1 Problem Text

The 5-digit number, $16807 = 7^5$, is also a fifth power. Similarly, the 9-digit number, $134217728 = 8^9$, is a ninth power.

How many n-digit positive integers exist which are also an nth power?

2 Solution

A number of the form x^y has $\lfloor \log x^y \rfloor + 1$ digits in base 10. We are asked for how many $x, y \in \mathbb{Z}^+$ does the following equation hold:

$$\lfloor \log x^y \rfloor + 1 = y \tag{1}$$

Firstly, extract the exponent from the logarith:

$$\lfloor \log x^y \rfloor + 1 \equiv \lfloor y \cdot \log x \rfloor + 1 \tag{2}$$

Secondly, get rid of the floor function. By this function's definition:

$$\lfloor y \cdot \log x \rfloor \leq y \cdot \log x < \lfloor y \cdot \log x \rfloor + 1 \tag{3}$$

Note that by (1), $\lfloor y \cdot \log x \rfloor = y - 1$; likewise $\lfloor y \cdot \log x \rfloor + 1 = y$. Combined with equation (3), we have:

$$y - 1 \leq y \cdot \log x < y \tag{4}$$

Dividing the whole expresion by y:

$$1 - \frac{1}{y} \leq \log x < 1 \tag{5}$$

Finally, we have an upper bound for x, namely $\log x < 1 \iff x < 10$.

Regarding the exponent y, for all integers $0 < x < 10$, (5) is satisfied if and only if $1 - \frac{1}{y} \leq \log x$. Solving for y, $y < \frac{1}{1 - \log x}$. Given that the exponent must be a positive integer, $\lfloor \frac{1}{1-\log x} \rfloor$ represents the number of exponents resulting in an n-digit number for $0 < x < 10$.

Python one-liner The following Python code calculates the result based on the previous observations.

```
from math import log10, floor;
sum( (floor(1 / (1-log10(i))) for i in xrange(1,10) ) )
```

The result is 49.