5. Write down a 5×5 matrix P such that multiplication of another matrix by P on the left causes rows 2 and 5 to be exchanged.

6. (a) Write down the 4×4 matrix P such that multiplying a matrix on the left by P causes the second and fourth rows of the matrix to be exchanged. (b) What is the effect of multiplying on the *right* by P? Demonstrate with an example.

7. Change four entries of the leftmost matrix to make the matrix equation correct:

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 5 & 6 \\ 5 & 6 & 7 & 8 \\ 7 & 8 & 9 & 0 \end{bmatrix} = \begin{bmatrix} 5 & 6 & 7 & 8 \\ 3 & 4 & 5 & 6 \\ 7 & 8 & 9 & 0 \\ 1 & 2 & 3 & 4 \end{bmatrix}.$$

8. Find the PA = LU factorization of the matrix A in Exercise 2.3.15. What is the largest multiplier l_{ij} needed?

9. (a) Find the PA = LU factorization of $A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ -1 & 1 & 0 & 1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & -1 & 1 \end{bmatrix}$. (b) Let A be the $n \times n$ matrix of the same form as in (a). Describe the entries of each matrix of its PA = LU factorization.

10. (a) Assume that A is an $n \times n$ matrix with entries $|a_{ij}| \le 1$ for $1 \le i, j \le n$. Prove that the matrix U in its PA = LU factorization satisfies $|u_{ij}| \le 2^{n-1}$ for all $1 \le i, j \le n$. See Exercise 9(b). (b) Formulate and prove an analogous fact for an arbitrary $n \times n$ matrix A.

Reality Check 2 The Euler–Bernoulli Beam

The Euler–Bernoulli beam is a fundamental model for a material bending under stress. Discretization converts the differential equation model into a system of linear equations. The smaller the discretization size, the larger is the resulting system of equations. This example will provide us an interesting case study of the roles of system size and ill-conditioning in scientific computation.

The vertical displacement of the beam is represented by a function $y(x)$, where $0 \le x \le L$ along the beam of length L. We will use MKS units in the calculation: meters, kilograms, seconds. The displacement $y(x)$ satisfies the Euler–Bernoulli equation

$$EIy'''' = f(x) \tag{2.27}$$

where E, the Young's modulus of the material, and I, the area moment of inertia, are constant along the beam. The right-hand-side $f(x)$ is the applied load, including the weight of the beam, in force per unit length.

Techniques for discretizing derivatives are found in Chapter 5, where it will be shown that a reasonable approximation for the fourth derivative is

$$y''''(x) \approx \frac{y(x-2h) - 4y(x-h) + 6y(x) - 4y(x+h) + y(x+2h)}{h^4} \tag{2.28}$$

for a small increment h. The discretization error of this approximation is proportional to h^2 (see Exercise 5.1.21.). Our strategy will be to consider the beam as the union of many segments of length h, and to apply the discretized version of the differential equation on each segment.

For a positive integer n, set $h = L/n$. Consider the evenly spaced grid $0 = x_0 < x_1 < \ldots < x_n = L$, where $h = x_i - x_{i-1}$ for $i = 1, \ldots, n$. Replacing the differential equation (2.27) with the difference approximation (2.28) to get the system of linear equations for the displacements $y_i = y(x_i)$ yields

$$y_{i-2} - 4y_{i-1} + 6y_i - 4y_{i+1} + y_{i+2} = \frac{h^4}{EI} f(x_i). \tag{2.29}$$

We will develop n equations in the n unknowns $y_1, \ldots, y_n$. The coefficient matrix, or structure matrix, will have coefficients from the left-hand side of this equation. However, notice that we must alter the equations near the ends of the beam to take the boundary conditions into account.

A diving board is a beam with one end clamped at the support, and the opposite end free. This is called the **clamped-free** beam or sometimes the **cantilever** beam. The boundary conditions for the clamped (left) end and free (right) end are

$$y(0) = y'(0) = y''(L) = y'''(L) = 0.$$

In particular, $y_0 = 0$. Note that finding y_1, however, presents us with a problem, since applying the approximation (2.29) to the differential equation (2.27) at x_1 results in

$$y_{-1} - 4y_0 + 6y_1 - 4y_2 + y_3 = \frac{h^4}{EI} f(x_1), \tag{2.30}$$

and y_{-1} is not defined. Instead, we must use an alternate derivative approximation at the point x_1 near the clamped end. Exercise 5.1.22(a) derives the approximation

$$y''''(x_1) \approx \frac{16y(x_1) - 9y(x_1 + h) + \frac{8}{3}y(x_1 + 2h) - \frac{1}{4}y(x_1 + 3h)}{h^4} \tag{2.31}$$

which is valid when $y(x_0) = y'(x_0) = 0$.

Calling the approximation "valid," for now, means that the discretization error of the approximation is proportional to h^2, the same as for equation (2.28). In theory, this means that the error in approximating the derivative in this way will decrease toward zero in the limit of small h. This concept will be the focal point of the discussion of numerical differentiation in Chapter 5. The result for us is that we can use approximation (2.31) to take the endpoint condition into account for $i = 1$, yielding

$$16y_1 - 9y_2 + \frac{8}{3}y_3 - \frac{1}{4}y_4 = \frac{h^4}{EI} f(x_1).$$

The free right end of the beam requires a little more work because we must compute y_i all the way to the end of the beam. Again, we need alternative derivative approximations at the last two points x_{n-1} and x_n. Exercise 5.1.22 gives the approximations

$$y''''(x_{n-1}) \approx \frac{-28y_n + 72y_{n-1} - 60y_{n-2} + 16y_{n-3}}{17h^4} \tag{2.32}$$

$$y''''(x_n) \approx \frac{72y_n - 156y_{n-1} + 96y_{n-2} - 12y_{n-3}}{17h^4} \tag{2.33}$$

which are valid under the assumption $y''(x_n) = y'''(x_n) = 0$.

Now we can write down the system of n equations in n unknowns for the diving board. This matrix equation summarizes our approximate versions of the original differential equation (2.27) at each point $x_1, \ldots, x_n$, accurate within terms of order h^2:

$$\begin{bmatrix} 16 & -9 & \frac{8}{3} & -\frac{1}{4} & & & & & & \\ -4 & 6 & -4 & 1 & & & & & & \\ 1 & -4 & 6 & -4 & 1 & & & & & \\ & 1 & -4 & 6 & -4 & 1 & & & & \\ & & \ddots & \ddots & \ddots & \ddots & \ddots & & & \\ & & & & 1 & -4 & 6 & -4 & 1 & \\ & & & & & 1 & -4 & 6 & -4 & 1 \\ & & & & & & \frac{16}{17} & -\frac{60}{17} & \frac{72}{17} & -\frac{28}{17} \\ & & & & & & -\frac{12}{17} & \frac{96}{17} & -\frac{156}{17} & \frac{72}{17} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ \\ \vdots \\ \\ \vdots \\ y_{n-1} \\ y_n \end{bmatrix} = \frac{h^4}{EI} \begin{bmatrix} f(x_1) \\ f(x_2) \\ \vdots \\ \\ \vdots \\ \\ \vdots \\ f(x_{n-1}) \\ f(x_n) \end{bmatrix}. \tag{2.34}$$

The structure matrix A in (2.34) is a **banded matrix**, meaning that all entries sufficiently far from the main diagonal are zero. Specifically, the matrix entries $a_{ij} = 0$, except for $|i - j| \le 3$. The **bandwidth** of this banded matrix is 7, since $i - j$ takes on 7 values for nonzero a_{ij}.

Finally, we are ready to model the clamped-free beam. Let us consider a solid wood diving board composed of Douglas fir. Assume that the diving board is $L = 2$ meters long, 30 cm wide, and 3 cm thick. The density of Douglas fir is approximately 480 kg/m^3. One Newton of force is 1 kg-m/sec^2, and the Young's modulus of this wood is approximately $E = 1.3 \times 10^{10}$ Pascals, or Newton/m^2. The area moment of inertia I around the center of mass of a beam is $wd^3/12$, where w is the width and d the thickness of the beam.

You will begin by calculating the displacement of the beam with no payload, so that $f(x)$ represents only the weight of the beam itself, in units of force per meter. Therefore $f(x)$ is the mass per meter $480wd$ times the downward acceleration of gravity $-g = -9.81$ m/sec^2, or the constant $f(x) = f = -480wdg$. The reader should check that the units match on both sides of (2.27). There is a closed-form solution of (2.27) in the case f is constant, so that the result of your computation can be checked for accuracy.

Following the check of your code for the unloaded beam, you will model two further cases. In the first, a sinusoidal load (or "pile") will be added to the beam. In this case, there is again a known closed-form solution, but the derivative approximations are not exact, so you will be able to monitor the error of your modeling as a function of the grid size h, and see the effect of conditioning problems for large n. Later, you will put a diver on the beam.

Suggested activities:

1. Write a MATLAB program to define the structure matrix A in (2.34). Then, using the MATLAB \ command or code of your own design, solve the system for the displacements y_i using $n = 10$ grid steps.

2. Plot the solution from Step 1 against the correct solution $y(x) = (f/24EI)x^2(x^2 - 4Lx + 6L^2)$, where $f = f(x)$ is the constant defined above. Check the error at the end of the beam, $x = L$ meters. In this simple case the derivative approximations are exact, so your error should be near machine roundoff.

3. Rerun the calculation in Step 1 for $n = 10 \cdot 2^k$, where $k = 1, \ldots, 11$. Make a table of the errors at $x = L$ for each n. For which n is the error smallest? Why does the error begin to increase with n after a certain point? You may want to make an accompanying table of the

condition number of A as a function of n to help answer the last question. To carry out this step for large k, you may need to ask MATLAB to store the matrix A as a sparse matrix to avoid running out of memory. To do this, just initialize A with the command `A=sparse(n,n)`, and proceed as before. We will discuss sparse matrices in more detail in the next section.

4. Add a sinusoidal pile to the beam. This means adding a function of form $s(x) = -pg \sin \frac{\pi}{L}x$ to the force term $f(x)$. Prove that the solution

$$y(x) = \frac{f}{24EI}x^2(x^2 - 4Lx + 6L^2) - \frac{pgL}{EI\pi}\left(\frac{L^3}{\pi^3}\sin\frac{\pi}{L}x - \frac{x^3}{6} + \frac{L}{2}x^2 - \frac{L^2}{\pi^2}x\right)$$

satisfies the Euler–Bernoulli beam equation and the clamped-free boundary conditions.

5. Rerun the calculation as in Step 3 for the sinusoidal load. (Be sure to include the weight of the beam itself.) Set $p = 100$ kg/m and plot your computed solutions against the correct solution. Answer the questions from Step 3, and in addition the following one: Is the error at $x = L$ proportional to h^2 as claimed above? You may want to plot the error versus h on a log–log graph to investigate this question. Does the condition number come into play?

6. Now remove the sinusoidal load and add a 70 kg diver to the beam, balancing on the last 20 cm of the beam. You must add a force per unit length of $-g$ times 70/0.2 kg/m to $f(x_i)$ for all $1.8 \le x_i \le 2$, and solve the problem again with the optimal value of n found in Step 5. Plot the solution and find the deflection of the diving board at the free end.

7. If we also fix the free end of the diving board, we have a "clamped-clamped" beam, obeying identical boundary conditions at each end: $y(0) = y'(0) = y(L) = y'(L) = 0$. This version is used to model the sag in a structure, like a bridge. Begin with the slightly different evenly spaced grid $0 = x_0 < x_1 < \ldots < x_n < x_{n+1} = L$, where $h = x_i - x_{i-1}$ for $i = 1, \ldots, n$, and find the system of n equations in n unknowns that determine $y_1, \ldots, y_n$. (It should be similar to the clamped-free version, except that the last two rows of the coefficient matrix A should be the first two rows reversed.) Solve for a sinusoidal load and answer the questions of Step 5 for the center $x = L/2$ of the beam. The exact solution for the clamped-clamped beam under a sinusoidal load is

$$y(x) = \frac{f}{24EI}x^2(L - x)^2 - \frac{pgL^2}{\pi^4 EI}\left(L^2 \sin\frac{\pi}{L}x + \pi x(x - L)\right).$$

8. Ideas for further exploration: If the width of the diving board is doubled, how does the displacement of the diver change? Does it change more or less than if the thickness is doubled? (Both beams have the same mass.) How does the maximum displacement change if the cross-section is circular or annular with the same area as the rectangle? (The area moment of inertia for a circular cross-section of radius r is $I = \pi r^4/4$, and for an annular cross-section with inner radius r_1 and outer radius r_2 is $I = \pi(r_2^4 - r_1^4)/4$.) Find out the area moment of inertia for I-beams, for example. The Young's modulus for different materials are also tabulated and available. For example, the density of steel is about 7850 kg/m^3 and its Young's modulus is about 2×10^{11} Pascals.

The Euler–Bernoulli beam is a relatively simple, classical model. More recent models, such as the Timoshenko beam, take into account more exotic bending, where the beam cross-section may not be perpendicular to the beam's main axis. ✔