Ghosts and Phantoms: A Correction to an Exploratory Framework for Completing Graph Arithmetic

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This note serves as a brief correction and clarification to ideas developed in *Graph Reals: An Exploratory Framework for Completing Graph Arithmetic* [1]. That manuscript was explicitly presented as provisional and exploratory: it compiled several constructions and heuristics surrounding the completion of graph arithmetic into a field-like and metric setting, with the stated goal of understanding what kinds of "continuous" graph objects might emerge. As part of that exploration, we pursued a limiting construction intended to produce "ghost edges"—objects with vanishing vertex count but residual edge structure—motivated by the analytic behaviour of the extended vertex and edge invariants near singular loci. In hindsight, this was an instance of working too hard: we were carried away by the excitement of the completion framework, failed to double-check convergence assumptions carefully, and overlooked the simpler algebraic fact that the objects in question already exist at a much earlier stage of the theory. This note records that hiccup and corrects the narrative.

Ghosts and Phantoms in the Graph Integers

Let $\mathbb{Z}_G$ denote the ring of Graph Integers, obtained as the Grothendieck completion of finite graphs under disjoint union and Cartesian product. There are two canonical ring homomorphisms

$$v, e : \mathbb{Z}_G \longrightarrow \mathbb{Z},$$

extending the usual vertex and edge counts by

$$v([G] - [H]) = v(G) - v(H), \qquad e([G] - [H]) = e(G) - e(H).$$

These maps provide coarse numerical invariants of graph arithmetic, but their kernels already contain structurally nontrivial elements.

We distinguish two classes of such elements.

Ghosts. A *ghost* is a nonzero element $g \in \mathbb{Z}_G$ with

$$v(g) = 0.$$

Equivalently, g is represented by a formal difference of graphs with the same total vertex count. Ghosts may have nonzero edge count, and indeed the simplest examples arise from differences of graphs on the same vertex set but with different edge sets. From the point of view of (v, e), ghosts occupy the $v = 0$ hyperplane, and they form an ideal of $\mathbb{Z}_G$ as the kernel of the ring homomorphism v.

Phantoms. A *phantom* is a nonzero element $p \in \mathbb{Z}_G$ satisfying

$$v(p) = 0 \quad \text{and} \quad e(p) = 0.$$

Thus phantoms lie in the intersection $\ker v \cap \ker e$. Concretely, a phantom can be realized as the difference of two non-isomorphic graphs with the same number of vertices and edges. Although invisible to both of the basic numerical invariants v and e, a phantom is not the additive identity in $\mathbb{Z}_G$ and generally carries genuine combinatorial structure.

The presence of phantoms highlights an important limitation of the (v, e)-description: while v and e behave like the zeroth- and first-order terms of a valuation, they do not exhaust the information contained in a graph integer. Phantoms encode higher-order or purely geometric data that survives subtraction but is annihilated by both coarse counts.

Concrete examples

A few explicit elements make the definitions less abstract.

Example 1 (A ghost with nonzero edge count)**.** *Let K_2 be the graph on two vertices with a single edge, and let $2K_1$ denote the disjoint union of two isolated vertices. In $\mathbb{Z}_G$ consider*

$$g := [K_2] - [2K_1].$$

Then
$$v(g) = v(K_2) - v(2K_1) = 2 - 2 = 0, \qquad e(g) = e(K_2) - e(2K_1) = 1 - 0 = 1.$$

So g is a ghost: it has zero vertex count in the extended sense but carries one "unit" of edge information. As an element of $\mathbb{Z}_G$, g still remembers the distinction between a genuine edge and two isolated vertices; that information sits entirely beyond the reach of v and e.

Example 2 (A phantom)**.** *Let P_4 be the path on four vertices and $K_{1,3}$ the star on four vertices. Both graphs satisfy*
$$v(P_4) = v(K_{1,3}) = 4, \qquad e(P_4) = e(K_{1,3}) = 3,$$

but they are non-isomorphic. The element

$$p := [P_4] - [K_{1,3}] \in \mathbb{Z}_G$$

then has
$$v(p) = 4 - 4 = 0, \qquad e(p) = 3 - 3 = 0,$$

so p lies in $\ker v \cap \ker e$. By construction $p \neq 0$ in $\mathbb{Z}_G$, since $[P_4]$ and $[K_{1,3}]$ are distinct generators in the Grothendieck completion. Thus p is a phantom: the coarse invariants see nothing, yet the group element still encodes a very concrete geometric difference between a path and a star.

By taking differences of any two non-isomorphic graphs with the same number of vertices and edges, one obtains an infinite family of phantoms. For any fixed vertex number n, differences of graphs on n vertices with distinct edge counts give ghosts with prescribed e-values.

Multiplicative Structure and Stratification

The maps v and e satisfy the identities

$$v(gh) = v(g)v(h), \qquad e(gh) = v(g)e(h) + v(h)e(g),$$

From these formulas one obtains a natural stratification of $\mathbb{Z}_G$ (and of $\mathbb{Q}_G$ wherever the formulas apply) into three types:

regular elements $(v \neq 0)$, ghosts $(v = 0, \ e \neq 0)$, phantoms $(v = 0, \ e = 0)$.

Multiplication respects this stratification in a one-directional way. Products of regular elements remain regular. Multiplying a ghost by a regular element yields another ghost, while the product of two ghosts has vanishing (v, e) and therefore lands in the phantom stratum. Once an element lies in the phantom stratum, multiplication, at least within the Graph Integers, cannot lift it back out: products involving phantoms remain phantoms (or collapse to zero). In this sense, multiplication pushes elements "downward" through the filtration

regular $\supset$ ghost $\supset$ phantom.

This behaviour explains several features that were previously interpreted analytically. For example, ghost elements are nilpotent at the level of the (v, e)-calculus: if g is a ghost, then g^2 has $v(g^2) = e(g^2) = 0$, so it is phantom from the point of view of the coarse invariants, even though it need not vanish in $\mathbb{Z}_G$ or $\mathbb{Q}_G$.

Passage to Rationals and Reals

When passing from $\mathbb{Z}_G$ to the Graph Rationals $\mathbb{Q}_G$ and further to the Graph Reals $\mathbb{R}_G$, the algebraic distinction between ghosts and phantoms remains meaningful, but the ideal language no longer applies: fields admit no nontrivial ideals. Instead, ghosts and phantoms should be regarded as distinguished additive subgroups or strata defined by the vanishing of the invariants v and e. Proper extension of e and v over to the Graph Rationals and Graph Reals needs more consideration than what was done in the initial exploration paper.

Conclusion

The corrected picture is straightforward. Ghosts and phantoms are intrinsic elements of the Graph Integers. There is no need to even reach for Graph Reals and a functional theory of limits to find them. Their behaviour under multiplication reveals a natural stratification of graph arithmetic, while their invisibility to (v, e) explains why naive analytic reasoning can be misleading near $v = 0$. The completion to Graph Reals remains valuable for geometric and metric questions, but it should be understood as refining an already rich algebraic landscape, not as the source of these phenomena.

A   Well-definedness of Vertex and Edge Counts

Proposition 1 (Well-definedness of v and e)**.** *Let $\mathbb{Z}_G$ be the Grothendieck completion of finite graphs under disjoint union. If $[G] - [H] = [G'] - [H']$ in $\mathbb{Z}_G$, then*

$$v(G) - v(H) = v(G') - v(H') \quad and \quad e(G) - e(H) = e(G') - e(H').$$

Hence the functions $v, e : \mathbb{Z}_G \to \mathbb{Z}$ given by

$$v([G] - [H]) := v(G) - v(H), \qquad e([G] - [H]) := e(G) - e(H)$$

are well defined.

Proof. Since disjoint union of finite graphs is cancellative up to isomorphism, the Grothendieck relation is:

$$(G, H) \sim (G', H') \quad \Longleftrightarrow \quad G \sqcup H' \cong G' \sqcup H.$$

Assume $[G] - [H] = [G'] - [H']$ in $\mathbb{Z}_G$. Then

$$G \sqcup H' \cong G' \sqcup H.$$

Graph isomorphism preserves vertex and edge counts, and both v and e are additive under disjoint union. Therefore

$$v(G) + v(H') = v(G') + v(H), \qquad e(G) + e(H') = e(G') + e(H).$$

Rearranging gives

$$v(G) - v(H) = v(G') - v(H'), \qquad e(G) - e(H) = e(G') - e(H'),$$

which is exactly the statement that v and e are independent of the chosen representative. $\square$

Proposition 2 (Ghosts form an ideal). *Let $\mathbb{Z}_G$ be the Grothendieck ring of finite graphs with addition given by disjoint union and multiplication given by Cartesian product, and let*

$$v : \mathbb{Z}_G \to \mathbb{Z}$$

be the vertex-count map. Define the ghost ideal *by*

$$\mathcal{G} := \ker(v) = \{g \in \mathbb{Z}_G : v(g) = 0\}.$$

Then $\mathcal{G}$ is a (two-sided) ideal of $\mathbb{Z}_G$. Moreover, $g \in \mathcal{G}$ if and only if g admits a representative $g = [G] - [H]$ with $v(G) = v(H)$.

Proof. For the final statement, write $g = [G] - [H]$. By definition of v on $\mathbb{Z}_G$,

$$v(g) = v(G) - v(H).$$

Hence $v(g) = 0$ holds exactly when $v(G) = v(H)$ for (equivalently, for every) representative.

To show $\mathcal{G}$ is an ideal, first note that $\mathcal{G}$ is an additive subgroup of $\mathbb{Z}_G$ because v is a group homomorphism: if $g_1, g_2 \in \mathcal{G}$ then

$$v(g_1 + g_2) = v(g_1) + v(g_2) = 0, \qquad v(-g_1) = -v(g_1) = 0,$$

so $g_1 + g_2, -g_1 \in \mathcal{G}$.

Finally, let $x \in \mathbb{Z}_G$ and $g \in \mathcal{G}$. Since v is a ring homomorphism,

$$v(xg) = v(x)\, v(g) = v(x) \cdot 0 = 0,$$

so $xg \in \mathcal{G}$. Since $\mathbb{Z}_G$ is commutative, this also gives $gx \in \mathcal{G}$. Therefore $\mathcal{G}$ is an ideal of $\mathbb{Z}_G$. $\square$

References

[1] Daniel Goldman. *Graph Reals: An Exploratory Framework for Completing Graph Arithmetic.* Unpublished, 2025. `https://www.researchgate.net/doi/10.13140/RG.2.2.30709.03040`.