Applying Conformal Geometric Algebra Algorithms to 3-D Survey Plan Boundary Topology Problems

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SUMMARY

This paper applies Conformal Geometric Algebra algorithms and operational techniques, along with 3-D point-point distance evaluations and geometric concepts such as collinearity and coplanarity to test data modelling processes that are designed for the purpose of classifying geometrical and topological relationships between 3-D spatial objects. The paper reports on how these computational processes can be applied to the boundaries of 3-D cadastral units registered to a survey plan for the purpose of solving 3-D cadastral boundary problems. These problems are related to survey plan validation, specifically for verifying that adjacent units have the correct shared boundaries as intended on the plan prior to registration.

The approach to classifying relationships between two 3-D cadastral units A and B was to first evaluate the relationships between all boundary components in unit A with all boundary components in unit B. These relationships were then combined and interpreted together to describe the relationships between two 3-D cadastral units A and B. Six sets of data flow processing algorithms were developed to determine the relationship classifications that occur between the point-point, line-point, line-line, plane-point, plane-line, and plane-plane boundary component pair sets between the cadastral units. These algorithms were coded in MATLAB.

Several experiments using simulated datasets were presented in a previous paper to validate the relationship classification processes that were applied in this paper. The experiment presented here consists of a cadastral dataset that was derived from a condominium survey plan registered in Alberta, Canada. Relationships between all sets of 3-D cadastral units tested were known *a priori* to running the experiments and relationship classification results were validated manually to check if the implementation program produced the correct results. Results show how relationship classifications between 3-D cadastral units that have shared boundary points, lines and planes can be verified using data flow processes prior to registering a survey plan. These processes could be leveraged by land surveyors and land administration professionals when analysing 3-D survey plan boundaries.

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1. INTRODUCTION

This paper reports on research that applied methodological processes that were developed for the purpose of performing topological relationship analysis on 3-D legal boundaries. It applies data flow processes that use established mathematical theory of Conformal Geometric Algebra (CGA) objects and operations in combination with various 3-D point-point distance evaluations to achieve this. Specifically, it investigates how these processes could be applied to solve 3-D cadastral boundary topology problems between the boundaries of two adjacent 3-D cadastral units that were derived from a condominium survey plan registered in Alberta, Canada.

1.1 Problem Context (Background)

As municipalities and other jurisdictions grow and develop, there is an increased need for mapping and managing complex legal structures that exist above and below the ground such as rights, restrictions, and responsibilities associated with buildings, underground facilities, and utilities, among other right spaces such as those that are shown by El-Mekawy et al. (2015) and Kitsakis et al. (2016). Currently 3-D property can be registered to a title with a supplementary condominium or strata subdivision survey plan in many jurisdictions, including Alberta, Canada. In these scenarios, cadastral boundaries are recorded and registered as 2-D drawings, vertical profiles, cross sections, and 3-D isometric drawings. An example layout describing an isometric view can be seen in Figure 1 below. Boundary analysis can be performed by visual inspection of the survey plan drawings, but this often requires professional expertise such as a land surveyor or lawyer. In more complex layouts, investigating these plan drawings becomes more difficult and can be time consuming.

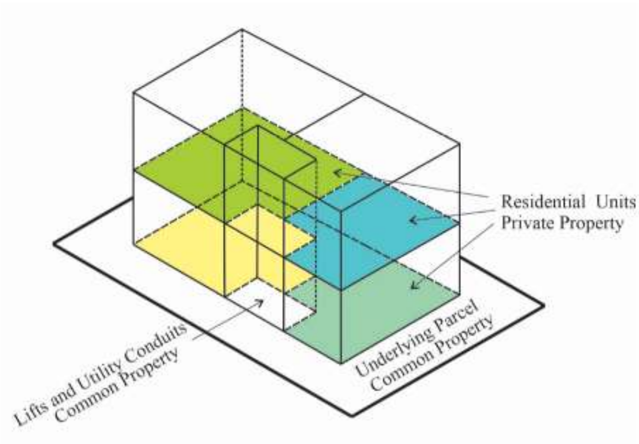

Figure 1: Common 3-D Plan Schema (Source Barry, M. 2015)

1.2 Objectives

This research applied relationship classification algorithms to solve a practical 3-D boundary problem example in the land surveying field, specifically towards validating a shared boundary between two adjacent 3-D cadastral units as intended on the survey plan prior to registering it.

The algorithms that were applied to achieve this consisted of methodological data flow processes that were designed to classify geometrical and topological relationships between the 3-D boundary components of two 3-D cadastral units by applying existing CGA mathematical algorithms and operational techniques along with 3-D point-point distance evaluations as was presented by Pullano & Barry (2020). The main objectives were as follows.

- Generate digital parameters for the points, lines, and planes defining the boundary of two 3-D cadastral units using measurements included in the survey plan drawings
- Test topological relationship algorithms on the generated cadastral dataset to validate the shared boundary relationship and geometry between the two 3-D cadastral units

1.3 Scope and Application

This is the second part of a research project. The first part presented by Pullano & Barry (2020) validated boundary relationship classification algorithms using seven simulated 3-D datasets. This paper applies these algorithms to classify topological relationships between the boundaries of two 3-D cadastral units registered by a condominium survey plan where the boundaries of units are defined by the centre of the walls and floors between adjacent units. The work was performed to show how the relationship classification algorithms can be applied for 3-D survey plan boundary validation purposes. The experimental work in this paper is limited to classifying relationships between 3-D points, lines, and planes (i.e. 3-D boundary components) only. We do not cover relationships between volumetric spaces.

2. THEORY

Theory is presented in three sections. Section 2.1 presents theory related to defining and registering 3-D property in a 3-D cadastral system. It describes the ways that 3-D property can be registered in Alberta, Canada and presents the approach chosen in this paper for representing 3-D boundaries. Section 2.2 reviews some of the foundations for organizing topological relationships between spatial objects and discusses how relationships can be organized for objects modelled in 3-D space. Section 2.3 reviews mathematical theory associated with CGA. It briefly explains how the algebra is set up and presents important equations that were used in this paper to generate 3-D boundary component parameters and to perform geometric and topological operations between them.

2.1 Cadastral Property and 3-D Boundary Definitions

Stoter (2004) claims that a 3-D cadastre should record the geometries, ownership information, and other interests related to land and 3-D property. Among these other interests are Rights, Restrictions, and Responsibilities (RRRs). El-Mekawy et al. (2015) say that these RRRs will be different depending on the legal system for land and property registration in a specific country or jurisdiction. Kitsakis et al. (2016) performed a study that reviewed how several jurisdictions are already implementing some form of a 3-D cadastral system. They found that there were varying degrees to which 3-D property was recorded and 3-D rights were able to be officially registered in the existing cadastral systems (Kitsakis et al., 2016). This suggests that any solution related to modelling 3-D property in a cadastral system should consider the 3-D property types and boundary formats used to register them for a specific jurisdiction being considered. This research considers 3-D property that can be registered in Alberta, Canada.

In Alberta's cadastral system, the survey plan which includes geometric drawings with angles and distance measurements is what determines the extent of the real property's boundary that is linked to each title of ownership. There are two main subdivision survey plan types that can be used to register 3-D property in Alberta, Canada, being condominium and strata survey plans. A condominium survey plan in Alberta subdivides a parcel into at least two units and usually defines common property. The exterior boundary of the condominium building is georeferenced to known ground control points and the building footprint needs to be contained within the underlying 2-D parcel. Condominium unit boundaries can be calculated when their boundaries are registered using the centreline of walls and floors between adjacent units. A strata survey plan in Alberta subdivides a parcel into volumetric units located above or below the earth's surface or occupied in whole or in part by any structure. An example of a scenario in which this is needed would be for a mixed-use development involving a parkade and various commercial offices. Strata boundaries are defined by georeferenced coordinates that are referenced to a known boundary beacon and network. The example presented in this paper is derived from a centerline condominium survey plan. A similar process can be followed to model strata units.

The surveyed objects that this research deals with are derived using vector-based datasets. The legal boundaries are surveyed and represented using 3-D boundary point coordinates, straight boundary lines derived from plan angles and distances, and polylines that are combined to create flat boundary planes. For this reason, and to reflect the accuracy of cadastral boundaries recorded in condominium and strata survey plans, a vector-based storage method is used similar to the one proposed by Ying et al. (2014). The main geometric primitives (boundary components) used by Ying et al. (2014) to represent 3-D cadastral objects are node, edge, and face. A node (point) is defined by a 3-D [x, y, z] coordinate, an edge (line) is defined by two (start and end) nodes, and a face (plane) is defined by a closed polygon consisting of at least 3 edges. A 3-D cadastral boundary is closed by at least 4 boundary faces, each having an indicated normal plane direction vector to represent the outside direction with respect to the 3-D cadastral boundary being modelled. This structure ensures topological consistency between the boundary components of each cadastral object.

2.2 Topology Frameworks and Characteristics

Ellul & Haklay (2006) reviewed several research articles to identify a general list of common requirements for topology in 3-D GIS applications. Ellul & Haklay (2006) note that while topology frameworks need to be able to examine relationships between 3-D objects, to be comprehensive they also need to examine topological relationships between lower dimensional (0-D, 1-D, and 2-D) boundary components that might be part of a 3-D object (i.e. points, lines, and planes). It is important that each topological element should have some form of geometrical representation so that the results can be visualised.

The dimensional model (DM) is a topology framework presented by Billen et al. (2002) to provide insight on spatial relationships that can exist between objects. The model is presented using the concepts of dimensional elements and dimensional relationships. Dimensional elements are organized as having extensions and limits of dimensionally ordered boundary elements. Figure 2 shows common spatial elements that can exist in 3-D Euclidean space ($\mathbf{R}^3$) as a composition of their interior extension (bolded) and boundary limit (non-bolded) elements. The DM describes the relationship between two 3-D objects by combining the relationships that exist between the (0D, 1D, 2D, 3D) dimensional elements of both objects. Billen (2002) suggests that relationships could be computed using collinearity and coplanarity algorithms.

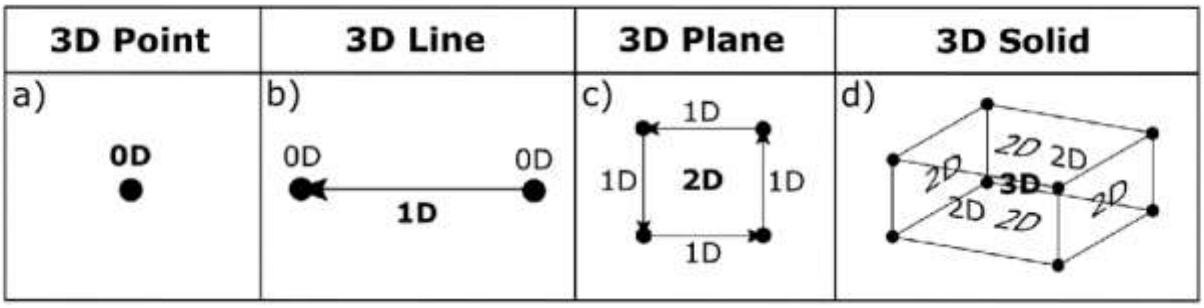

Figure 2: Cadastral Elements with Extensions and Boundaries with Respect to the DM (after Billen et al., 2002)

Fu et al. (2018) studied the characteristics of topological relationships that can exist between 3-D cadastral datasets and described an approach to classify them. They organized topology between 3-D spatial objects using disjoint, touch, overlap, contain, or equal relationships which are represented in Figure 3 below. This research considers overlap, contain, and equal relationships as the same topological relationship type (intersection/overlap) and can classify disjoint, touch, and intersection/overlap relationships between 3-D boundaries.

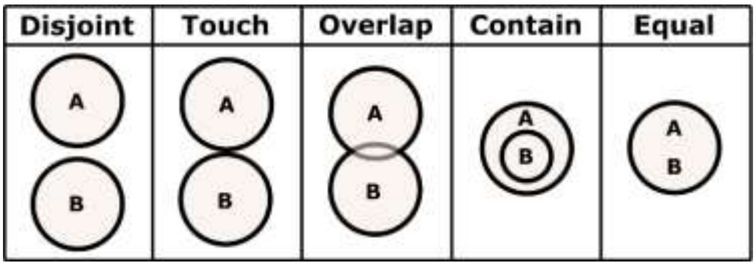

Figure 3: Visual Representation of Topological Relationships (after Fu et al., 2018)

Fu et al. (2018) identified various disjoint and touch relationships that can exist theoretically between 3-D cadastral objects by distinguishing between them using projected relationships such as the boundary lines and boundary planes between two 3-D cadastral units being collinear or coplanar. For example, in the physical world a building may contain several strata units that share the same wall. In this scenario, these boundary planes between units should be coplanar. The majority, however, will not share a boundary. The algorithms used in this research follow a similar approach in that relationships between units are first organised using collinearity and coplanarity.

2.3 Conformal Geometric Algebra Theory

Conformal Geometric Algebra is a mathematical framework that can be used to model objects that exist in 3-D Euclidean space ($\mathbf{R^3}$) such as points, lines, and planes using mathematical concepts such as the inner product, outer product, geometric product, and vector subspaces (Dorst, 2007, p. 10). It has been proven to be useful for many applications in engineering and computer science. The algorithms used in this research apply CGA theory to assist in classifying topological relationships between sets of two 3-D boundary components.

2.3.1 Geometric Algebra and the Conformal Model

The basic algebraic elements in an n-dimensional Vector Algebra are (e_1, e_2, e_3, …, e_n), or e_1 (x), e_2 (y), and e_3 (z) in 3-D Vector Algebra. In comparison, Hildenbrand & Oldenburg (2015) say that k-blades are the basic algebraic elements in an n-dimensional Geometric Algebra. They describe these blades as having grades (k = 0, 1, 2, 3, …, n) representing the dimensionality of different subspace. A scalar is a 0-blade (blade of grade k = 0) and the basis vectors (e_1, e_2, e_3, …, e_n) are 1-blades. 2-blades are spanned by two 1-blades (e.g. e_1^e_2), 3-blades are spanned by three 1-blades (e.g. e_1^e_2^e_3), and so on. The single n-blade is called a pseudoscalar (e_1^e_2^e_3…^e_n or I_n) that represents the entire space (Hildenbrand & Oldenburg, 2015).

The main product used in Geometric Algebra is called the geometric product (Hildenbrand & Oldenburg, 2015), which is defined as the summation of the inner (dot) product and the outer (cross) product (see Equation 1). Applying the geometric product (***ab***) between two vectors ***a*** and ***b*** can produce a multivector consisting of different k-blade parameters (Dorst et al., 2007, p. 151). The definition of the inner product ($\boldsymbol{a} \cdot \boldsymbol{b}$) and outer product ($\boldsymbol{a} \wedge \boldsymbol{b}$) that are derived from the geometric product (***ab***) are presented below in Equations 2 and 3 respectively.

Geometric Product: $$\boldsymbol{ab} = \boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{a} \wedge \boldsymbol{b} \qquad [1]$$

Inner (dot) Product: $$\boldsymbol{a} \cdot \boldsymbol{b} = \frac{1}{2}(\boldsymbol{ab} + \boldsymbol{ba}) \qquad [2]$$

Outer (cross) Product: $$\boldsymbol{a} \wedge \boldsymbol{b} = \frac{1}{2}(\boldsymbol{ab} - \boldsymbol{ba}) \qquad [3]$$

These equations show that the inner and outer products are the symmetric and antisymmetric components of the geometric product. The inner and outer products are both commonly used in 3-D Vector Algebra to describe metrics between two vectors. The inner product calculates scalar information such as distances or angles. The outer product can be used span two vectors to create higher dimensional spatial objects such as lines and planes (see Equations 4-6).

The conformal model of geometric algebra (CGA) embeds 3-D Euclidean (x, y, z) space $\mathbf{R}^3$ into an orthogonal 5-D conformal vector space (n_o, e_1, e_2, e_3, n_i) using an $\mathbf{R}^{4,1}$ metric where e_1, e_2, and e_3 are the x, y, and z vectors respectively. The two added vectors are the point at the origin (n_o) and the point at infinity (n_i) (Dorst et al., 2007, p. 9). The point at the origin is the origin of the system and point at infinity is defined by Dorst et al. (2007, p.356, 359) as a point that has infinite distance to all finite points, is common to all lines and planes, and is invariant under Euclidean transformations. The Conformal model of Geometric Algebra refers to the 1-blades as vectors, the 2-blades as bivectors, the 3-blades as trivectors, and the 4-blades as quadvectors. There are 32 k-blades associated with CGA that can be seen in Table 1 below.

Table 1: 32 Subspace Blade Parameters of CGA R4,1

Vector Subspace Blade Parameters of CGA (R4,1)					
Scalar (0-blades)	**Vector (1-blades)**	**Bivector (2-blades)**	**Trivector (3-blades)**	**Quadvector (4-blades)**	**Pseudoscalar (5-blade)**
1	no	no^e1	no^e1^e2	no^e1^e2^e3	no^e1^e2^e3^ni
	e1	no^e2	no^e1^e3	no^e1^e2^ni	
	e2	no^e3	no^e1^ni	no^e1^e3^ni	
	e3	no^ni	no^e2^e3	no^e2^e3^ni	
	ni	e1^e2	no^e2^ni	e1^e2^e3^ni	
		e1^e3	no^e3^ni		
		e1^ni	e1^e2^e3		
		e2^e3	e1^e2^ni		
		e2^ni	e1^e3^ni		
		e3^ni	e2^e3^ni		

A variety of geometric objects can be represented using the k-blade parameters presented in Table 1. Yu et al. (2016) make a distinction between CGA objects as either being rounds, flats, Euclidean blades, free blades, and tangent blades. Rounds can have finite length, area, or volume while flats include an (n_i) component and can stretch outwards to infinity. Flat objects will be referred to as 'projected objects' for the rest of this paper. The CGA spatial objects used in this research to help represent 3-D boundary components (null vectors, projected lines, and projected planes) can be seen in Figure 4 below.

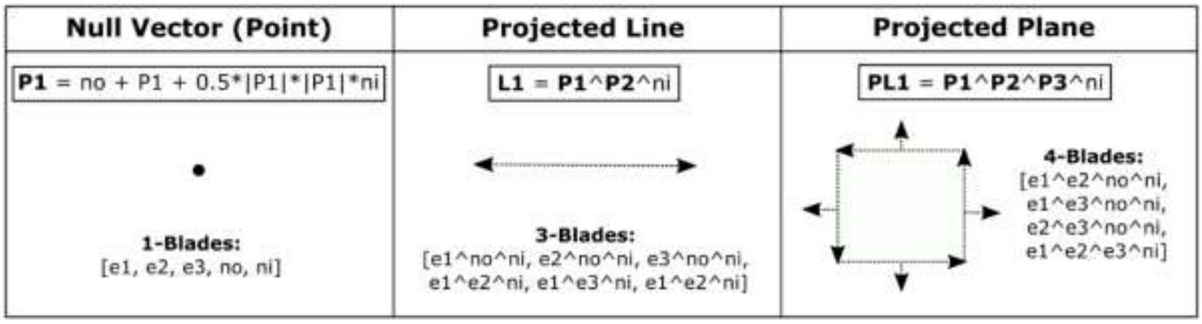

Figure 4: CGA Objects of Interest (with Blade Parameters)

The equation used to generate the CGA parameters for a null vector (point) $\boldsymbol{P}$ is presented in Equation 4. A 3-D point of $\mathbf{R}^3$ is embedded into the 5-D conformal space $\mathbf{R}^{4,1}$ by setting the n_o, e_1, e_2, and e_3 parameters to 1, x, y, z respectively, and the n_i parameter to $0.5(x^2 + y^2 + z^2)$.

Null Vector: $$\boldsymbol{P} = \boldsymbol{n_0} + (x)\,\boldsymbol{e_1} + (y)\,\boldsymbol{e_2} + (z)\,\boldsymbol{e_3} + \frac{1}{2}(x^2 + y^2 + z^2)\,\boldsymbol{n_i} \qquad [4]$$

The equations used to generate the CGA parameters for a projected line ($\boldsymbol{L}$) and plane ($\boldsymbol{PL}$) are presented in Equations 5 and 6 respectively.

Projected Line: $$\boldsymbol{L} = \boldsymbol{P_1} \wedge \boldsymbol{P_2} \wedge n_i \qquad [5]$$
Projected Plane: $$\boldsymbol{PL} = \boldsymbol{P_1} \wedge \boldsymbol{P_2} \wedge \boldsymbol{P_3} \wedge n_i \qquad [6]$$

Equation 5 generates a projected line $\boldsymbol{L}$ that connects 2 points ($\boldsymbol{P_1}$ and $\boldsymbol{P_2}$) while Equation 6 generates a projected plane $\boldsymbol{PL}$ that contains 3 points ($\boldsymbol{P_1}$, $\boldsymbol{P_2}$, and $\boldsymbol{P_3}$). There are six 3-blade parameters that represent a projected line $\boldsymbol{L}$ and there are four 4-blade parameters that represent a projected plane $\boldsymbol{PL}$ in $\mathbf{R}^{4,1}$. All these blades include an 'n_i' component.

Equations 7, 8, and 9 can be used to derive geometric and topological information between CGA objects. Equation 7 is used to calculate the closest distance $\boldsymbol{D_{L\text{-}P}}$ between a projected line $\boldsymbol{L}$ and a null vector point $\boldsymbol{P}$ and is used in this research to determine if a point and line are collinear. The norm of a multivector is defined as its 0-blade scalar components.

Line & Point Distance $$\boldsymbol{D_{L\text{-}P}} = norm(\boldsymbol{L} \cdot \boldsymbol{P}) \qquad [7]$$
Dualization: $$\boldsymbol{M}^* = \boldsymbol{M} \rfloor I_n^{-1} \qquad \textit{(Dorst et al., 2007, p. 80)} \qquad [8]$$
Intersect/Meet: $$\boldsymbol{Meet(A, B)} = \boldsymbol{B}^* \rfloor \boldsymbol{A} \qquad \textit{(Dorst et al., 2007, p. 131)} \qquad [9]$$

Equation 8 and Equation 9 use the left contraction (⌋) which is a generalization of the inner product for multivectors. Dorst et al. (2007, p. 75-77) say that the left contraction between two multivectors $\boldsymbol{A}$ and $\boldsymbol{B}$ can be described as the subspace of $\boldsymbol{A}$ that can be taken out of $\boldsymbol{B}$. The dualization $\boldsymbol{M}^*$ of a multivector $\boldsymbol{M}$ (Equation 8) can be viewed as the perpendicular component of a multivector with respect to the space (I_n) it resides in (Dorst et al., 2007, p. 80). Equation 9 can be applied as a general intersection/meet operation between two multivectors $\boldsymbol{A}$ and $\boldsymbol{B}$. The meet operation is applied to sets of projected CGA boundary component representations in this research to initially categorize the projected relationship that is occurring between two 3-D boundary components (see Figure 6) by interpreting the format of the meet return geometry.

2.3.2 Modelling Cadastral Objects using CGA

There have been multiple studies in recent years that use CGA to model and perform analysis on cadastral objects. Yu et al. (2016) presented a framework to compute geometry oriented topological relations between 3-D objects being rendered using CGA. They used the concepts of an object being inside, on, or outside another object to create a general topology operator for determining 8 relationships between multidimensional objects. Results were presented between sets of disjoint 3-D objects (Yu et al., 2016). Zhang et al. (2016) used the intersection (meet) results of CGA projected lines and planes along with various other CGA operation checks to create specific judgement rules for testing if two 3-D boundary components intersect or not. Six sets of judgement rules were determined for the point-point, line-point, polygon-point, line-line, polygon-line, and polygon-polygon boundary object pairs. By first performing intersections between CGA projected objects, the complexity of the 3-D object intersection problem was reduced to several smaller problems. These problems were approached differently based on the return geometry type of the projected CGA intersections.

3. METHODS

The methods are presented in two sections. Section 3.1 describes the processes that were followed to generate digital parameters for the boundary components associated with the two 3-D cadastral units that were registered through a condominium survey plan. Section 3.2 describes the general processes that were followed to classify the topological relationships between the two 3-D cadastral units and their lower dimensional boundary components.

3.1 Generating 3-D Boundary Component Parameters from Survey Plan Drawings

The 3-D cadastral units modelled in this study are defined as having a 3-D interior extension, as well as 2-D, 1-D, and 0-D boundary limits with respect to the Dimensional Model. Portrayed in Figure 5, this section presents the formal definitions for the 3-D point (BP), 3-D line (BL), and 3-D plane (BPL) boundary components that were used to represent the boundaries of 3-D cadastral units. An overview of these 3-D boundary components, including the point sets and CGA parameters that are used to define them are shown in Figure 5 below. With respect to the dimensional model, Figure 5a shows a 3-D boundary point that has a 0-D interior extention and no limit, Figure 5b shows a 3-D boundary line that has a 1-D interior extention and two 0-D (start and end) boundary limits, and Figure 5c shows a 3-D boundary plane that has a 2-D interior extention, four 1-D boundary limits, and four 0-D boundary limits.

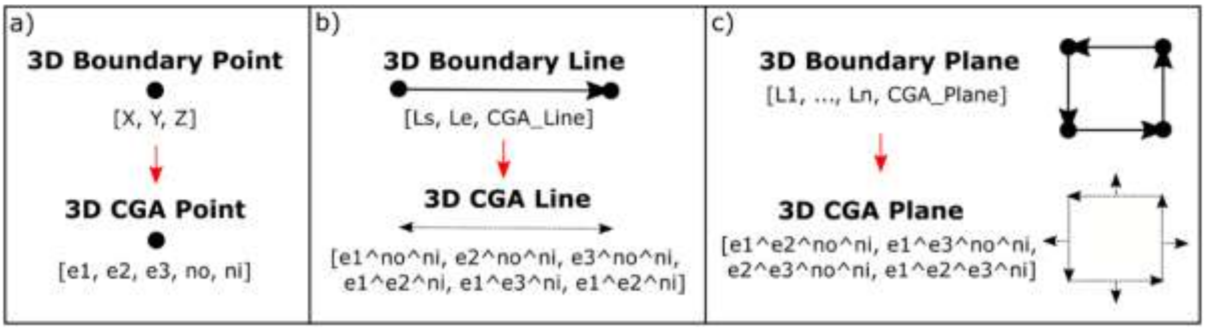

Figure 5: Boundary Component Limits, Extensions, and CGA Representations

The geometry for each 3-D cadastral unit modelled here is defined by a set of georeferenced 3-D boundary points with [x, y, z] coordinates. To generate parameters for each boundary point, 3-D point coordinates are first calculated by applying direct (polar) calculations that use distance and angle measurements included in the survey plan drawings, which are common to those in the surveying profession. Additional CGA parameters are generated using Equation 4. Boundary lines are defined on a survey plan by the line that connects two 3-D boundary points. The geometry of each boundary line in this study is indirectly represented through the start and end 3-D point coordinates (Ls, Le) that bound it. The interior extension of each boundary line is represented by six CGA 3-blade parameters generated using Ls, Le, and Equation 5. Boundary planes are represented on a survey plan by identifying a closed set of coplanar points and lines that define it. In this study, four sequential boundary lines (L1, …, Ln) define a boundary plane. The interior extension of each boundary plane is represented by four CGA 4-blade parameters generated using 3 boundary points that define the plane and Equation 6.

3.2 Topological Relationship Classification Processes

The relationship classification process relies on evaluating relationships between all sets of boundary component pairs, determining projected relationships between them using CGA operations, and classifying final relationships using a series of additional CGA operations and 3-D point-point distance evaluations as is shown by Pullano & Barry (2020).

When determining the relationship between two 3-D cadastral units, relationships between all pairs of their lower dimensional boundary components are first classified individually and then combined to describe the relationships that are occurring between both 3-D units, if any exist. Table 2 below shows 9 sets of boundary component pairs that are evaluated and interpreted together to describe the relationship that occurs between two 3-D cadastral units A and B. There are six classification algorithms (point-point, line-point, line-line, plane-point, plane-line, plane-plane) that are applied to evaluate all 9 sets of boundary component pairs.

Table 2: Boundary Component Pairs between Cadastral Units A and B

Boundary Component Pairs between Units A and B			
	B Points	**B Lines**	**B Planes**
A Points	$Point_A$-$Point_B$	$Point_A$-$Line_B$	$Point_A$-$Plane_B$
A Lines	$Line_A$-$Point_B$	$Line_A$-$Line_B$	$Line_A$-$Plane_B$
A Planes	$Plane_A$-$Point_B$	$Plane_A$-$Line_B$	$Plane_A$-$Plane_B$

Relationships between boundary components are initially categorized using the concepts of projected CGA objects being parallel, collinear, coplanar, or intersecting. This follows the approach of Zhang et al. (2016). The projected relationship categories that can exist between five of the 6 types of boundary component pairs is presented in Figure 6 below using disjoint scenarios. There are no projected relationship categories between point-point component pairs.

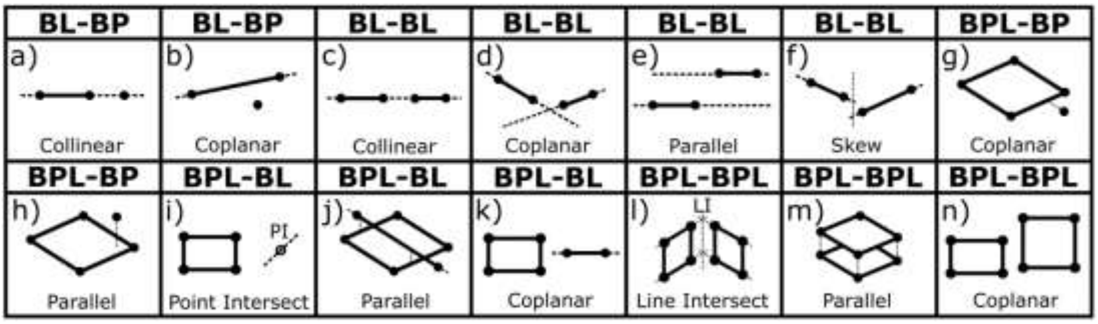

Figure 6: Projected CGA Relationship Categories Shown using Disjoint Examples

These categories are determined by calculating and interpreting CGA intersection operations (see Equations 7 and 9) between boundary component pairs as can be seen in Table 3 below. These categorizations allow for each relationship classification problem to be simplified into a set of smaller problems.

Table 3: Projected CGA Intersection Interpretations

CGA Object Intersections and Return Geometries							
Boundary Component Pair	**CGA Intersection operator**	**Not Parallel**	**Parallel**	**Collinear**	**Coplanar**	**Intersection Point**	**Intersection Line**
Line-Point	Equation 7	N/A	Scalar (+)	Scalar (0)	N/A	N/A	N/A
Plane-Point	Equation 9	N/A	Scalar (+/-)	N/A	Point	N/A	N/A
Line-Line	Equation 9	Free Scalar	Free Vector	Line	Flat Point	N/A	N/A
Plane-Line	Equation 9	N/A	Free Vector	N/A	Line	Flat Point	N/A
Plane-Plane	Equation 9	N/A	Free Bivector	N/A	Plane	N/A	Line

Table 3 shows that the geometry of the intersection results between boundary component pairs can be scalars, free scalars, free vectors, free bivectors, flat points, points, lines, or planes. These data types are identified by evaluating what k-blade parameters exist in each intersection result.

Final overlap relationships between boundary component pairs are described using four characteristics that are additional to the projected relationship category they were placed in. The first and second characteristics are the dimensionalities of the overlap on boundary components one and two respectively. The third characteristic is the geometry type (point, line, or plane) of the overlap between the two boundary components being considered. The fourth characteristic is the topological relationship description of the overlap and can be 'Disjoint', 'Touch' or 'Intersect' (see section 2.2). Disjoint relationships do not use the first three characteristics.

Six main MATLAB functions with additional secondary functions were presented by Pullano & Barry (2020) to evaluate the relationship classifications between all point-point, line-point, line-line, plane-point, plane-line, and plane-plane boundary component pair sets. The six main functions first determined which projected relationship was occurring between each boundary component pair set and then passed the necessary data from each boundary component pair into the required secondary function. The secondary functions evaluated a series of 3-D point-point distance calculations along with additional CGA operations in some cases to determine the final relationship classifications for each boundary component pair. The processes and evaluations for these main and secondary functions are described in detail by Pullano & Barry (2020).

4. RESULTS AND DISCUSSION

Results are presented in two sections. Section 4.1 provides an overview of the total number of relationships that can be classified using the methods that were put forward and validated by Pullano & Barry (2020) and were applied in this paper. Section 4.2 presents experimental results from a dataset consisting of two 3-D cadastral units that were derived from a condominium survey plan registered in Alberta, Canada.

4.1 Summary of Topological Relationship Classifications

Pullano & Barry (2020) created and tested seven simulated experimental datasets using the classification processes presented, each consisting of two cube-like 3-D cadastral units with the relationships between them known *a priori* to testing. Between these datasets, 38 distinct overlap relationships could be be classified. These are presented in Figure 7 below.

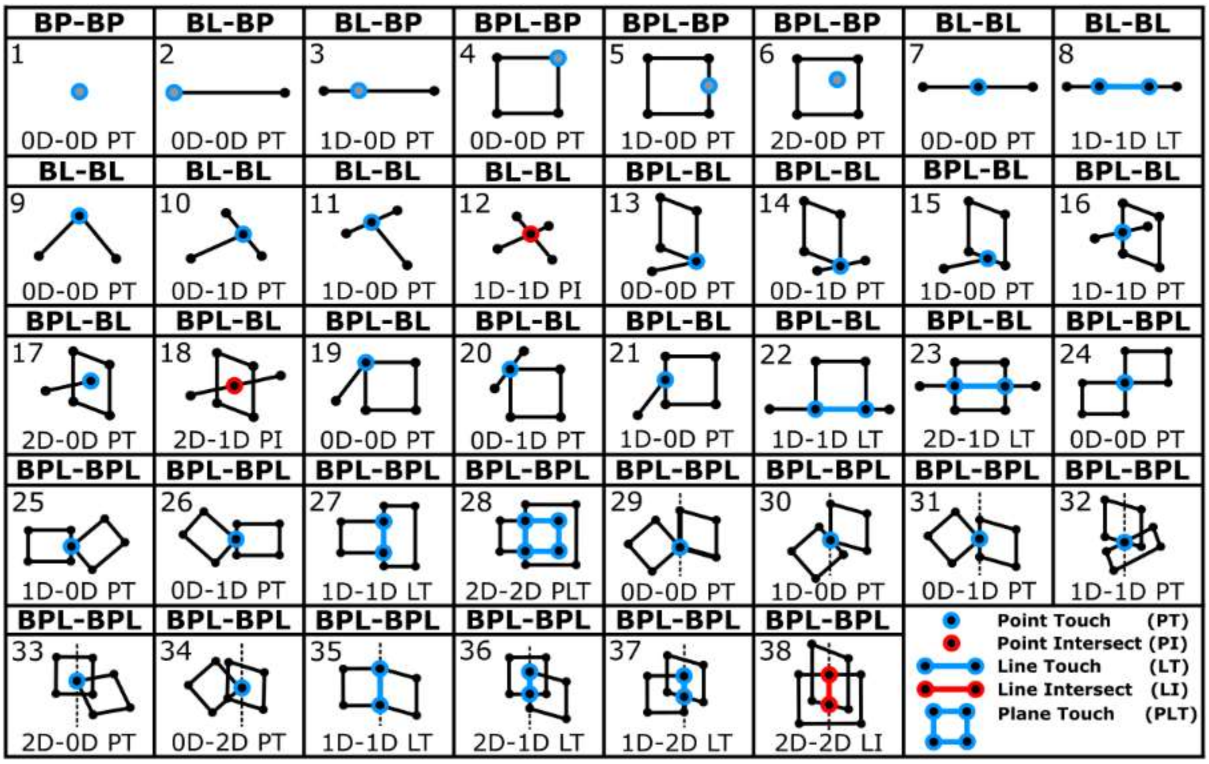

Figure 7: Overlap Relationship Classifications between Component Pairs

Figure 7 shows 38 relationship classifications describing that describe some form of overlap relationship existing between the boundary component pairs considered here. There are 1, 2, 3, 6, 11, and 15 distinct overlap relationships that can exist between point-point, line-point, plane-point, line-line, plane-line, and plane-plane boundary component pairs respectively. The methods put forward by Pullano & Barry (2020) were also able to distinguish 15 distinct disjoint relationships that are not considered in this paper.

4.2 Condominium Experimental Results

An experiment using two 3-D cadastral units derived from a registered condominium survey plan in Alberta, Canada was tested to show how the methods developed and tested by Pullano & Barry (2020) could be applied to analyse a practical 3-D cadastral boundary problem. For the example chosen, the condominium building footprint was confirmed to be contained within the 2-D parcel it was registered to by interpreting its site layout drawing. Figure 8 below includes two drawings that were used to calculate local 3-D boundary point coordinates for the two cadastral units considered here.

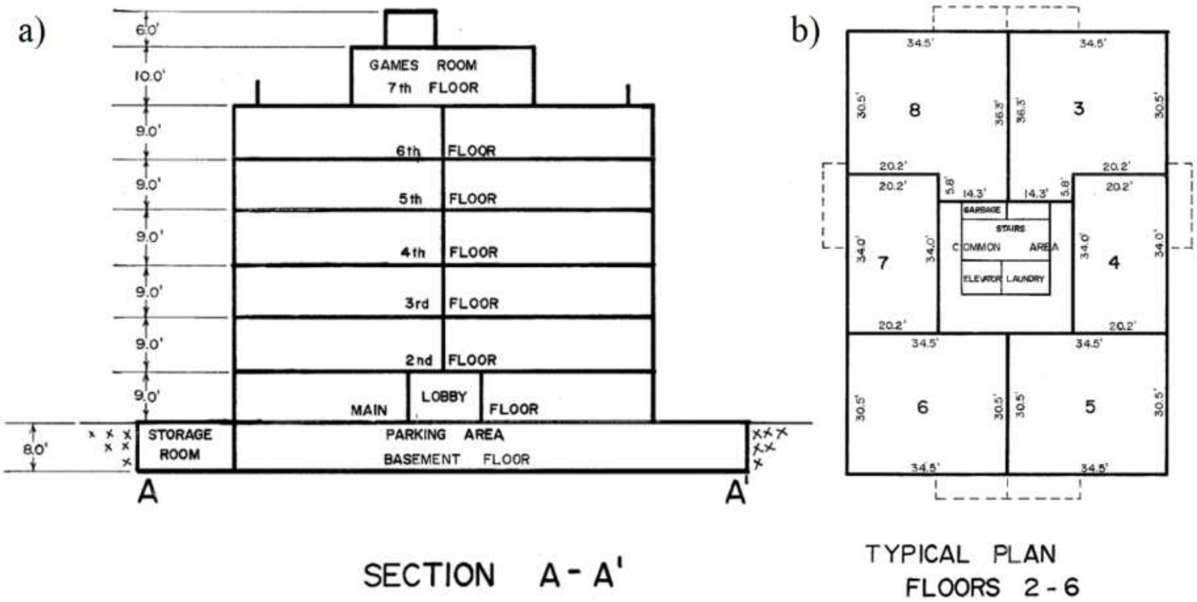

Figure 8: Condominium Building Cross Section and Typical Floor Plan

The first step in generating parameters for the boundary components of these units was to calculate the 3-D boundary points associated to each unit. A local coordinate system was used here where (0,0,0) was the x, y, z coordinate of the southwest corner of the 1^{st} floor of the building.

Figure 8a shows the side cross section view of the condominium building. Measurements on this drawing were used to derive (z) elevations for the boundary points of both 3-D cadastral units. Figure 8b shows the layout for floors 2 – 6 of the condominium building. Measurements on this drawing were used to calculate the (x, y) horizontal boundary point coordinates of both 3-D cadastral units. 3-D cadastral units 5 and 6 were chosen for this experiment from the second floor of the condominium building.

Boundary lines and planes were generated so that plane lines had clockwise rotation when looking at each plane from the inside of each unit. Unit 6 was run through the code as 3-D cadastral unit A and unit 5 was ran through the code as 3-D cadastral unit B. Visualizations for the boundary points, lines, and planes of each unit can be seen in Figure 9 below.

After the point coordinate geometry and point sets used to define boundary lines and planes were determined, digital CGA parameters for the 3-D boundary points, lines, and planes associated with 3-D cadastral units A and B were generated using the methods described in section 3.1 above.

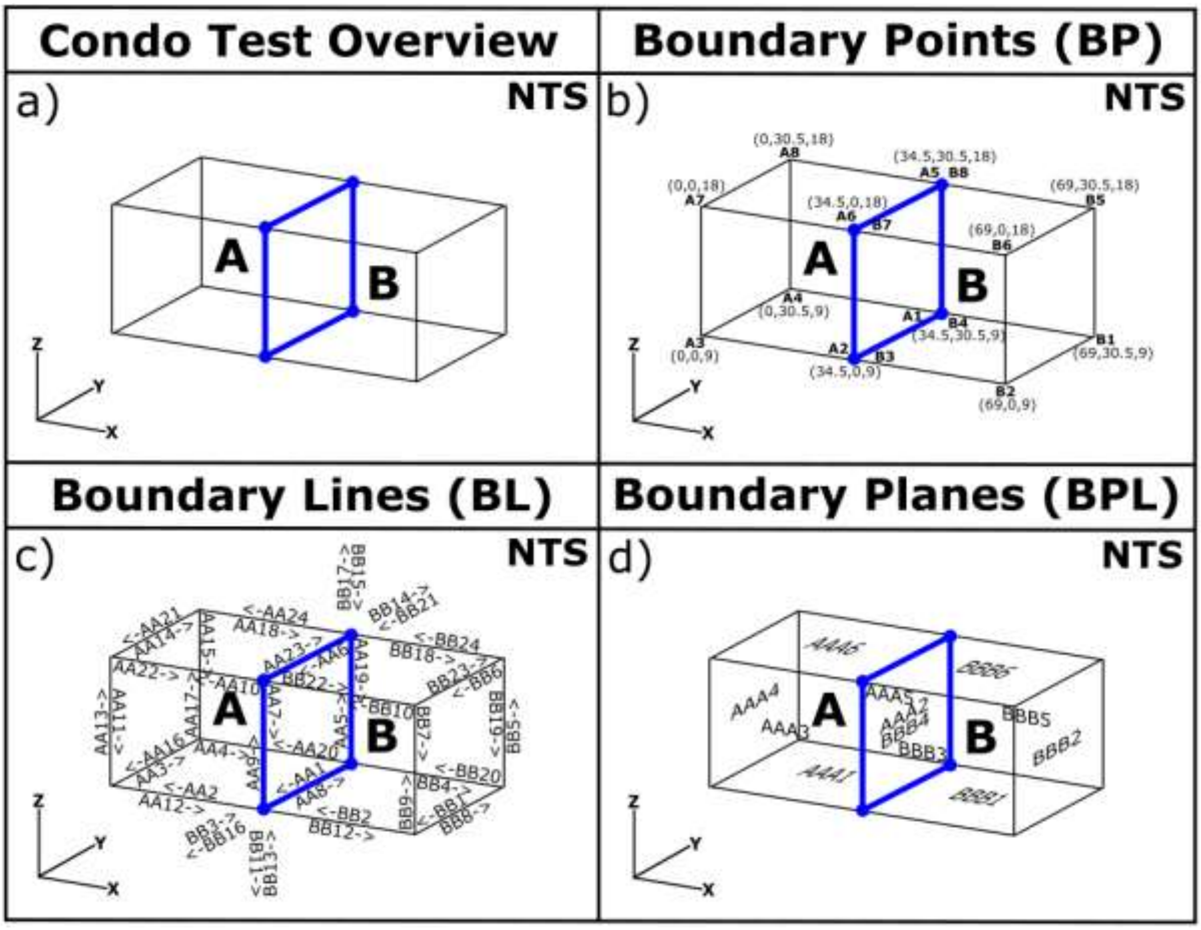

Figure 9: Condominium Experiment Dataset Visualization

The visualization drawings in Figure 9 are not drawn to scale. Figure 9a shows an overview of simulated 3-D cadastral units A and B used in the condominium experimental test. Figure 9b shows the names (A#/B#) and 3-D coordinates for all boundary points, Figure 9c shows the names (AA#/BB#) and topological directions for all boundary lines, and Figure 9d shows the names (AAA#/BBB#) for all boundary planes. The main overlap between the two 3-D cadastral units in this experiment can be described by the full plane overlap existing between them. A boundary plane of Unit A (AAA2) fully touches a boundary plane of unit B (BBB4) at a plane segment. Figure 10 below shows the 13 different overlap relationship classifications between 3-D condominium units A and B that were known to exist *a priori* to testing.

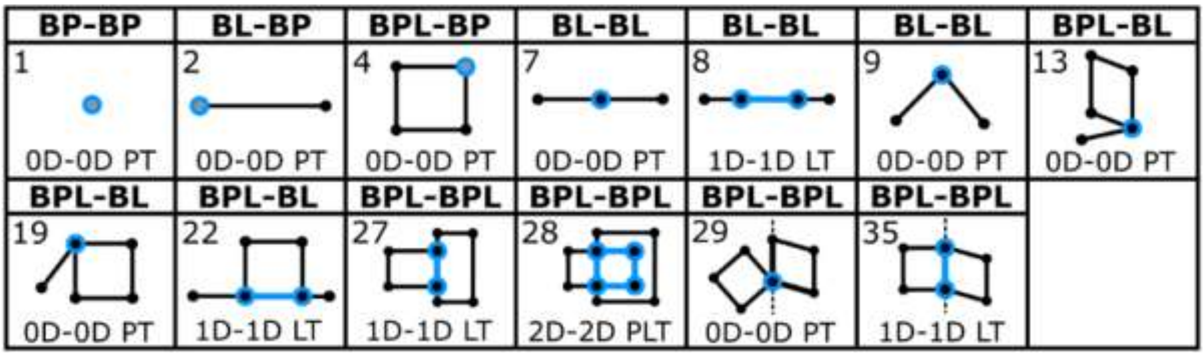

Figure 10: Condominium Experiment - Existing Overlap Relationship Classifications

This condominium experiment resulted in numerous occurrences of each of the relationship classifications shown in Figure 10 above. Table 4 below shows a compiled sample of the MATLAB classification program output (1 of each relationship classified) for the condominium dataset that was tested. It is split up into relationships between the six component pair types. Each component pair producing an overlap relationship classification was manually verified through visual inspection to be one of the *a priori* relationships shown in Figure 10 above by comparing the program output from Table 4 with the visualizations presented in Figure 9. The program correctly classified all boundary component pair relationships that were known *a priori* to running the experiment.

Table 4: Condominium Experiment – Sample of Program Output

Condominium Test - Component Pair Overlap Classification Results

BP-BP

Component Pair	Projected Relationship	Relationship Description	Overlap Classification	Geometry
A1-B4	N/A	0D-0D PT	1	[34.5,30.5,9]

BL-BP

Component Pair	Projected Relationship	Relationship Description	Overlap Classification	Geometry
BB11-A2	Collinear	0D-0D PT	2	[34.5,0,9]

BPL-BP

Component Pair	Projected Relationship	Relationship Description	Overlap Classification	Geometry
A5-BBB4	Coplanar	0D-0D PT	4	[34.5,30.5,18]

BPL-BL

Component Pair	Projected Relationship	Relationship Description	Overlap Classification	Geometry
AA10-BBB4	P Intersect	0D-0D PT	13	[34.5,0,18]
BB18-AAA5	Coplanar	0D-0D PT	19	[34.5,30.5,18]
AA19-BBB4	Coplanar	0D-0D PT	–	[34.5,30.5,18]
		1D-1D LT	22	[34.5,30.5,18] | [34.5,30.5,9]
		0D-0D PT	–	[34.5,30.5,9]

BL-BL

Component Pair	Projected Relationship	Relationship Description	Overlap Classification	Geometry
AA10-BB10	Collinear (C)	0D-0D PT	7	[34.5,0,18]
AA19-BB15	Collinear (C)	1D-1D LT	8	[34.5,30.5,18] | [34.5,30.5,9]
AA19-BB16	Coplanar	0D-0D PT	9	[34.5,30.5,9]

BPL-BPL

Component Pair	Projected Relationship	Relationship Description	Overlap Classification	Geometry
AAA1-BBB1	Coplanar	0D-0D PT	–	[34.5,0,9]
		1D-1D LT	27	[34.5,0,9] | [34.5,30.5,9]
		0D-0D PT	–	[34.5,30.5,9]
AAA2-BBB4	Coplanar	0D-0D PT	–	[34.5,0,9]
		1D-1D LT	22 (28)	[34.5,0,9] | [34.5,0,18]
		0D-0D PT	–	[34.5,0,18]
		1D-1D LT	22 (28)	[34.5,0,18] | [34.5,30.5,18]
		0D-0D PT	–	[34.5,30.5,18]
		1D-1D LT	22 (28)	[34.5,30.5,18] | [34.5,30.5,9]
		0D-0D PT	–	[34.5,30.5,9]
		1D-1D LT	22 (28)	[34.5,30.5,9] | [34.5,0,9]
AAA5-BBB1	L Intersect	0D-0D PT	29	[34.5,30.5,9]
AAA2-BBB6	L Intersect	0D-0D PT	–	[34.5,30.5,18]
		1D-1D LT	35	[34.5,30.5,18] | [34.5,0,18]
		0D-0D PT	–	[34.5,0,18]

Examining Table 4, the program identified 4 BP-BP pairs that produced the ‘0D-0D Point Touch’ relationship (classification 1). The program identified 48 BL-BP pairs that produced the ‘Collinear 0D-0D Point Touch’ relationship (classification 2). The program identified 24 BPL-BP pairs that produced the ‘Coplanar 0D-0D Point Touch’ relationship (classification 4).

The program identified 16 BL-BL pairs that produced the ‘Collinear 0D-0D Point Touch’ relationship (classification 7), 16 BL-BL pairs that produced the ‘Collinear 1D-1D Line Touch’ relationship (classification 8), and 96 BL-BL pairs that produced the ‘Coplanar 0D-0D Point Touch’ relationship (classification 9).

The program identified 48 BPL-BL pairs that produced the ‘Point Intersect 0D-0D Point Touch’ relationship (classification 13), 32 BPL-BL pairs that produced the ‘Coplanar 0D-0D Point Touch’ relationship (classification 19), and 32 BPL-BL pairs that produced the ‘Coplanar 1D-1D Line Touch’ relationship (classification 22).

The program identified 4 BPL-BPL pairs that produced the ‘Coplanar 1D-1D Line Touch’ relationship (classification 27) and 1 BPL-BPL pair that produced the ‘2D-2D Plane Touch’ relationship (classification 28) that consisted of 4 ‘Coplanar 1D-1D Line Touch’ segments (classification 22). The program identified 8 BPL-BPL pairs that produced the ‘Line Intersect 0D-0D Point Touch’ relationship (classification 29) and 8 BPL-BPL pairs that produced the ‘Line Intersect 1D-1D Line Touch’ relationship (classification 35).

5. CONCLUSIONS

This study applied data modelling processes that used CGA theory, objects, and operations, as well as 3-D point-point distance evaluations to classify geometrical and topological relationships that exist between the (point, line, and plane) components defining the boundaries of two 3-D cadastral units. These processes were initially developed to identify various types of disjoint, touch, and intersect relationships between six sets of 3-D boundary component pairs. The six classification algorithms were applied in this paper to validate the shared plane overlap between two cadastral units registered with a condominium survey plan in Alberta, Canada.

The theoretical contribution of this research is that methodological data flow processes and algorithms were developed and tested for the purpose of classifying 53 distinct geometric and topological relationships between six sets of 3-D boundary component pairs. The experimental results from the datasets that were tested support how the methods that were developed using CGA along with 3-D point-point distance evaluations can be applied to correctly classify topological relationships between 3-D cadastral boundaries.

The practical contribution of this research is that it showed how the methodological theory developed by Pullano & Barry (2020) can be applied to solving a practical 3-D cadastral boundary problem example in the land surveying field, specifically towards validating a shared boundary between two adjacent 3-D cadastral units as intended on the survey plan prior to registering it. An experiment using a real cadastral example was derived using measurements included in a 3-D condominium survey plan to validate the shared plane boundary between two adjacent 3-D cadastral units. While this type of relationship analysis can be done through visual inspection of survey plans, the methods developed here are more mathematically rigorous.

None of the results from the condominium dataset tested show the relationship classification methods applied here to be false. Experimental results still require validation through visual inspection as an independent method to verify that the results were correct. The methods applied here were not verified as being universally correct with all 3-D cadastral units, but rather have been validated with the simple cube-like condominium experiment chosen.

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