Consider the matrices

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

What is the orthogonal projector P onto $\mathscr{R}(A)$, and what is the image under P of $[1\ 2\ 3]^T$? Repeat for B.

11. (Adapted from Demmel) Let A_k be the best rank-k approximation of the matrix A, as defined in the so-called messy theorem. Let σ_i be the ith singular value of A. Show that A_k is unique if $\sigma_k > \sigma_{k+1}$.

12. (Adapted from Demmel) Let A have the form

$$A = \begin{bmatrix} R \\ S \end{bmatrix},$$

where R is n by n and upper triangular , and S is m by n and dense. Describe an algorithm using Householder triangulations for reducing A ti upper triangular form.

13. (Adapted from Axler) Suppose n is a positive integer. Define $T \in \mathscr{L}(\mathbb{F}^n)$ by $T(z_1, \ldots z_n) = (0, z_1, \ldots z_{n-1})$. Find a formula for T^*.

14. (Adapted from Strang) Find the best least squares solutions to

$$\begin{bmatrix} 2 & -1 \\ 2 & 2 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$$

15. (Adapted from Demmel) In this problem we will consider how to solve the *Sylvester* or *Lyapunov* equation $AX - XB = C$ where X and C are m-by-n, A is m-by-m, and B is n-by-n. This is a system of mn linear equations for the entries of X. If you wish you may assume that all matrix entries are real numbers.

(a) Given the Schur decomposition of A and B, show how $AX - XB = C$ can be transformed into a similar system $A'Y - YB' = C'$, where A' and B' are upper triangular.

(b) Show how to solve for the entries of Y one at a time by a process similar to back substitution. What conditions on the eigenvalues of A and B guarantees that the system of equations is nonsingular?

(c) Show how to transform Y to get the solution X.

16. (Adapted from Quarteroni) Prove that, if A is a symmetric and positive definite matrix, solving the linear system $A\mathbf{x} = \mathbf{b}$ amounts to computing $\mathbf{x} = \sum_{i=1}^n (c_i/\lambda_i)\mathbf{v}_i$, where λ_i are the eigenvalues of A and $\mathbf{v}_i$ are the corresponding eigenvectors.

17. Suppose A is an $m \times n$ matrix and B is the $n \times m$ matrix obtained by rotating ninety degrees clockwise on paper (not exactly a standard transformation :)). Do A and B have the same singular values? Prove that the answer is yes or give a counter example.

18. Using the SVD, prove that any matrix in $M_{m\times n}(\mathbb{F})$ is the limit of a sequence of matrices of full rank.