Theorem 1

We can explicitly construct a dimension d, $d^2 \times d^2$ matrices A, B, C and D, and a rational number $\beta > 0$, which can be chosen to be as small as desired, such that

(i) A is Hermitian, with matrix elements in $\mathbb{Z} + \beta\mathbb{Z} + \frac{\beta}{\sqrt{2}}\mathbb{Z}$;

(ii) B and C have integer matrix elements; and

(iii) D is Hermitian, with matrix elements in $\{0, 1, \beta\}$.

For each positive integer n, define the local interactions of a translationally invariant, nearest-neighbour Hamiltonian $H(n)$ on a 2D square lattice as

$$h_1(n) = \alpha(n)\Pi$$

$$h_{\text{row}} = D$$

$$h_{\text{col}} = A + \beta\left(e^{i\pi\varphi(n)}B + e^{-i\pi\varphi(n)}B^\dagger + e^{i\pi 2^{-|\varphi(n)|}}C + e^{-i\pi 2^{-|\varphi(n)|}}C^\dagger\right)$$

where $\varphi(n) = n/2^{|n|-1}$ is the rational number whose binary fraction expansion contains the binary digits of n after the decimal point, $|\varphi(n)|$ denotes the number of digits in this expansion, $\alpha(n) \leq \beta$ is an algebraic number that is computable from n, Π is a projector and the daggers denote Hermitian conjugation. Then

(i) the local interaction strength is ≤ 1 (that is, $\|h_1(n)\|, \|h_{\text{row}}\|, \|h_{\text{col}}(n)\| \leq 1$);

(ii) if the universal Turing machine halts on input n, the Hamiltonian $H(n)$ is gapped with $\gamma \geq 1$; and

$$h(\varphi)^{(i,j)} = |0\rangle\langle 0|^{(i)} \otimes (\mathbb{1} - |0\rangle\langle 0|)^{(j)} + h_{\text{u}}^{(i,j)}(\varphi) \otimes \mathbb{1}_{\text{d}}^{(i,j)} + \mathbb{1}_{\text{u}}^{(i,j)} \otimes h_{\text{d}}^{(i,j)} \tag{1}$$

The spectrum of the new Hamiltonian H is

$$\text{spec}H = \{0\} \cup \{\text{spec}H_{\text{u}}(\varphi) + \text{spec}H_{\text{d}}\} \cup S \tag{2}$$

with $S \geq 1$ (see Supplementary Information for details). Recalling that we chose H_{d} to be gapless, we see immediately from equation (2) that if the ground state energy density of H_{u} tends to zero from below (so that $\lambda_0(H_{\text{u}}) < 0$), then $H(\varphi)$ is gapless; if H_{u} has a strictly positive ground state energy density (so that $\lambda_0(H_{\text{u}})$ diverges to $+\infty$), then it has a spectral gap ≥ 1, as required (see Fig. 2).

This construction is rather general: by choosing different h_{d}, we obtain undecidability of any physical property that distinguishes a Hamiltonian from a gapped system with a unique product ground state.

Encoding computation in ground states

To construct the Hamiltonian $H_{\text{u}}(\varphi)$, we encode the halting problem into the local interactions $h_{\text{u}}(\varphi)$ of the Hamiltonian. The halting problem concerns the dynamics of a classical system—a Turing machine. To relate it to the ground state energy density—a static property of a quantum system—we construct a Hamiltonian whose ground state encodes the entire history of the computation carried out by the Turing